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Symmetrised Doubly Non Central, Nonsingular Matrix Variate Beta Distribution

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Abstract. In this paper, we determine the symmetrised density of a nonsingular doubly noncentral matrix variate beta type I and II distributions under different definitions.

Keywords. Random matrices; noncentral distribution; symmetrised doubly noncentral distribution; matrix variate beta.

1 Introduction

In the univariate case, the doubly non-central beta type II distribution (also termed doubly non-central $\mathcal{F}$ distribution) has been studied by Searle (1971) and Tiku (1965). This distribution has been utilized to find power functions for the analysis of variance tests in the presence of an interaction for the two-way model layout with one observation per cell (see Bulgren, 1971). It has also been used in engineering problems in the context of information theory to calculate the error probability for a particular binary signalling system in which the receiver tries to learn the state of a multiple parallel link noise perturbed channel (see Price, 1962). Doubly non-central distributions have also been applied to problems in communications, in signals captured through radar, and pattern recognition where quadratic forms on Normal data are involved (see, for example, (Turin, 1959), Kailath (1961), Sebestyen (1961) and Wishner (1962)).

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In the multivariate case, the matrix variate beta type I and II distributions for the central, non-central and doubly non-central cases have been studied by different authors from diverse approaches, see Olkin and Rubin (1964), Khatri (1970), Chikuse (1980), Muirhead (1982), Cadet (1996), Gupta and Nagar (2000) and Díaz-García and Gutiérrez-Jáimez (2001), among many others. In particular, doubly non-central distributions play a very important role in testing the power of hypotheses in the context of multivariate analysis, such as canonical correlation analysis and general linear hypothesis in MANOVA, see Muirhead (1982) and Srivastava (1968). Moreover, the univariate problems mentioned above in the context of the theory of information and communication have recently been studied in the multivariate case, and doubly non-central matrix variate distributions have again been featured in these studies, see Ting, et al. (2004), Ratnarajah and Vaillancourt (2005), among others.

In general, the use of non-central, doubly non-central and, especially, beta-type distributions has not been developed as much as could be desired, due particularly to the fact that these distributions depend on hypergeometric functions with matrix argument, zonal or invariant polynomials. Until very recently, such functions were quite complicated to evaluate. Studies have recently appeared describing algorithms that are very efficient for the calculations, involving both of zonal polynomials and of hypergeometric functions with a matrix argument. These algorithms enable a broader and more efficient use of non-central distributions in general, see Gutiérrez, et al. (2000), Sáez (2004), Demmel and Koev (2006), Koev (2004), Koev and Edelman (2006) and Dimitria, et al. (2005).

In statistical literature, as well as in classification of the beta distribution as beta type I and type II (see Gupta and Nagar, 2000 and Srivastava and Khatri, 1979), two alternative definitions have been proposed for each of the latter. Initially, let us consider to the beta type I distribution. If A and B have a central Wishart distribution, i.e., $A \sim \mathcal{W}_m(r, I)$ and $B \sim \mathcal{W}_m(s, I)$ independently, then the beta matrix U can be defined as

$$U = \begin{cases} (A+B)^{-\frac{1}{2}} A((A+B)^{-\frac{1}{2}})', & \text{Definition 1 or,} \\ A^{\frac{1}{2}}(A+B)^{-1}(A^{\frac{1}{2}})', & \text{Definition 2,} \end{cases} \tag{1}$$

where $C^{1/2}(C^{1/2})' = C$ is a reasonable non-singular factorisation of C, see Gupta and Nagar (2000), Sirvastava and Khatri (1979) and Muirhead (1982). It is apparent that under Definitions 1 and 2, its density function is given by

$$\mathcal{B}I_m\left(U; \frac{r}{2}, \frac{s}{2}\right) = \frac{1}{\beta_m\left[\frac{r}{2}, \frac{s}{2}\right]} |U|^{\frac{r-m-1}{2}} |I_m - U|^{\frac{s-m-1}{2}}, \tag{2}$$

with $0 < U < I_m$, and is denoted as $U \sim \mathcal{B}I_m(r/2, s/2)$, where $r \geqslant m$ and

$s \geqslant m$; and $\beta_m[r/2, s/2]$ denotes the multivariate beta function defined by

$$\beta_m[b,a] = \int_{0<S<I_m} |S|^{\frac{a-(m+1)}{2}} |I_m - S|^{\frac{b-(m+1)}{2}} (dS) = \frac{\Gamma_m[a]\Gamma_m[b]}{\Gamma_m[a+b]},$$

where $\Gamma_m[a]$ denotes the multivariate gamma function and is defined as

$$\Gamma_m[a] = \int_{R>0} \operatorname{etr}(-R)|R|^{\frac{a-(m+1)}{2}} (dR),$$

$\operatorname{Re}(a) > (m-1)/2$, $\operatorname{etr}(\cdot) \equiv \exp(\operatorname{tr}(\cdot))$ and (dR) denote the exterior product of the $m(m+1)/2$ distinct differentials (dr_{ij}) of matriz dR, see Muirhead (1982).

An alternative definition for the beta type I matrix was proposed by Srivastava and Khatri (1979), Srivastava (1968), Muirhead (1982) and Gupta and Nagar (2000); it is formulated as follows:

Let $B \sim \mathcal{W}_m(s, I)$ and let us state $A = Y'Y$ where $Y \sim \mathcal{N}_{r\times m}(0, I_r \otimes I_m)$, $m > r$, independently of B, then, letting $U = Y(Y'Y + B)^{-1}Y' = Y(A + B)^{-1}Y'$, $U \sim \mathcal{B}I_r(m/2, (s+r-m)/2)$.

On the other hand, observe that if A has a noncentral Wishart distribution with matrix of noncentrality parameters Ω_1, i.e., $A \sim \mathcal{W}_m(r, I, \Omega_1)$ (or under the alternative definition, $Y \sim \mathcal{N}_{r\times m}(\mu, I_r \otimes I_m)$ with $\Omega_1 = \mu'\mu/2$), then U has a noncentral matrix variate beta type $I(B)$ distribution. Similarly, when $B \sim \mathcal{W}_m(s, I, \Omega_2)$, then U has a noncentral matrix variate beta type $I(A)$ distribution, see Greenacre (1973) and Gupta and Nagar (2000).

Note that in the central and non-central cases, the density, the properties and the associated distributions can be obtained from the definitions in (1) by replacing m by r, r by m and s by $s+r-m$, i.e., by making the substitutions

$$m \to r, \quad r \to m, \quad s \to s+r-m, \tag{3}$$

see Srivastava and Khatri (1979) or Muirhead (1982). For this reason, we shall focus our attention on the definitions stated in (1).

In an analogous way for the beta type II distributions, the following definitions have been proposed:

$$V = \begin{cases} B^{-\frac{1}{2}}A(B^{-\frac{1}{2}})', & \text{Definition 1,} \\ A^{\frac{1}{2}}B^{-1}(A^{\frac{1}{2}})', & \text{Definition 2,} \\ Y^{\frac{1}{2}}B^{-1}Y', & \text{Definition 3.} \end{cases} \tag{4}$$

The distribution is denoted by $V \sim \mathcal{B}II_m(r/2, s/2)$. In a similar way to the case of the beta type I distribution, the results under Definition 3 can be obtained from the results of Definition 2 and applying the transforms (3), see James (1964) and Muirhead (1982).

In this case, the central beta type II density under Definitions 1 and 2 is denoted and defined as

$$\mathcal{B}II_m\left(V;\frac{r}{2},\frac{s}{2}\right)=\frac{1}{\beta[\frac{r}{2},\frac{s}{2}]}|V|^{\frac{r-m-1}{2}}|I+V|^{\frac{-(r+s)}{2}},\qquad V>0.$$

In a similar way to the matrix variate beta type I distribution, if $A\sim\mathcal{W}_m(r,I,\Omega_1)$ (or under the alternative definition, $Y\sim\mathcal{N}_{r\times m}(\mu,I_r\otimes I_m)$ with $\Omega_1=\mu'\mu/2$), then U has a noncentral matrix variate beta type $II(B)$ distribution. And, if $B\sim\mathcal{W}_m(s,I,\Omega_2)$, then U has a noncentral matrix variate beta type $II(A)$ distribution, see Greenacre (1973), Gupta and Nagar (2000).

When these ideas are extended to the doubly non-central case, i.e. when $A\sim\mathcal{W}_m(r,I,\Omega_1)$ and $B\sim\mathcal{W}_m(s,I,\Omega_2)$, strictly speaking, we have not found the densities of the matrix variate beta types I and II distributions under Definitions 1 or 2. Rather, for the case of the beta type II distribution, Davis (1980) found the distribution of $\tilde{V}=\tilde{B}^{-1/2}\tilde{A}(\tilde{B}^{-1/2})'$ where $\tilde{A}=H'AH$ and $\tilde{B}=H'BH$, $H\in\mathcal{O}(m)$, with $\mathcal{O}(m)=\{H\in\Re^{m\times m}|HH'=H'H=I_m\}$. It is straightforward to show that the procedure proposed by Chikuse (1980) is equivalent to finding the symmetrised density defined by Greenacre (1973), see also Roux (1975).

In this paper, we obtain the symmetrised density function of the doubly noncentral matrix variate beta types I and II under the three definition proposed in the literature. Moreover, we determine the densities corresponding to the eigenvalues of the beta distribution types I and II. Clearly, the central and non-central distributions are obtained as particular cases of the distributions being studied. We propose this as a solution to the problem of determining the non-central beta densities, as described by Constantine (1963), Khatri (1970) and reconsidered in Farrell (1985) and Gupta and Nagar (2000), see also Díaz-García and Gutiérrez-Jáimez (2006).

2 Preliminary Results

In multivariate analysis we find several unsolved integrals which are involved in some noncentral and doubly noncentral distributions. One attempt for solving this problem is given by Greenacre (1973), who proposes a certain class of distributions called symmetrised distributions. The symmetrised distributions have some special properties which can solve most of the integrations in the cited distributions. In particular, the distributions of the eigenvalues in the nonsymmetrised and symmetrised cases coincide. This implies that any function of the eigenvalues, as the trace, the determinant, etc., can be determined from the symmetrised distribution. The approach of Greenacre is based on the following idea:

Given a function $f(X)$, $X : m \times m$, $X > 0$, Greenacre (1973) (see also Roux, 1975) proposes the following definition:

$$f_s(X) = \int_{\mathcal{O}(m)} f(HXH')(dH), \quad H \in \mathcal{O}(m) \tag{5}$$

where $\mathcal{O}(m) = \{H \in \Re^{m \times m} | HH' = H'H = I_m\}$ and (dH) denotes the normalised invariant measure on $\mathcal{O}(m)$ Muirhead (1982). This function $f_s(X)$ is termed the symmetrised function. We adopt the approach developed by Greenacre (1973) to determine te densities of the symmetrised doubly noncentral beta distributions. To do so, let us consider the following:

Theorem 1 *Let $X > 0$, $E > 0$ be $m \times m$ matrices, $a \geqslant (m-1)/2$, $b \geqslant (m-1)/2$ and*

$$g(X) = \int_{E>0} |E|^{\frac{a+b-(m+1)}{2}} \operatorname{etr}(-Q(X)E)\, C_\kappa \left\{\Theta E^{\frac{1}{2}} R(X) \left(E^{\frac{1}{2}}\right)'\right\} \times C_\lambda \left(\Xi E^{\frac{1}{2}} S(X) \left(E^{\frac{1}{2}}\right)'\right)(dE)$$

where $Q(X) > 0$, $R(X) > 0$ and $S(X) > 0$ are $m \times m$ matrix functions of the matrix X such that, $Q(HXH') = HQ(X)H'$, $H \in \mathcal{O}(m)$, with the same property for $R(X)$ and $S(X)$; $C_\kappa(M)$ is the zonal polynomial of M corresponding to the partition $\kappa = (k_1, \dots, k_m)$ of k with $\sum_{i=1}^m k_i = k$ and $C_\lambda(N)$ is the zonal polynomial of N corresponding to the partition $\lambda = (l_1, \dots, l_m)$ of l with $\sum_{i=1}^m l_i = l$. Then

$$g_s(X) = \sum_{\varphi \in \kappa \cdot \lambda} \frac{\Gamma_m[a+b](a+k)_\varphi}{|Q(X)|} \cdot \frac{C_\varphi^{\kappa,\lambda}(\Theta, \Xi) C_\varphi^{\kappa,\lambda}\{R(X)Q(X)^{-1}, S(X)Q(X)^{-1}\}}{C_\varphi(I)},$$

where $Q(X)^{-1}$ denotes the inverse of matrix $Q(X)$ (not the inverse function of $Q(\cdot)$), $C_\varphi^{\kappa,\lambda}(\cdot)$ is the invariant polynomial with two matrix arguments, $(t)_\tau$ is the generalised hypergeometric coefficient or product of Poch_ hammer symbols.

Proof: We have

$$g(X) = \int_{E>0} |E|^{\frac{a+b-(m+1)}{2}} \operatorname{etr}(-Q(X)E)\, C_\kappa \left(\Theta E^{\frac{1}{2}} R(X) \left(E^{\frac{1}{2}}\right)'\right) \times C_\lambda \left(\Xi E^{\frac{1}{2}} S(X) \left(E^{\frac{1}{2}}\right)'\right)(dE).$$

Consider the symmetrised function g and the transform $E = HEH'$. Noting that $(dE) = (dHEH')$, one has

$$g_s(X) = \int_{E>0} |E|^{\frac{a+b-(m+1)}{2}} \operatorname{etr}\left(-Q(X)E\right) \int_{\mathcal{O}(m)} C_\kappa \left(\Theta H E^{\frac{1}{2}} R(X) \left(E^{\frac{1}{2}}\right)' H'\right) \times C_\lambda \left(\Xi H E^{\frac{1}{2}} S(X) \left(E^{\frac{1}{2}}\right)' H'\right) (dH)(dE),$$

from Davis (1980) (see also Chikuse, 1980) and thus

$$g_s(X) = \sum_{\varphi \in \kappa\cdot\lambda} \int_{E>0} |E|^{\frac{a+b-(m+1)}{2}} \operatorname{etr}(-Q(X)E) \times \frac{C_\varphi^{\kappa,\lambda}(\Theta, \Xi) C_\varphi^{\kappa,\lambda}(R(X)E, S(X)E)}{C_\varphi(I)} (dE).$$

Now, from Davis (1980),

$$g_s(X) = \sum_{\varphi \in \kappa\cdot\lambda} \frac{\Gamma[(a+b), \varphi]_m}{|Q(X)|^{a+b}} \cdot \frac{C_\varphi^{\kappa,\lambda}(\Theta, \Xi) C_\varphi^{\kappa,\lambda}(R(X)Q(X)^{-1}, S(X)Q(X)^{-1})}{C_\varphi(I)},$$

where $\Gamma_m[(a+b), \varphi] = (a+b)_\varphi \Gamma_m[(a+b)]$, see Constantine (1963).

3 Doubly Noncentral Beta Type I Distribution

Theorem 2 *Suppose that U has a doubly non-central matrix variate beta type I under the definition 1, which is denoted as $U \sim \mathcal{B}I_{1m}(r/2, s/2, \Omega_1, \Omega_2)$. Then, using the notation for the operator sum as in Davis (1980), we have that its symmetrised density function is*

$$f_s(U) = \mathcal{B}I_m\left(U; \frac{r}{2}, \frac{s}{2}\right) \operatorname{etr}\left(-\tfrac{1}{2}(\Omega_1 + \Omega_2)\right) \times \sum_{\kappa,\lambda;\ \varphi}^{\infty} \frac{\left(\frac{1}{2}(r+s)\right)_\varphi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\ l!} \cdot \frac{C_\varphi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2) C_\varphi^{\kappa,\lambda}(U, (I-U))}{C_\varphi(I)}, \quad 0 < U < I.$$

Proof: By independence, the joint density of A and B is

$$f_{A,B}(A,B) = c|A|^{\frac{r-m-1}{2}} |B|^{\frac{s-m-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}(A+B)\right) \times {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 A\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 B\right), \tag{6}$$

where

$$c = \frac{\operatorname{etr}\left(-\frac{1}{2}\left(\Omega_1 + \Omega_2\right)\right)}{2^{\frac{m(r+s)}{2}} \Gamma_m\left[\frac{r}{2}\right] \Gamma_m\left[\frac{s}{2}\right]}. \tag{7}$$

By performing the transformation $C = A + B$ with $(dA) \wedge (dB) = (dA) \wedge (dC)$ and then the transformation $A = C^{1/2}U\left(C^{1/2}\right)'$ with $(dA) \wedge (dC) = |C|^{(m+1)/2}(dC) \wedge (dU)$, we find that the joint density of C and U is given by

$$f_{C,U}(C,U) = c|U|^{\frac{r-m-1}{2}}|I-U|^{\frac{s-m-1}{2}}|C|^{\frac{r+s-m-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}C\right)$$
$$\times {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 C^{\frac{1}{2}}U\left(C^{\frac{1}{2}}\right)'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 C^{\frac{1}{2}}(I-U)\left(C^{\frac{1}{2}}\right)'\right),$$

From which, by expanding the hypergeometric functions in infinite series of zonal polynomials and integrating with respect to C, the desired result now follows with the assistance of Theorem 1. Note that in this case $Q(\cdot) = \frac{1}{2}I$, $R(\cdot) = U$ and $S(\cdot) = (I - U)$ in Theorem 1.

Similarly, under Definition 2, we have:

Theorem 3 *Suppose that U has a doubly non-central matrix variate beta type I distribution under Definition 2, which shall be denoted as $U \sim \mathcal{B}I_{2m}(\frac{r}{2}, \frac{s}{2}, \Omega_1, \Omega_2)$. Then, its symmetrised density function is the same as in Theorem 2.*

Proof: By independence, the joint density of A and B is given by (6). Let $C = A + B$ with $(dA) \wedge (dB) = (dA) \wedge (dC)$ and consider the transform $C = (A^{1/2})'U^{-1}A^{1/2}$ with $(dA) \wedge (dC) = |A|^{(m+1)/2}|U|^{-(m+1)}(dA) \wedge (dU)$. Then, the joint density of A and U is given by

$$f_{A,U}(A,U) = c|U|^{-\frac{s+m+1}{2}}|I-U|^{\frac{s-m-1}{2}}|A|^{\frac{r+s-m-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}AU^{-1}\right)$$
$$\times {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 A)'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 A^{\frac{1}{2}}(I-U)U^{-1}\left(A^{\frac{1}{2}}\right)'\right).$$

The result follows from integrating with respect to A, taking $Q(\cdot) = \frac{1}{2}U^{-1}$, $R(\cdot) = I$ and $S(\cdot) = (I-U)U^{-1}$ and $C = A$ in Theorem 2.

Corollary 1 *Let $U \sim \mathcal{B}I_{jm}(s/2, r/2, \Omega_1, \Omega_2)$, $j = 1$ or 2, then the joint density function of the eignvalues $u_1, \ldots, u_m$ of U is*

$$f(u_1, \ldots, u_m) = \frac{\pi^{\frac{m^2}{2}} \operatorname{etr}\left(-\frac{1}{2}(\Omega_1 + \Omega_2)\right)}{\Gamma_m\left[\frac{m}{2}\right]\beta_m\left[\frac{r}{2}, \frac{s}{2}\right]} \prod_{i=1}^{m}\left\{u_1^{\frac{r-m-1}{2}}(1-u_i)^{\frac{s-m-1}{2}}\right\}$$
$$\times \prod_{i<j}^{m}(u_i - u_j) \sum_{\kappa,\lambda;\varphi}^{\infty} \frac{\frac{1}{2}(r+s)_\varphi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\, l!} \cdot \frac{C_\varphi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2) C_\varphi^{\kappa,\lambda}(\Lambda, (I-\Lambda))}{C_\varphi(I)},$$

where $1 > u_1 > \cdots > u_m > 0$ *and* $\Lambda = \operatorname{diag}(u_1, \ldots, u_m)$.

Proof: The proof follows immediately by applying Theorem 3.2.17 in Muirhead (1982) to the beta type I density in Theorem 2, using the Equation (3.12) in Chikuse (1980).

4 Doubly Noncentral Beta Type II Distribution

Theorem 4 *Suppose that $F > 0$ has a doubly non-central matrix variate beta type II distribution under Definition 1, denoted as*

$$F \sim \mathcal{B}II_{1m}\left(\frac{r}{2}, \frac{s}{2}, \Omega_1, \Omega_2\right).$$

Then, using the notation for the operator sum as in Davis (1980), we have that its symmetrised density function is

$$g_s(F) = \mathcal{B}II_m\left(F; \frac{r}{2}, \frac{s}{2}\right) \operatorname{etr}\left(-\tfrac{1}{2}(\Omega_1 + \Omega_2)\right)$$
$$\times \sum_{\kappa,\lambda;\varphi}^{\infty} \frac{\frac{1}{2}(r+s)_\varphi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\, l!} \cdot \frac{C_\varphi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2) C_\varphi^{\kappa,\lambda}((I+F)^{-1}F, (I+F)^{-1})}{C_\varphi(I)}.$$

Proof: The joint density function of A and B is given by (6). Transforming $F = B^{-1/2}A\left(B^{-1/2}\right)'$ and noting that $(dA) \wedge (dB) = |B|^{(m+1)/2}(dB) \wedge (dF)$, the joint density of B and F is

$$g_{F,B}(F, B) = c|F|^{\frac{r-m-1}{2}}|B|^{\frac{r+s-m-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}B^{\frac{1}{2}}(I+F)\left(B^{\frac{1}{2}}\right)'\right)$$
$$\times\, {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 B^{\frac{1}{2}} F \left(B^{\frac{1}{2}}\right)'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 B\right),$$

From which, by expanding the hypergeometric functions in infinite series of zonal polynomials and integrating with respect to B, and then taking $Q(\cdot) = 1/2(I+F)$, $R(\cdot) = F$ and $S(\cdot) = I$ in Theorem 1, the result is obtained.

From Definition 2, we have:

Theorem 5 *Suppose that F has a doubly non-central matrix variate beta type II distribution under Definition 2, which shall be denoted as $F \sim \mathcal{B}I_{2m}(r/2, s/2, \Omega_1, \Omega_2)$. Then, its symmetrised density function is the same as in Theorem 4.*

Proof: By independence the joint density function of A and B is given by (6). Now, we make the change of variable $F = (A^{1/2})'B^{-1}A^{1/2}$ observing that $(dA) \wedge (dB) = |A|^{(m+1)/2}|F|^{-(m+1)}(dA) \wedge (dF)$. The joint density of A, F is then

$$g_{A,F}(A, F) = c|F|^{-\frac{s+m+1}{2}}|A|^{\frac{r+s-m-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}A(I+F^{-1})\right)$$
$$\times\, {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 A)'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 A^{\frac{1}{2}} F^{-1}\left(A^{\frac{1}{2}}\right)'\right).$$

Integrating with respect to A using Theorem 2 with $Q(\cdot) = 1/2(I + F^{-1})$, $R(\cdot) = I$ and $S(\cdot) = F^{-1}$ gives the stated marginal density for F.

Corollary 2 *Let* $F \sim \mathcal{B}II_{jm}(s/2, r/2, \Omega_1, \Omega_2)$, $j = 1$ *or* 2, *then the joint density function of the eignvalues* $f_1, \dots, f_m$ *of* F *is*

$$g(f_1, \dots, f_m) = \frac{\pi^{\frac{m^2}{2}} \operatorname{etr}\left(-\frac{1}{2}(\Omega_1 + \Omega_2)\right)}{\Gamma_m\left[\frac{m}{2}\right] \beta_m\left[\frac{r}{2}, \frac{s}{2}\right]} \prod_{i=1}^{m} \left\{ f_1^{\frac{r-m-1}{2}} (1 + f_i)^{-\frac{s+r}{2}} \right\}$$
$$\times \prod_{i<j}^{m} (f_i - f_j) \sum_{\kappa,\lambda;\varphi}^{\infty} \frac{\frac{1}{2}(r+s)_\varphi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\, l!} \cdot \frac{C_\varphi^{\kappa,\lambda}((I+\Upsilon)^{-1}\Upsilon, (I-\Upsilon)^{-1})}{\left[C_\varphi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2)\right]^{-1} C_\varphi(I)},$$

where $f_1 > \cdots > f_m > 0$ *and* $\Upsilon = \operatorname{diag}(f_1, \dots, f_m)$.

Proof: The proof follows immediately by applying Theorem 3.2.17 in Muirhead (1982, p. 104) to the beta type II density in Theorem 4, using equation (3.12) in Chikuse (1980).

Remark 1 Note that when $\Omega_1 = \Omega_2 = 0$ in Theorem 2, the central matrix variate beta type I symmetrised or nonsymmetrised distribution is obtained (in this case, the two coincide).

Similarly, note that when $\Omega_1 = 0$, we obtain the symmetrised noncentral matrix variate beta type I(A) distribution, see Greenacre (1973) and Gupta and Nagar (2000), given by

$$f_s(U) = \mathcal{B}I_m\left(U; \frac{r}{2}, \frac{s}{2}\right) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_2\right) {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2, (I-U)\right). \quad (8)$$

However, from (8) it is possible to propose an expression for the nonsymmetrised density of U; this is done by inversely applying the definition of symmetrised density given by Greenacre (1973). That is, observing that

$$\int_{H \in \mathcal{O}(m)} {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2 H(I-U)H'\right)(dH)$$
$$= {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2, (I-U)\right)$$

we obtain

$$f_U(U) = \mathcal{B}I_m\left(U; \frac{r}{2}, \frac{s}{2}\right) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_2\right) {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2(I-U)\right). \quad (9)$$

Of course, densities (8) and (9) are still invariant under definitions 1 and 2. Note, moreover, that from (9) we have indirectly obtained a solution to the integral proposed by Constantine (1963), Khatri (1970) and reformulated by Farrell (1985) and Gupta and Nagar (2000); see also Díaz-García and Gutiérrez-Jáimez (2006).

Analogous particular results are straightforwardly obtained for the noncentral matrix variate type I(B) distribution from Theorem 2 and for the noncentral matrix variate type II(A) and II(B) distributions from Theorem 4, see Gupta and Nagar (2000) and Greenacre (1973). Finally, note that the densities of the eigenvalues of the central and noncentral beta type I and II distributions in all their variates are found as particular cases of Corollaries 1 and 2.

5 Conclusions

In this paper, we show that the densities of symmetrised doubly non-central matrix variate beta type I distributions, obtained under Definitions 1 and 2, coincide. An analogous result is obtained for the case of the symmetrised doubly non-central beta type II distribution, and therefore we need not concern ourselves with which definition to adopt, as either will serve our purpose. Note, furthermore, that both in the case of the beta type I distribution and in that of type II, when we take $\Omega_1 = \Omega_2 = 0$, the corresponding central distributions are obtained (these being symmetrised or non-symmetrised, as in this case, they coincide). In addition, when we assume $\Omega_1 = 0$, we obtain beta type I(A) and II(A) non-central distributions (symmetrised and non-symmetrised), see Gupta and Nagar (2000) or Greenacre (1973). Otherwise, if $\Omega_2 = 0$, the distributions obtained are beta type I(B) and beta type II(B) symmetrised and non-symmetrised, see Gupta and Nagar (2000) or Greenacre (1973). The non-central distributions that were obtained intrinsically solve the problem presented by Constantine (1963), Khatri (1970), Farrell (1985) and Gupta and Nagar (2000), see also Díaz-García and Gutiérrez-Jáimez (2006).

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