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پژوهشکده‌ی آمار

Numerical Methods of Option Pricing for Two Specific Models of Electricity Prices

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Abstract. In this work, two models are proposed for electricity prices as energy commodity prices which in addition to mean-reverting properties have jumps and spikes, due to non-storability of electricity. The models are simulated using an Euler scheme, and then the Monte-Carlo method is used to estimate the expectation of the discounted cash-flow under historical probability, which is considered as the option price. A so called random variable simulation and a control variate method are then used to decrease, the discretization error and the Monte-Carlo error, respectively. As the option prices satisfy PDE's associated with the models, by solving these PDE's, numerically, we can find the option prices by a second method, thereby being able to make comparisons.

Keywords. Electricity prices; mean-reversion; spikes; Monte-Carlo method.

1 Introduction

In this work, two models are proposed for electricity prices as energy commodity prices with mean reverting properties and with jumps and spikes. The general forms of these models were introduced initially by Shijie Deng (2000), in his paper entitled: *Stochastic Models of Energy Commodity Prices and their Applications: Mean-reversion with Jumps and Spikes*, February 2000, in the series of POWER, PWP-073, see also Deng (2000). The models are proposed in such a way to express the mean-reverting property, and especially to mimic the spikes present in the prices, so that some jump processes, and switching regime are present in the models. Before Deng (2000), Kaminski (1997), and Barz, and Johnson (1998) had also provide models for electricity prices. Kaminski used jumps and stochastic volatility and Barz and Johnson (1998) suggested a price model which combined a mean-reverting process with a single jump process. Deng examined a broader class of stochastic models which can be used to model behaviors in commodity prices including jumps and stochastic volatility, as well as stochastic convenience yield. In 2005 Deng (2005) used Levy processes in mean-reverting electricity price models (see Cont and Tankov (2003) for a complete reference for Levy processes).

In this paper, we consider two of his three models in Deng (2000), i.e. mean-reverting process with two types of jumps and mean-reverting regime-switching process. The problem is finding the option prices on electricity, or assuming that the option price is given by the expectation of the discounted cash-flow under historical probability (assuming that the market is complete!), to calculate this expectation. To this end we proceed in two ways. In the first one, we simulate the processes whose dynamics are given by the models and then use the Monte-Carlo method to find the desired price. In this procedure two types of error, namely discretization error (in each simulation) and Monte-Carlo error (due to the number of simulations) are present. Using a so called random variable simulation and a control variate method we will decrease these two types of error respectively. In the second way we solve the fundamental partial differential equation (PDE), which the price function satisfies numerically. In this work, we focus rather on the first method and go through the second one briefly, mainly to use its results as values which measure the accuracy of the Monte-Carlo method in our special cases. The numerical results and comparisons are represented in tables. Complete tables could be find in Zamani (2001).

2 The Models Studied and Their Simulations

In this section we introduce the various models for electricity prices we consider and are going to simulate. Our discretization procedure for the simulation of $\{X_t\}_{0\leqslant t\leqslant T}$ is essentially based on Euler schemes. A complete reference for simulation of stochastic processes is Ross (2002).

2.1 Model 1, Mean-Reverting Process with Two Types of Jumps

Let $X_t = \ln S_t$ denote the logarithm of the electricity price, then Model 1 corresponds to the following Stochastic Differential Equation (SDE) under the risk-neutral measure Q (see for example Björk (1998))

$$dX_t = \alpha(\beta - X_t)dt + \sigma_1 dW(t) + \sum_{i=1}^{2} \Delta Z_t^i, \qquad X_0 = 0, \quad 0 \leqslant t \leqslant T. \tag{1}$$

Here W_t is a $\mathcal{F}_t$-adapted standard Brownian motion under Q in $\mathbb{R}$; Z^j is a compound Poisson process in $\mathbb{R}$ with Poisson arrival intensity $\lambda_j (j = 1, 2)$, and the random jump size ΔZ^1 (resp. ΔZ^2) is deterministic and equal to σ_2 (resp. $-\sigma_2$).

Discretization procedure for $\{X_t\}_{0 \leqslant t \leqslant T}$: To simulate the trajectories of the process X_t satisfying equation (1), we use the Euler Scheme combined with a Bernoulli approximation for the Poisson process. Thus, we rewrite the equation in the form of the following difference equation:

$$\Delta X_{t_i} = \alpha(\beta - X_{t_{i-1}})\Delta t_i + \sigma_1 \Delta W_{t_i} + \Delta Z_{t_i}^1 + \Delta Z_{t_i}^2,$$

where

$$\begin{aligned} \Delta X_{t_i} &= X_{t_i} - X_{t_{i-1}}, \\ \Delta t_i &= t_i - t_{i-1} = \frac{T}{n}, \\ \Delta W_{t_i} &= W_{t_i} - W_{t_{i-1}}, \end{aligned}$$

and

$$\Delta Z_{t_i}^1 = \begin{cases} \sigma_2 & \text{w.p. } \lambda_1 \Delta t_i \\ 0 & \text{w.p. } 1 - \lambda_1 \Delta t_i \end{cases} \qquad \Delta Z_{t_i}^2 = \begin{cases} -\sigma_2 & \text{w.p. } \lambda_2 \Delta t_i \\ 0 & \text{w.p. } 1 - \lambda_2 \Delta t_i \end{cases}.$$

So we simulate X_{t_i} as the following random variable,

$$\begin{aligned} X_{t_i} &= X_{t_{i-1}} + \alpha(\beta - X_{t_{i-1}})\Delta t_i + \sigma_1 \sqrt{\Delta t_i} \mathcal{N}(0,1) \\ &\quad + 1_{\{u_i^1 < \lambda_1 \Delta t_i\}} \sigma_2 - 1_{\{u_i^2 < \lambda_2 \Delta t_i\}} \sigma_2, \end{aligned} \tag{2}$$

where $\sqrt{\Delta(t_i)}\mathcal{N}(0,1)$ is used to simulate the Brownian increments ΔW_{t_i}, while for $\Delta Z_{t_i}^j$, we use the random variables u_i^j, which are independent and uniformly distributed on $[0, 1]$.

2.2 Model 2, Mean-reverting Regime-Switching Process

In this model the logarithm of electricity price satisfies the following SDE under the risk-neutral measure Q

$$dX_t = \alpha(\beta - X_t)dt + \sigma_1 dW_t + \eta(U_{t-})dM_t, \qquad X_0 = 0, \quad 0 \leqslant t \leqslant T. \tag{3}$$

As before W_t is a $\mathcal{F}_t$-adapted standard Brownian motion under Q in $\mathbb{R}$, and the size of the random jumps in state variables when regime-switching occurs is assumed to be constant $\eta(0) = \eta(1) = \sigma_3$, $N_t^{(i)}$ is a poisson process with parameter $\lambda(i)$ $(i = 0, 1)$, and $\delta(0) = -\delta(1) = 1$. Here M_t is a continuous-time Markov chain related to the continuous-time two-state Markov chain

$$dU_t = 1_{\{U_{t-}=0\}}\delta(U_{t-})dN_t^{(0)} + 1_{\{U_{t-}=1\}}\delta(U_{t-})dN_t^{(1)}.$$

The Markov chain M_t could be considered non-compensated:

$$dM_t = dU_t,$$

or compensated:

$$dM_t = -\lambda(U_{t-})\delta(U_{t-})dt + dU_t.$$

We are going to study the model in these two cases separately.

Actually, the difference between the compensated model and the non-compensated model is rather linked to a model choice. If we think of the evolution X_t as "signal + centered noise" as usual, the choice of M_t as a compensated process (with zero mean) may be more coherent.

2.2.1 Model 2.1, Mean-Reverting with Non-Compensated Regime-Switching Process

The dynamics of $X_t = \ln S_t$ is described by the following SDE:

$$\begin{aligned} dX_t &= \alpha(\beta - X_t)dt + \sigma_1 dW(t) \\ &\quad + \sigma_3\big(1_{\{U_{t-}=0\}}dN_t^{(0)} - 1_{\{U_{t-}=1\}}dN_t^{(1)}\big), \qquad X_0 = 0, \quad 0 \leqslant t \leqslant T. \end{aligned} \tag{4}$$

Discretization procedure of $\{X_t\}_{0\leqslant t\leqslant T}$: By Euler scheme we consider the following difference equation to simulate X_t,

$$\begin{aligned} \Delta X_{t_i} &= \alpha(\beta - X_{t_{i-1}})\Delta t_i + \sigma_1 \Delta W_{t_i} \\ &\quad + \sigma_3\big(1_{\{U_{t_{i-1}}=0\}}1_{\{u_i<\lambda(0)\Delta t_i\}} - 1_{\{U_{t_{i-1}}=1\}}1_{\{u_i<\lambda(1)\Delta t_i\}}\big), \end{aligned}$$

or

$$X_{t_i} = X_{t_{i-1}} + \alpha(\beta - X_{t_{i-1}})\Delta t_i + \sigma_1 \Delta W_{t_i} + \sigma_3 \big(1_{\{U_{t_{i-1}}=0\}} 1_{\{u_i < \lambda(0)\Delta t_i\}} - 1_{\{U_{t_{i-1}}=1\}} 1_{\{u_i < \lambda(1)\Delta t_i\}}\big), \qquad (5)$$

where u_i's are uniform random variables on $[0, 1]$ as before.

2.2.2 Model 2.2, Mean-Reverting with Compensated Regime-Switching Process

The dynamics of $X_t = \ln S_t$ is described by the following SDE:

$$dX_t = \alpha(\beta - X_t)dt + \sigma_1 dW(t) + \sigma_3 \big(1_{\{U_{t-}=0\}} dN_t^{(0)} - 1_{\{U_{t-}=1\}} dN_t^{(1)} - \lambda(U_{t-})\delta(U_{t-})dt\big), \qquad X_0 = 0, \quad 0 \leqslant t \leqslant T. \qquad (6)$$

Discretization procedure of $\{X_t\}_{0 \leqslant t \leqslant T}$: The difference equation corresponding to (4) is the following,

$$X_{t_i} = X_{t_{i-1}} + \alpha(\beta - X_{t_{i-1}})\Delta t_i + \sigma_1 \Delta W_{t_i} + \sigma_3 \big(1_{\{U_{t_{i-1}}=0\}} 1_{\{u_i < \lambda(0)\Delta t_i\}} - 1_{\{U_{t_{i-1}}=1\}} 1_{\{u_i < \lambda(1)\Delta t_i\}} - \lambda(U_{t_{i-1}})\delta(U_{t_{i-1}})\Delta t_i\big), \qquad (7)$$

or

$$X_{t_i} = X_{t_{i-1}} + \alpha(\beta - X_{t_{i-1}})\Delta t_i + \sigma_1 \Delta W_{t_i} + \sigma_3 1_{\{U_{t_{i-1}}=0\}} \times \big(1_{\{u_i < \lambda(0)\Delta t_i\}} - \lambda(0)\Delta t_i\big) - \sigma_3 1_{\{U_{t_{i-1}}=1\}} \big(1_{\{u_i < \lambda(1)\Delta t_i\}} - \lambda(1)\Delta t_i\big),$$

with u_i's uniformly distributed on $[0, 1]$.

In Models 2.1 and 2.2, the simulations suit the expected behavior (with spikes) of the electricity spot price better than in Model 1.

3 The Monte-Carlo Method

In this section we are going to use the Monte-Carlo (MC) method to calculate the expectation $E_Q[\exp\{-rT\}(\exp\{X_T\} - K)_+]$ ($A_+ = \max(A, 0)$) for all of the models presented before, and for various values of the strikes K. If X is a random variable with values in $\mathbb{R}$, then by Monte-Carlo method we can estimate $E(f(X))$ for each function f on X by

$$\overline{E} f(X) \simeq \frac{1}{M} \sum_{m=1}^{M} f(X_m), \qquad (8)$$

where $X_1, X_2, ..., X_M$ are independent simulations of X. According to the strong law of large numbers the summation converges almost surely to $E(f(X))$ as M increases. In our case, the random variable we are interested in is X_T, the final value of the stochastic process $\{X_t\}_{0 \leqslant t \leqslant T}$ satisfying equations (2) or (5) or (7). So to apply Monte-Carlo method in order to calculate $\overline{E}f(X) = \overline{E}[\exp\{-rT\}(\exp\{X_T\} - K)_+]$ we need as many as possible independent simulations of X_T. To this purpose we perform many times the simulation program of the process $\{X_t\}_{0 \leqslant t \leqslant T}$, satisfying equations (2), (5) and (7).

It is well known that this approach leads to two types of errors: first, a discretization error because of the time discretization scheme; second, a Monte-Carlo error due to the number of trajectories. Our objective is to make these errors as small as possible and the two next paragraphs are devoted to this goal.

3.1 The Discretization Error

Here, we focus our attention on Model 2 with spikes, which raises unusual issues for the simulation.

An Exact Simulation Procedure

To test the accuracy of our method, we can compare our algorithm with a random variable simulation (RVS) procedure, which is available only in the special case of Models with linear drift coefficients, the other coefficients being constant.

The following algorithm is also interesting by itself, since it is easy and quick to implement. In this algorithm we simulate only

$$\{X_{\tau_1-}, X_{\tau_1}, X_{\tau_2-}, X_{\tau_2}, ..., X_{\tau_n-}, X_{\tau_n}\},$$

where $\{\tau_1, \tau_2, ..., \tau_n\}$ are the jump times of the Poisson processes $N_t^{(0)}$ and $N_t^{(1)}$ in Models 2.1 and 2.2.

Remember that for *Models 2*, X_t is given by the following SDE

$$dX_t = \alpha(\beta - X_t)dt + \sigma_1 dW_t + \eta(U_{t-})dM_t, \qquad X_0 = 0,\ 0 \leqslant t \leqslant T,$$

where in Model 2.1

$$dM_t = dU_t,$$

and in Model 2.2

$$dM_t = -\lambda(U_{t-})\delta(U_{t-})dt + dU_t,$$

with

$$dU_t = 1_{\{U_{t-}=0\}}\delta(U_{t-})dN_t^{(0)} + 1_{\{U_{t-}=1\}}\delta(U_{t-})dN_t^{(1)}.$$

Now suppose that $U_0 = 0$, and let τ_1 be the time at which the first jump of the Poisson process N_t^0 takes place. This jump time τ_1 could be simulated as

$$\tau_1 \sim -\frac{1}{\lambda(0)} \ln u_1,$$

where u_1 is a uniform random variable on $[0,1]$. Then on $[0,\tau_1)$ in Model 2.1

$$dX_t = \alpha(\beta - X_t)dt + \sigma_1 dW(t),$$

and in Model 2.2

$$dX_t = (\alpha(\beta - X_t) - \sigma_3\lambda(0))dt + \sigma_1 dW(t).$$

So that X_t is an Ornstein-Uhlenbeck (OU) process on $[0,\tau_1)$. Set

$$\beta' = \begin{cases} \beta & \text{Model 2.1} \\ \beta - \frac{\lambda(0)\sigma_3}{\alpha} & \text{Model 2.2} \end{cases}$$

Then $V_t = X_t - \beta'$ satisfies

$$dV_t = -\alpha V_t dt + \sigma_1 dW_t,$$

and

$$V_t \sim \mathcal{N}\left(V_0 \exp\{-\alpha t\},\ \frac{\sigma_1^2}{2\alpha}(1 - \exp\{-2\alpha t\})\right).$$

Hence

$$X_t \sim \mathcal{N}\left(\beta' + (X_0 - \beta')\exp\{-\alpha t\},\ \frac{\sigma_1^2}{2\alpha}(1 - \exp\{-2\alpha t\})\right) = \mathcal{N}(\mu_t, V_t),$$

and $X_{\tau_1 -}$ and X_{τ_1} could be simulated as

$$X_{\tau_1 -} = \mu_{\tau_1} + \sqrt{V_{\tau_1}}\mathcal{N}(0,1),$$
$$X_{\tau_1} = X_{\tau_1 -} + \sigma_3.$$

This step will be repeated switching to $U_{\tau_1} = 1$, and thereby taking the first jump time for the Poisson process $N_t^{(1)}$, and so on. Finally we will reach to a jump time τ_{n+1}, for which $\tau_n < T < \tau_{n+1}$, or $T = \tau_{n+1}$, in which cases

$$X_T = \mu_{T-\tau_n} + \sqrt{V_{T-\tau_n}}\mathcal{N}(0,1),$$

or

$$X_{T-} = \mu_T + \sqrt{V_T}\mathcal{N}(0,1),$$
$$X_T = X_{T-} + \sigma_3,$$

respectively.

By this, we reach to the simulated values of X_T without going along the procedure of time discretization. So that this method of simulating X_T, takes much shorter time compared to the simulation of the whole process $\{X_t\}$.

Empirical Analysis of the Discretization Error

As we mentioned before, when representing MC results, there is an important quantity which should be also taken into consideration, namely the Monte-Carlo error, the amount by which the empirical mean (8) differs from the true mean $E(f(X))$. Set $V = Var(f(X))$ for the variance of $f(X)$, we know that, $E(f(X))$ lies asymptotically in the following confidence interval with probability 95%:

$$\left[\overline{E}f(X) - 1.96\sqrt{\frac{V}{M}}, \overline{E}f(X) + 1.96\sqrt{\frac{V}{M}}\right].$$

In the following tables we represent the Monte-Carlo results for $E[\exp\{-rT\}(\exp\{X_T\} - K)_+]$ obtained for the process simulation (PS), and also the value obtained by the random variable simulation (RVS) for 10^8 simulations, for which there is no discretization error.

Monte-Carlo for PS and RVS of Model 2.1 and 2.2

In the following tables the values are obtained for $\{X_t\}$ in Model 2.1 and 2.2, satisfying equations (5) and (7) with the following parameters:

$$T = 1, \quad \alpha = 10, \ \beta = 0, \ \sigma_1 = 0.25, \ \sigma_3 = 2 \text{ and } \lambda(0) = 2, \ \lambda(1) = 50.$$

The MC error $1.96\sqrt{\frac{V}{M}}$ is also represented (V is also evaluated by MC, using the same simulations as for $E(f(X))$).

The option strikes below have been chosen close to the forward electricity spot price $E(\exp\{-rT\}\exp\{X_T\}) \approx 1.0565$, in Model 2.1 and $E(\exp\{-rT\}\exp\{X_T\}) \approx 3.8880$ in Model 2.2. For the sake of briefness here and in all the coming tables, in addition to results for $E[\exp\{-rT\}\exp\{X_T\}]$ we represent, the call option price $E[\exp\{-rT\}(\exp\{X_T\} - K)_+]$ only for three values of K among the five values we have investigated.

Table 1. Mc for PS and RVS of Model 2.1 (Non-Compensated Regime-Switching)

$E[\exp\{-rT\}\exp\{X_T\}]$	PS, $n = 250$	PS, $n = 500$	PS, $n = 1000$
MC, 10^4 sim.	1.5025 ± 0.0551	1.5092 ± 0.0485	1.5359 ± 0.0501
MC, 10^5 sim.	1.5409 ± 0.0223	1.5257 ± 0.0246	1.5186 ± 0.0193
MC for RVS, 10^8 sim.		1.0565 ± 0.0002	

$E[\exp\{-rT\}(\exp\{X_T\}-1)_+]$	**PS, $n=250$**	**PS, $n=500$**	**PS, $n=1000$**
MC, 10^4 sim.	0.1761 ± 0.0170	0.1767 ± 0.0169	0.1686 ± 0.0160
MC, 10^5 sim.	0.1810 ± 0.0054	0.1749 ± 0.0052	0.1667 ± 0.0050
MC for RVS, 10^8 sim.		0.1680 ± 0.0002	

$E[\exp\{-rT\}(\exp\{X_T\}-1.05)_+]$	**PS, n = 250**	**PS, $n=500$**	**PS, $n=1000$**
MC, 10^4 sim.	0.1685 ± 0.0195	0.1595 ± 0.0160	0.1729 ± 0.0167
MC, 10^5 sim.	0.1666 ± 0.0053	0.1594 ± 0.0051	0.1586 ± 0.0051
MC for RVS, 10^8 sim.		0.1562 ± 0.0002	

$E[\exp\{-rT\}(\exp\{X_T\}-1.1)_+]$	**PS, $n=250$**	**PS, $n=500$**	**PS, $n=1000$**
MC, 10^4 sim.	0.1647 ± 0.0169	0.1540 ± 0.0160	0.1503 ± 0.0157
MC, 10^5 sim.	0.1629 ± 0.0053	0.1502 ± 0.0050	0.1531 ± 0.0050
MC for RVS, 10^8 sim.		0.1515 ± 0.0002	

For Model 2.1, take $n = 500$, as the number of time discretization steps is enough to get to an error of order 2.5%.

Table 2. MC for PS and RVS of Model 2.2 (Compensated Regime-Switching)

$E[\exp\{-rT\}\exp\{X_T\}]$	**PS, $n=250$**	**PS, $n=500$**	**PS, $n=1000$**
MC, 10^4 sim.	3.6614 ± 0.9336	3.8867 ± 0.8654	3.7825 ± 0.9137
MC, 10^5 sim.	3.0037 ± 0.1873	3.4159 ± 0.2340	3.6586 ± 0.2313
MC for RVS, 10^8 sim.		3.8880 ± 0.0091	

$E[\exp\{-rT\}(\exp\{X_T\}-3.75)_+]$	**PS, $n=250$**	**PS, $n=500$**	**PS, $n=1000$**
MC, 10^4 sim.	3.2762 ± 0.5940	3.2862 ± 0.4850	3.6423 ± 0.6720
MC, 10^5 sim.	3.2045 ± 0.2335	3.6846 ± 0.2786	3.6997 ± 0.2786
MC for RVS, 10^8 sim.		2.8621 ± 0.0091	

$E[\exp\{-rT\}(\exp\{X_T\}-3.9)_+]$	**PS, $n=250$**	**PS, $n=500$**	**PS, $n=1000$**
MC, 10^4 sim.	4.2374 ± 1.7530	4.1040 ± 1.2310	3.5513 ± 0.7584
MC, 10^5 sim.	3.2298 ± 0.2318	3.5104 ± 0.2574	3.6091 ± 0.2188
basic MC, 10^8 sim.		2.8637 ± 0.0091	

$E[\exp\{-rT\}(\exp\{X_T\} - 4.05)_+]$	PS, $n = 250$	PS, $n = 500$	PS, $n = 1000$
MC, 10^4 sim.	3.8249 ± 1.3599	4.0678 ± 0.9443	3.5865 ± 0.6678
MC, 10^5 sim.	3.1136 ± 0.1828	3.7886 ± 0.2988	3.6287 ± 0.2425
MC for RVS, 10^8 sim.		2.8365 ± 0.0090	

At least a number of 1000 time discretization steps is necessary to get an acceptable error, which is much worse than it is in the case of Model 2.1. This comes form the compensation drift $\lambda(U_{t-})\delta(U_{t-})$ in M_t. Indeed, this term could be equal to 50 (in our tests) and this leads to a dramatic increase in the discretization error.

3.2 The Monte-Carlo Error

As we mentioned before, taking $V = Var(f(X))$, $E(f(X))$ lies in the confidence interval:

$$\left[\bar{E}f(X) - 1.96\sqrt{\frac{V}{M}}, \bar{E}f(X) + 1.96\sqrt{\frac{V}{M}}\right],$$

with probability 95%. That means that $\overline{E}f(X) - Ef(X)$ is of order $\sqrt{\frac{V}{M}}$, so that in order to decrease the precision by 10 times, we should decrease M by 100 times, that is to do the simulations 100 times longer. An alternative way for obtaining a more precise value by Monte-Carlo method is to reduce the variance V.

Control Variate Method

The method of *control variates* is a classical method of variance reduction which is adapted to the SDE problem. A control variate is a secondary variate which is simulated along with the primary variate of the Monte-Carlo method (here $f(X)$). The secondary variate should be positively correlated with the P.V. and its mean should be known. By subtracting it from the primary variate one obtains a new variate which has a lower variance than $f(X)$, and whose mean differs from that of $f(X)$ by a known amount, see El Karoui et al., (2000).

Control Variate in Model 1

To use the control variate method, we need to know the exact value of $E(\exp\{-rT\}\exp\{X_T\})$. In Model 1, $E(\exp\{-rT\}\exp\{X_T\})$ can be computed using an analytical expression. Indeed, it can be driven explicitly by solving the corresponding PDE to the model, See Oksendal (1998), pp 153-158 for different versions of Girsanov Theorem.

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 f}{\partial x^2} + \alpha(\beta - x)\frac{\partial f}{\partial x} - rf + \lambda_1 \Delta_1 f + \lambda_2 \Delta_2 f = 0, \tag{9}$$

$$f(T, x) = h(x).$$

where

$$\Delta_1 f(t,x) = f(t, x + \sigma_2) - f(t,x),$$
$$\Delta_2 f(t,x) = f(t, x - \sigma_2) - f(t,x).$$

For $h(x) = \exp\{x\}$, $f(t,x) = E[\exp\{-r(T-t)\}\exp\{X_T\}|X_t = x]$, is given explicitly by:

$$f(t,x) = \exp\Bigg(\int_t^T \Big\{\alpha\beta\exp\{-\alpha(T-s)\} + \frac{1}{2}\sigma_1^2\exp\{-2\alpha(T-s)\} - r + \lambda_1(H_1(s)-1) + \lambda_2(H_2(s)-1)\Big\}ds + \exp\{-lpha(T-t)\}x\Bigg), \tag{10}$$

where

$$H_1(t) = \int_t^T \exp(\sigma_2 \exp\{-\alpha u\})du \quad \text{and} \quad H_2(t) = \int_t^T \exp(-\sigma_2\exp\{-\alpha u\})du.$$

Note that however $H_1(t)$ and $H_2(t)$ could not be calculated analytically, but numerically. The coefficients of equation (2) are taken as

$$T = 1,\ \alpha = 10,\ \beta = 0,\ \sigma_1 = 0.25,\ \sigma_2 = 2 \text{ and } \lambda_1 = \lambda_2 = 2.$$

For these parameters from (10) $f(0,0) = E(\exp\{-rT\}\exp\{X_T\})$ is calculated as

$$E(\exp\{-rT\}\exp\{X_T\}) \approx 1.5288,$$

where the integrals H_1 and H_2 has been calculated numerically using 10^7 time steps. In the following tables the estimated values of $E[\exp\{-rT\}(\exp\{X_T\} - K)_+]$ for various strikes K, different meshes of time discretization for the SDE, and different numbers of simulations are represented, by the basic method of Monte-Carlo and the Monte-Carlo with control variate. The last two rows in the first table represent the calculations we derived numerically in the end of the previous section. The strike prices K are chosen close to the forward price $E(\exp\{-rT\}\exp\{X_T\})$ to be more realistic. The confidence intervals derived by these methods could be compared in the tables.

Table 3. Basic MC, and MC with control variate for Model 1

$E[\exp\{-rT\}\exp\{X_T\}]$	$n = 250$	$n = 500$	$n = 1000$
basic MC, 10^4 sim.	1.5025 ± 0.0551	1.5092 ± 0.0485	1.5359 ± 0.0501
basic MC, 10^5 sim.	1.5409 ± 0.0223	1.5257 ± 0.0246	1.5186 ± 0.0193

$E[e^{-rT}(e^{X_T} - 1.5)_+]$	$n = 250$	$n = 500$	$n = 1000$
basic MC, 10^4 sim.	0.5741 ± 0.0482	0.5653 ± 0.0508	0.6696 ± 0.1042
basic MC, 10^5 sim.	0.6054 ± 0.0209	0.6020 ± 0.0196	0.5843 ± 0.0208
MC with CV, 10^4 sim.	0.6030 ± 0.0074	0.5977 ± 0.0075	0.6053 ± 0.0076
MC with CV, 10^5 sim.	0.5990 ± 0.0028	0.5984 ± 0.0023	0.5973 ± 0.0025

$E[\exp\{-rT\}(\exp\{X_T\} - 1.55)_+]$	$n = 250$	$n = 500$	$n = 1000$
basic MC, 10^4 sim.	0.6062 ± 0.0567	0.5865 ± 0.0628	0.6023 ± 0.0538
basic MC, 10^5 sim.	0.5738 ± 0.0178	0.6096 ± 0.0306	0.5903 ± 0.0208
MC with CV, 10^4 sim.	0.5826 ± 0.0079	0.5886 ± 0.0076	0.5901 ± 0.0080
MC with CV, 10^5 sim.	0.5904 ± 0.0031	0.5904 ± 0.0028	0.5889 ± 0.0042

Control Variate in Model 2.1

For Model 2.1, the derived values of $E[\exp\{-rT\}(\exp\{X_T\} - K)_+]$ for various strikes K, different meshes n of time discretization for the SDE, and different number of simulations, by the basic method of Monte-Carlo and the Monte-Carlo with control variate are represented in the following tables. As before the strike prices K take values close to $E(\exp\{-rT\}\exp\{X_T\})$. For this model, $E(\exp\{-rT\}\exp\{X_T\})$ in the representation of C_1 is derived by applying basic Monte-Carlo to the exact simulation of X_T, which was described in the previous section, for 10^8 simulations. The derived value is

$$E(\exp\{-rT\}\exp\{X_T\}) \approx 1.0566 \pm 0.0002.$$

The coefficients in equation (5) are taken as

$$T = 1, \alpha = 10,\ \beta = 0,\ \sigma_1 = 0.25,\ \sigma_3 = 2 \text{ and } \lambda(0) = 2,\ \lambda(1) = 50.$$

Table 4. Basic MC, and MC with control variate for Model 2.1

$E[\exp\{-rT\}\exp\{X_T\}]$	$n = 250$	$n = 500$	$n = 1000$
basic MC, 10^4 sim.	1.0629 ± 0.0170	1.0606 ± 0.0168	1.0621 ± 0.0167
basic MC, 10^5 sim.	1.0659 ± 0.055	1.0570 ± 0.0052	1.0566 ± 0.0052

$E[\exp\{-rT\}(\exp\{X_T\} - 1)_+]$	$n = 250$	$n = 500$	$n = 1000$
basic MC, 10^4 sim.	0.1761 ± 0.0170	0.1767 ± 0.0169	0.1686 ± 0.0160
basic MC, 10^5 sim.	0.1810 ± 0.0054	0.1749 ± 0.0052	0.1667 ± 0.0050
MC with CV, 10^4 sim.	0.1706 ± 0.0020	0.1679 ± 0.0019	0.1689 ± 0.0019
MC with CV, 10^5 sim.	0.1702 ± 0.0006	0.1689 ± 0.0006	0.1686 ± 0.0005

$E[\exp\{-rT\}(\exp\{X_T\} - 1.05)_+]$	$n = 250$	$n = 500$	$n = 1000$
basic MC, 10^4 sim.	0.1685 ± 0.0195	0.1595 ± 0.0160	0.1729 ± 0.0167
basic MC, 10^5 sim.	0.1666 ± 0.0053	0.1594 ± 0.0051	0.1586 ± 0.0051
MC with CV, 10^4 sim.	0.1605 ± 0.0021	0.1592 ± 0.0021	0.1586 ± 0.0021
MC with CV, 10^5 sim.	0.1589 ± 0.0006	0.1569 ± 0.0006	0.1570 ± 0.0007

$E[\exp\{-rT\}(\exp\{X_T\} - 1.1)_+]$	$n = 250$	$n = 500$	$n = 1000$
basic MC, 10^4 sim.	0.1647 ± 0.0169	0.1540 ± 0.0160	0.1503 ± 0.0157
basic MC, 10^5 sim.	0.1629 ± 0.0053	0.1502 ± 0.0050	0.1531 ± 0.0050
MC with CV, 10^4 sim.	0.1538 ± 0.0021	0.1521 ± 0.0021	0.1508 ± 0.0021
MC with CV, 10^5 sim.	0.1543 ± 0.0007	0.1529 ± 0.0007	0.1520 ± 0.0006

Control Variate in Model 2.2

Finally for the same series of parameters of Model 2.1, the same values are compared in the tables below. As in Model 2.1, $E(\exp\{-rT\}\exp\{X_T\})$ in C_1 is derived by applying basic Monte-Carlo to the direct simulation of X_t for 10^8 simulations. It is equal to

$$E(\exp\{-rT\}\exp\{X_T\}) \approx 3.8920 \pm 0.0091.$$

Table 5. Basic MC, and MC with control variate for Model 2.2

$E[\exp\{-rT\}\exp\{X_T\}]$	$n = 250$	$n = 500$	$n = 1000$
basic MC, 10^4 sim.	3.6614 ± 0.9336	3.8867 ± 0.8654	3.7825 ± 0.9137
basic MC, 10^5 sim.	3.0037 ± 0.1873	3.4159 ± 0.2340	3.6586 ± 0.2313

$E[\exp\{-rT\}(\exp\{X_T\}-3.75)_+]$	$n=250$	$n=500$	$n=1000$
basic MC, 10^4 sim.	3.2762 ± 0.5940	3.2862 ± 0.4850	3.6423 ± 0.6720
basic MC, 10^5 sim.	3.2045 ± 0.2335	3.6846 ± 0.2786	3.6997 ± 0.2786
MC with CV, 10^4 sim.	2.8589 ± 0.0204	2.8587 ± 0.0230	2.8555 ± 0.0168
MC with CV, 10^5 sim.	2.8505 ± 0.0050	2.8494 ± 0.0053	2.8505 ± 0.0099

$E[\exp\{-rT\}(\exp\{X_T\}-3.90)_+]$	$n=250$	$n=500$	$n=1000$
basic MC, 10^4 sim.	4.2374 ± 1.7530	4.1040 ± 1.2310	3.5513 ± 0.7584
basic MC, 10^5 sim.	3.2298 ± 0.2318	3.5104 ± 0.2574	3.6091 ± 0.2188
MC with CV, 10^4 sim.	2.8322 ± 0.0168	2.8537 ± 0.0169	2.8395 ± 0.0358
MC with CV, 10^5 sim.	2.8348 ± 0.0062	2.8491 ± 0.0052	2.8502 ± 0.0057

$E[\exp\{-rT\}(\exp\{X_T\}-4.05)_+]$	$n=250$	$n=500$	$n=1000$
basic MC, 10^4 sim.	3.8249 ± 1.3599	4.0678 ± 0.9443	3.5865 ± 0.6678
basic MC, 10^5 sim.	3.1136 ± 0.1828	3.7886 ± 0.2988	3.6287 ± 0.2425
MC with CV, 10^4 sim.	2.7956 ± 0.0242	2.8418 ± 0.0211	2.8190 ± 0.0211
MC with CV, 10^5 sim.	2.8251 ± 0.0058	2.8290 ± 0.0092	2.8449 ± 0.0054

Conclusion

To evaluate $E(\exp\{-rT\}(\exp\{X_T\}-K)_+)$ for the models studied, we have used control variate method, this leads to highly significant reduction of variance. It is often observed (especially for in the money options) that the confidence interval is 10 times smaller; in other words, for a given accuracy level, the method speeds up by a factor 100.

4 The PDE Equations

The PDE equations corresponding to the models we consider, are the PDE's which are satisfied by the functions of the form

$$f(t,x) = E[\exp\{-r(T-t)\}h(X_T)|X_t = x],$$

where h is a real function. For suitable choices of h like, $h(x) = (\exp\{x\}-K)_+$, $K \geqslant 0$, $f(t,x)$ could be interpreted as the price of the call option on $S_t = \exp\{X_t\}$ with strike price K, at time t, so that the PDE's are the equations which the option price function on S_t satisfies. In the following we represent briefly the PDE's associated with each model and the corresponding finite difference (FD) equations. For associated PDE's see Oksendal (1998), pp 153-158 for different versions of Girsanov Theorem. Here we are not going through technicalities.

The PDE and FD for Model 1

As we mentioned before the PDE of Model 1, is given as

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial x^2}+\alpha(\beta-x)\frac{\partial f}{\partial x}-rf+\lambda_1\Delta_1 f+\lambda_2\Delta_2 f=0,\ x\in\mathbb{R},\ 0<t<T,$$
$$f(T,x)=h(x). \tag{11}$$

where

$$\Delta_1 f(t,x)=f(t,x+\sigma_2)-f(t,x),$$
$$\Delta_2 f(t,x)=f(t,x-\sigma_2)-f(t,x).$$

Now let us apply the change of variable $t=T-t$ to the PDE (13) and get to the following PDE

$$\frac{\partial f}{\partial t}=\frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial x^2}+\alpha(\beta-x)\frac{\partial f}{\partial x}-rf+\lambda_1\Delta_1 f+\lambda_2\Delta_2 f,$$
$$f(0,x)=h(x),\qquad x\in\mathbb{R},\ 0<t<T. \tag{12}$$

or the following FD,

$$\frac{\partial f}{\partial t}=Af+Bf,\quad f(0,x)=h(x),\qquad x\in\mathbb{R},\ 0<t<T \tag{13}$$

for

$$Af=\frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial x^2}+\alpha(\beta-x)\frac{\partial f}{\partial x}-rf,$$

and

$$Bf=\lambda_1\Delta_1 f+\lambda_2\Delta_2 f.$$

In (13), the operator A is a classical partial differential operator in space whereas B represents the jump terms. The time splitting algorithm will consist of solving equation

$$\frac{\partial u}{\partial t}=Au,$$

by an implicit scheme (backward Euler scheme or sometimes a Crank-Nilcoson scheme) over a time step δt and then solving

$$\frac{\partial u}{\partial t}=Bu,$$

by an explicit scheme.

The PDE's and FD's for Model 2.1

In this model depending on whether the two-state Markov chain U_t starts in the position 0, that is $U_0 = 0$, or it starts in 1, $U_0 = 1$, two functions are considered

$$f_0(t,x) = E[\exp\{-r(T-t)\}h(X_T)|X_t = x, U_t = 0],$$

$$f_1(t,x) = E[\exp\{-r(T-t)\}h(X_T)|X_t = x, U_t = 1]. \tag{14}$$

These two functions satisfy a coupled system of PDE's.

$$\begin{gathered}\frac{\partial f_0}{\partial t} + \frac{1}{2}\sigma_1^2\frac{\partial^2 f_0}{\partial x^2} + \alpha(\beta - x)\frac{\partial f_0}{\partial x} - rf_0 + \lambda(0)\Delta_0(f_0, f_1) = 0,\\ \frac{\partial f_1}{\partial t} + \frac{1}{2}\sigma_1^2\frac{\partial^2 f_1}{\partial x^2} + \alpha(\beta - x)\frac{\partial f_1}{\partial x} - rf_1 + \lambda(1)\Delta_1(f_0, f_1) = 0,\\ f_0(T,x) = f_1(T,x) = h(x).\end{gathered} \tag{15}$$

where

$$\begin{aligned}\Delta_0(f_0, f_1)(t,x) &= f_1(t, x+\sigma_2) - f_0(t,x),\\ \Delta_1(f_0, f_1)(t,x) &= f_0(t, x-\sigma_2) - f_1(t,x).\end{aligned}$$

Again by applying the change of variable $t = T - t$, we are lead to the following system of PDE's

$$\begin{aligned}\frac{\partial f_0}{\partial t} &= Af_0 + \lambda(0)\Delta_0(f_0, f_1),\\ \frac{\partial f_1}{\partial t} &= Af_1 + \lambda(1)\Delta_1(f_0, f_1),\\ f_0(0,x) = f_1(0,x) &= h(x),\end{aligned}$$

where

$$Af = \frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial x^2} + \alpha(\beta - x)\frac{\partial f}{\partial x} - rf.$$

The novelty compared to Model 1 is the coupling of two partial differential equations. The discretization scheme consists of a time splitting in the system. First both equations $\frac{\partial f_i}{\partial t} = Af_i$, $i = 1, 2$ are solved independently by an implicit scheme in time. Then the coupling terms corresponding to the jump terms are solved by an explicit scheme in time. The spatial discretizations and the truncature of the domain are identical to that of Model 1.

The PDE's and FD's for Model 2.2

For this model, only the drift terms in the system (15) are changed. The functions

$$f_0(t,x) = E[\exp\{-r(T-t)\}h(X_T)|X_t = x, U_t = 0],$$
$$f_1(t,x) = E[\exp\{-r(T-t)\}h(X_T)|X_t = x, U_t = 1],$$

satisfy the following system

$$\begin{aligned}\frac{\partial f_0}{\partial t} + \frac{1}{2}\sigma_1^2\frac{\partial^2 f_0}{\partial x^2} + \{\alpha(\beta - x) - \lambda(0)\sigma_2\}\frac{\partial f_0}{\partial x} - rf_0 + \lambda(0)\Delta_0(f_0, f_1) = 0,\\ \frac{\partial f_1}{\partial t} + \frac{1}{2}\sigma_1^2\frac{\partial^2 f_1}{\partial x^2} + \{\alpha(\beta - x) + \lambda(1)\sigma_2\}\frac{\partial f_1}{\partial x} - rf_1 + \lambda(1)\Delta_1(f_0, f_1) = 0,\\ f_0(T,x) = f_1(T,x) = h(x).\end{aligned} \quad (16)$$

where Δ_0 and Δ_1 are defined as in (15).

By following the same arguments as before we rewrite the original backward system (16) as the following forward system

$$\begin{aligned}\frac{\partial f_0}{\partial t} &= A_0 f_0 + \lambda(0)\Delta_0(f_0, f_1),\\ \frac{\partial f_1}{\partial t} &= A_1 f_1 + \lambda(1)\Delta_1(f_0, f_1),\\ f_0(0,x) &= f_1(0,x) = h(x),\end{aligned}$$

where

$$A_0 f = \frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial x^2} + \{\alpha(\beta - x) - \lambda(0)\sigma_2\}\frac{\partial f}{\partial x} - rf,$$

and

$$A_1 f = \frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial x^2} + \{\alpha(\beta - x) + \lambda(1)\sigma_2\}\frac{\partial f}{\partial x} - rf.$$

Then using the centered scheme or the non-centered scheme to solve the PDE's

$$\frac{\partial u}{\partial t} = A_0 u, \qquad \frac{\partial u}{\partial t} = A_1 u,$$

we get to the same equations, in which the term $\alpha(\beta - x_n)$, is now replaced by

$$\alpha(\beta - x_n) - \lambda(0)\sigma_2, \qquad \alpha(\beta - x_n) - \lambda(0)\sigma_2,$$

corresponding respectively to A_0 and A_1. The discretization scheme is very similar to that of model 2.1: a time splitting of the coupled system, an implicit scheme for the independent partial differential equations and an explicit scheme for the coupling terms which correspond to jump terms as well.

4.1 Comparing with the Monte-Carlo Results

Here we present only one table in each case comparing the Monte-Carlo and the finite-difference results for the option prices. The result presented as the Monte-Carlo result is the one with control variate, for 10^5 simulations, and for 1000 step of time descritizations. To obtain the finite-difference results we double the number of time desretizations systematically, finding the results which are compatible with this rule we calculate the FD result by extrapolation. Here we consider the two centered and (non-centered) Crank-Nicolson method to the PDE's.

- **Model 2.1** $K = 1.5$, MC with CV, 10^5 simulations, $nt = 1000$: 0.5973 ± 0.0025

$E[e^{-rT}(e^{X_T} - 1.5)_+]$	$nt, nx = 250, 450$	$500, 700$	$1000, 900$	$2000, 1350$	extrapolated
centered	0.6875	0.6420	0.6200	0.6092	0.5984
differences		0.0455	0.0220	0.0108	

$E[e^{-rT}(e^{X_T} - 1.5)_+]$	$nt, nx = 250, 120$	$500, 250$	$1000, 500$	$2000, 1000$	extrapolated
(non-centered) CN	0.7050	0.6490	0.6235	0.6110	0.5984
differences		0.0560	0.0267	0.0125	

- **Model 2.1** $K = 1.05$, MC with CV, 10^5 simulations, $nt = 1000$: 0.1570 ± 0.0007

$E[e^{-rT}(e^{X_T} - 1.05)_+]$	$nt, nx = 250, 450$	$500, 700$	$1000, 900$	$2000, 1350$	extrapolated
centered	0.1604	0.1586	0.1574	0.1568	0.1562
differences		0.0018	0.0012	0.0006	

$E[e^{-rT}(e^{X_T} - 1.05)_+]$	$nt, nx = 250, 120$	$500, 250$	$1000, 500$	$2000, 1000$	extrapolated
(non-centered) CN	0.1669	0.1630	0.1593	0.1578	
differences		0.0039	0.0037	0.0015	

- **Model 2.2** $K = 3.9$, MC with CV, 10^5 simulations, $nt = 1000$: 2.8502 ± 0.0057

$E[e^{-rT}(e^{X_T} - 3.9)_+]$	$nt, nx = 250, 450$	$500, 700$	$1000, 900$	$2000, 1350$	extrapolated
centered	3.9188	3.3618	3.1019	2.9766	2.8513
differences		0.5573	0.2599	0.1253	

$E[e^{-rT}(e^{X_T} - 3.9)_+]$	$nt, nx = 250, 120$	$500, 250$	$1000, 500$	$2000, 1000$	extrapolated
(non-centered) CN	3.9615	3.3591	3.0983	2.9743	2.8503
differences		0.6024	0.2608	0.1240	

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