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پژوهشکده‌ی آمار

On the Distribution of Discounted Collective Risk Model

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Abstract. We study the distribution of discounted collective risk model where the counting process is Poisson. For the model considered here, we obtain mean, variance and moment generating function (m.g.f) of the model. To do this, we use two approaches. In the first approach we use classical methods to obtain the mean and variance. In the second approach we introduce some proper martingale and then we obtain the m.g.f of total loss by features of martingales. Additionally, we use Fast Fourier Transform to numerically calculate the distribution of discounted collective risk model.

Keywords. Discounted collective risk model; Poisson process; Martingale.

1 Introduction

Modeling an insurance company's cash flows by a certain stochastic process and hence to quantify and measure the risk associated with the operation of insurance business are belong to the most important problems in risk theory. As Feng (2009) wrote, a risk model is typically built upon two assumptions on insurance business: Incoming cash flows, such as premium income, interest returns from financial markets; and Outgoing cash flows, such as its financial obligations to insurance claims, operating costs, business overhead costs, etc. There has been a great variety of stochastic processes used in literature to model outgoing cash flows, especially for insurance claims, due to their nature of randomness (see for example Paulsen, 1993; and Feng, 2009). Meanwhile,

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rising claim costs and higher interest rates began to motivate regulators to scrutinize cash flow models more carefully.

We notice that to calculate the outgoing cash flows that will result from any given insurance policy, the models look forward instead of backward. They typically multiply claim size by the discounted factor and add up to every considered period time. For example, if S_t is the claim size at time t and e^{-R_t} is the discounted factor at this time, then total outgoing cash flows at period $(0,\infty)$ is given by Z_∞:

$$Z_\infty = \int_0^\infty e^{-R_t} dS_t. \tag{1}$$

Gerber (1979) studied the distribution of random variable Z_∞ where he supposed that S_t be an independent compound Poisson process i.e. $S_t = \sum_{i=1}^{N(t)} X_i$ where $N(t)$ is a Poisson process and $\{X_i\}_{i\geqslant 1}$ a sequence of i.i.d, nonnegative random variables, independent of $N(t)$ and constant force of interest δ. He showed that when $t \to \infty$ and in addition the $\{X_i\}$ are exponentially distributed with mean β, then

$$Z_\infty \sim \Gamma\left(\frac{\lambda}{r}, \beta^{-1}\right).$$

Paulsen (1993) studied the distribution of the stochastic integral Z_∞ under the assumption that R_t and S_t are independent stochastic processes with independent stationary increments and with a finite number of jumps on each finite time interval. Under the assumption that $E[Z_\infty^2] < \infty$, it was shown that the characteristic function of Z_∞ can be found by solving an integro-differential equation. When $S_t = st + \sigma_p B_{p,t}$ and $R_t = rt + \sigma B_t$ with B_p and B independent Brownian motions, the density was found to be

$$f_{Z_\infty}(z) = \frac{f_0}{\left(\sigma_p^2 + \sigma^2 z^2\right)^{\frac{1}{2}+\frac{r}{\sigma^2}}} \exp\left\{\frac{2p}{\sigma\sigma_p} \arctan\left(\frac{\sigma}{\sigma_p} z\right)\right\} \tag{2}$$

provided $r > \sigma^2$. Here f_0 is a normalization constant. This result generalizes a previous result of Dufresne (1990). Assuming that $S_t = t$ and $R_t = rt + \sigma B_t$, he proved by quite different methods that $Z_\infty^{-1} \sim \Gamma(\frac{2r}{\sigma^2}, \frac{2}{\sigma^2})$.

Nilsen and Paulsen (1996), also considered compound Poisson process for random variable Z_∞, but with a stochastic Brownian motion interest rate. They showed that Z_∞ has the same distribution as that of a gamma

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distributed random variable divided by an independent beta distributed random variable. The quantity Z_∞, has also been studied by Paulsen (1997) for more general S_t and R_t, but with main focus on its first two moments. Hernandez and Pacheco (2008) also considered the present value surplus process when $R_t = \delta W_i$ and $t \to \infty$, where W_i is a renewal point process.

Now consider the quantity $Z_t = \int_0^t e^{-\delta t} dS_t$ in specific time horizon $(0, t]$. As above let S_t be an independent compound Poisson process. Let $W_i, i = 1, 2, \dots$ denote the ith claim arrival time. Then the stochastic integral (1) in finite time horizon can be written as

$$Z_t = \sum_{i=1}^{N(t)} X_i e^{-R_{W_i}}. \tag{3}$$

There are a few works on this quantity. Several papers provide the asymptotic tail probability of the discounted aggregate claim process over a finite time horizon under the assumption of independence between the claim amounts and the inter-claim times. Assuming a constant force of interest, Tang (2005, 2007) and Wang (2008) derived asymptotic results for both the classical compound Poisson and the renewal risk models. Ladoucette & Teugels (2006) studied a similar problem for a free interest risk model assuming a general claim arrival process. Asmussen (2000) showed that $Z_t \xrightarrow{a.s} Z_\infty$ as $t \to \infty$. He also obtained the mean and variance of Z_t by using the features of stochastic integrals. But the distribution of Z_t is still unknown in the literature whereas for many applications of the collective risk theory the knowledge of the aggregate claim density functions play an essential role. In this paper we will calculate the expected value and variance of Z_t when $R_{W_i} = \delta W_i$ as well as its moment generating function. Hence the results of this paper may be very useful in this matter.

We mention that the process $\{Z_t\}$ is an extension of the well-known *renewal reward processes*, and when $\delta = 0$ it is a particular instance of the so-called aggregate claim amount in the insurance jargon. These processes have renewal properties, feature that classifies them in a more general family called regenerative processes.

The rest of this paper is organized as follows. In section 2 we mention some notes about collective risk models. Section 3 derives our main results on the distribution of the discounted collective risk model. Finally some conclusion are presented in section 4.

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2 Collective Risk Model

The goal of insurance modeling is to develop a probability distribution for the total amount claim paid in specific period of time. Specially in Non-life insurance, actuaries usually consider the interaction between the distribution of the number of claims and the distribution of the individual claims. In fact, non-life insurance ratemaking is based on a claim frequency distribution and a loss distribution. Claim frequency is defined as the number of incurred claims per unit of risk exposure. The average loss severity is the average payment per incurred claim per risk exposure too.

Although it is not necessary to separate the insurance loss process into frequency and severity components, this collective risk models has several advantages in fulfilling the need for adequate model for different insurance purpose such as; separately modifying the components, estimating parameters of components from separate sources of information and adjusting inflation or other time independent factors. So, collective risk model has been developed for the total amount paid on all claims occurring in a fixed time period based on these two components.

The main advantage of a collective risk model is that it is a computationally efficient model, which is also rather close to reality. But, in collective models, some policy information is ignored. If a portfolio contains only one policy that could generate a high claim, this term will appear at most once in the individual model. In the collective model, however, it could occur several times. Moreover, in collective models we require the claim number and the claim amounts to be independent. This makes it somewhat less appropriate to model a car insurance portfolio, since for instance bad weather conditions will cause a lot of small claim amounts. In practice, however, the influence of these phenomena appears to be small (see, Kass et al. 2002).

Definition 1. A collective risk model, in specific period of time $(0, t]$, represents the total loss S_t as the sum of a random number claims, $N(t)$, of individual payment amounts $\left(X_1, X_2, \ldots, X_{N(t)}\right)$ as follows:

$$S_t = \sum_{i=1}^{N(t)} X_i, \tag{4}$$

Where:

1. Individual claims X_i are independent and identically distributed,

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2. $N(t)$ and X_i are independent, and

3. $S_t = 0$ when $N(t) = 0$.

The distribution of total losses, S_t, have a compound distribution and is named after its frequency distribution e.g. in case that $N(t)$ is Poisson distributed, S_t has a compound Poisson distribution. Collective risk model has several useful properties. In particular if μ_k denote the kth moments of claim severity, $E[X^k]$, then, It can be easily proved by using the rule of Iterated expectation that

$$E[S_t] = \mu_1 E[N(t)]. \tag{5}$$

Based on this model a reasonable estimate for pure premium of policyholder would be the expected value of S_t. In special case where $N(t) \sim \text{Poisson}(\lambda t)$, we have

$$E[S_t] = \mu_1 \lambda t. \tag{6}$$

Moreover, there are some algorithms to calculate the distribution of S_t (for example, please see Kass et al., 2002). Of course there is a significant amount of literature which address this model and its applications to casualty insurance. The primary source is probably the text by Beard et al. (1984). Other complete texts dealing with Collective Risk theory and its applications are those by Bühlmann (1970), Borch (1970) and Seal (1969).

3 Discounted Collective Risk Model

Although collective risk model seems to have many advantages, but one of its drawbacks is that it overlooks the arrival time of claims and the effect of interest rate. In property-liability insurance contracts there are always a time lag between the premium payment and claims arrival time. During this time lags, the insurer earns investment income on the unexpanded component of the premium which is not involved in collective risk model equation (4). So insured are eligible to have some of this investment profit during policy coverage period.

Definition 2. A discounted collective risk model in specific period of time $(0, t]$, represents the total loss, Z_t, as the sum of a random number claims, $N(t)$, of individual present value payment amounts $\left(X_1, X_2, \ldots, X_{N(t)}\right)$ respect to arrival times $\left(W_1, W_2, \ldots, W_{N(t)}\right)$ and constant force of interest δ

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as follows:

$$Z_t = \sum_{i=1}^{N(t)} X_i e^{-\delta W_i}, \tag{7}$$

Where:

1. Individual claims X_i are independent and identically distributed,
2. $N(t)$ and X_i are independent, and
3. $Z_t = 0$ when $N(t) = 0$.

The discounted collective risk models defined as above have several interesting and useful properties. At the first, it incorporates investment income into pricing model. Moreover, it provide better model for property and liability insurance in which the interval between premium payments and claim payments is a significant factor. Therefore, insurers can present long term insurance product in property and liability insurance market. One important quantity is the expected value of Z_t which can be interpret as the net premium amount needed to cover insurance liability on its becoming due without paying any expenses or contingent charges. We calculate this expected value in an important special case of discounted collective risk model where $N(t)$ has Poisson distribution.

The below lemma states some simple but fundamental features of the conditional arrival time of Poisson distribution.

Lemma 1. In the discount collective risk model (7), let $N(t) \sim \text{Poisson}(\lambda t)$, then density function of ith arrival time given $N(t) = k$ is

$$f_{W_i|N(t)}(s \mid k) = \frac{k-i+1}{t} \binom{k}{i-1} \left(\frac{s}{t}\right)^{i-1} \left(1 - \frac{s}{t}\right)^{k-i}, \qquad 0 \leqslant s \leqslant t. \tag{8}$$

Proof. Let $N(t) \sim \text{Poisson}(\lambda t)$ and W_i be the arrival time of the ith claim for $i = 1, \ldots, N(t)$, then distribution function of ith arrival time given $N(t) = k$ is

$$\begin{aligned} F_{W_i|N(t)}(s \mid k) &= \Pr[W_i \leqslant s \mid N(t) = k] \\ &= 1 - \Pr[N(s) \leqslant i-1 \mid N(t) = k] \\ &= 1 - \sum_{m=1}^{i-1} \frac{\Pr[N(s) = m, N(t) = k]}{\Pr[N(t) = k]} \end{aligned}$$

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$$= 1 - \sum_{m=1}^{i-1} \begin{pmatrix} k \\ m \end{pmatrix} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{k-m}, \qquad 0 \leqslant s \leqslant t. \tag{9}$$

Now the $f_{W_i|N(t)}$ can be found by taking derivatives of (9) with respect to s.

Using Lemma 1 for the lth moment of ith arrival time given $N(t) = k$, we have:

$$E\left[W_i^l \mid N(t) = k\right] = \int_0^t s^l \frac{k-i+1}{t} \begin{pmatrix} k \\ i-1 \end{pmatrix} \left(\frac{s}{t}\right)^{i-1} \left(1 - \frac{s}{t}\right)^{k-i} ds$$
$$= t^l \frac{k!\,(l+i-1)!}{(i-1)!\,(l+k)!}. \tag{10}$$

Applying the previous lemma, we now proceed to derive the net premium for discounted collective risk model under the Poisson assumption for claim frequency. The expected value of Z_t is stated in the following theorem.

***Theorem* 1.** *Suppose in the discounted collective risk model (7),* $N(t) \sim \text{Poisson}(\lambda t)$*, then*

$$E[Z_t] = \frac{\mu_1 \lambda}{\delta}\left(1 - e^{-\delta t}\right). \tag{11}$$

Proof. At the first we calculate $E[Z_t \mid N(t) = k]$ by using (10). We have:

$$E[Z_t|N(t) = k] = E\left[\sum_{i=1}^{N(t)} X_i e^{-\delta W_i} | N(t) = k\right]$$
$$= \sum_{i=1}^{k} \mu_1 E\left[e^{-\delta W_i} \mid N(t) = k\right]$$
$$= \sum_{i=1}^{k} \mu_1 E\left[\sum_{l=1}^{\infty} \frac{(-\delta)^l W_i^l}{l!} \mid N(t) = k\right]$$
$$= \mu_1 \sum_{l=0}^{\infty} (-\delta t)^l \frac{k!}{(k+l)!} \sum_{i=1}^{k} \begin{pmatrix} l+i-1 \\ l \end{pmatrix}. \tag{12}$$

(We notice that Fubini's Theorem allows the order of summation to be changed in the later expression). Moreover, it can be easily shown that $\sum_{i=1}^k \begin{pmatrix} l+i-1 \\ l \end{pmatrix} = \begin{pmatrix} l+k \\ l+1 \end{pmatrix}$. So, we can rewrite (12) as

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$$\mu_1 k \sum_{l=0}^{\infty} \frac{(-\delta t)^l}{(l+1)!} = \frac{\mu_1 k}{\delta t}\left(1-e^{-\delta t}\right). \tag{13}$$

Now we turn to calculation of $E\left[Z_t\right]$. From equation (13) we have,

$$E\left[Z_t\right] = E\left[\frac{\mu_1 N(t)}{\delta t}\left(1-e^{-\delta t}\right)\right] = \frac{\mu_1 \lambda}{\delta}\left(1-e^{-\delta t}\right).$$

Corollary 1. Process $\{A_t\}_{t>0} = \left\{Z_t - \frac{\mu_1\lambda}{\delta}\left(1-e^{-\delta t}\right)\right\}_{t>0}$ is a martingale.

Proof. It is enough to show that for all $h>0$:

$$E\left[A_{t+h} - A_t | A_t = a_t\right] = 0,$$

or equivalently:

$$E\left[Z_{t+h} - Z_t | Z_t = z_t\right] = \frac{\mu_1 \lambda}{\delta}\left\{e^{-\delta t} - e^{-\delta(t+h)}\right\}.$$

we have:

$$\begin{aligned} E\left[Z_{t+h} - Z_t | Z_t = z_t\right] &= E\left[\sum_{i=1}^{N(t+h)} X_i e^{-\delta W_i} - \sum_{i=1}^{N(t)} X_i e^{-\delta W_i} | Z_t = z_t\right] \\ &= E\left[\sum_{i=N(t)+1}^{N(t+h)} X_i e^{-\delta W_i}\right] \\ &= E\left[Z_{t+h}\right] - E\left[Z_t\right] \\ &= \frac{\mu_1 \lambda}{\delta}\left\{e^{-\delta t} - e^{-\delta(t+h)}\right\} \end{aligned}$$

Let us now consider the discrepancy between the obtained premiums based on the collective risk model equation (6), and by the discounted collective risk model equation (11). In fact equation (6) is a special case of relation (11) when the $\delta \to 0$. It is easy to see that,

$$\lim_{\delta \to 0} E\left[Z_t\right] = E\left[S_t\right].$$

Another special case is when $t \to \infty$. In this case:

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$$\lim_{t\to\infty} E\left[Z_t\right] = \lim_{t\to\infty} \frac{\mu_1\lambda}{\delta}\left(1-e^{-\delta t}\right) = \frac{\mu_1\lambda}{\delta}. \tag{14}$$

which can be interpreted as a net single premium for a perpetuity that continuously pays $\mu_1\lambda$.

Moreover, note that if $\delta\to\infty$, then $E\left[Z_t\right]\to 0$, and when $t\to 0$, then $E\left[Z_t\right]\to 0$, which are reasonable results.

Corollary 2. Suppose that in the discounted collective risk model (7), $N(t)\sim \text{Poisson}(\lambda t)$, then

$$E\left[\sum_{i,j:i<j}^{N(t)} X_iX_je^{-\delta W_i-\delta W_j} \mid N(t)=k\right] = \mu_1^2\frac{k(k-1)}{2}\left(\frac{1-e^{-\delta t}}{\delta t}\right)^2. \tag{15}$$

Proof. First one by going along the same lines of the proof of lemma 1 with some obvious modification it can be shown that

$$f_{W_i|W_j,N(t)}(s\mid u,k) = \frac{j-i+1}{u}\binom{j}{i-1}\left(\frac{s}{u}\right)^{i-1} \times\left(1-\frac{s}{u}\right)^{j-i}, \qquad 0\leqslant s\leqslant u\leqslant t. \tag{16}$$

Now by rule of Iterated expectation we have

$$E\left[\sum_{i,j:i<j}^{N(t)} X_iX_je^{-\delta W_i-\delta W_j} \mid N(t)=k\right] = E\left[E\left\{\sum_{i,j:i<j}^{N(t)} X_iX_je^{-\delta W_i-\delta W_j} \mid W_j=u, N(t)=k\right\} \mid W_j=u\right].$$

By using equation (16) we can get:

$$E\left[\sum_{i,j:i<j}^{N(t)} X_iX_je^{-\delta W_i-\delta u} \mid W_j=u, N(t)=k\right] = \mu_1^2\sum_{j=2}^{k} e^{-\delta u} \times E\left[\sum_{i=1}^{j-1} e^{-\delta W_j} \mid W_j=u, N(t)=k\right] = \mu_1^2\sum_{j=2}^{k} e^{-\delta u}\frac{(j-1)\left(1-e^{-\delta u}\right)}{\delta u}.$$

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Now we calculate the second expectation with respect to W_j,

$$
\begin{aligned}
E\left[\sum_{i,j:i<j}^{N(t)} X_i X_j e^{-\delta W_i-\delta W_j} \mid N(t)=k\right] &= \mu_1^2 E\left[\sum_{j=2}^{N(t)} e^{-\delta W_j} \right.\\
&\quad \left. \times \frac{(j-1)\left(1-e^{-\delta W_j}\right)}{\delta W_j} \mid N(t)=k\right]\\
&= \mu_1^2 \frac{1}{\delta} \sum_{j=2}^{k} (j-1) \left\{ \sum_{l=0}^{\infty} \frac{(-\delta)^l E\left[W_j^{l-1} \mid N(t)=k\right]}{l!} \right.\\
&\quad \left. - \sum_{n=0}^{\infty} \frac{(-2\delta)^n E\left[W_j^{n-1} \mid N(t)=k\right]}{n!} \right\}\\
&= \mu_1^2 \frac{1}{t\delta} \sum_{j=2}^{k} (j-1) \left\{ \sum_{l=0}^{\infty} \frac{(-t\delta)^l}{l!} \frac{(l+j-2)!k!}{(j-1)!(l+k-1)!} \right.\\
&\quad \left. - \sum_{n=0}^{\infty} \frac{(-2t\delta)^n}{n!} \frac{(n+j-2)!k!}{(j-1)!(n+k-1)!} \right\}\\
&= \mu_1^2 \frac{k(k-1)}{t\delta} \left\{ \left(\frac{1-e^{-\delta t}}{\delta t}\right) - \left(\frac{1-e^{-2\delta t}}{2\delta t}\right) \right\}\\
&= \mu_1^2 \frac{k(k-1)}{2} \left(\frac{1-e^{-\delta t}}{\delta t}\right)^2 .
\end{aligned}
$$

Based on this result, it is easy to calculate conditional second moment of Z_t given $N(t)=k$ as follows,

$$
\begin{aligned}
E\left[Z_t^2 \mid N(t)=k\right] &= E\left[\sum_{i=1}^{N(t)} X_i^2 e^{-2\delta W_i} \mid N(t)=k\right]\\
&\quad + 2E\left[\sum_{i,j:i<j}^{N(t)} X_i X_j e^{-\delta w_i-\delta w_j} \mid N(t)=k\right]
\end{aligned}
$$

From theorem 1 and relation (15) we can get:

$$
E\left[Z_t^2 \mid N(t)=k\right] = \mu_2 \frac{k}{\delta t}\left(1-e^{-2\delta t}\right) + \mu_1^2 k(k-1)\left(\frac{1-e^{-\delta t}}{\delta t}\right)^2 .
$$

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After some calculation we find that:

$$\operatorname{var}\left[Z_t \mid N(t)=k\right]=\mu_2 \frac{k}{\delta t}\left(1-e^{-2\delta t}\right)-\mu_1^2 k\left(\frac{1-e^{-\delta t}}{\delta t}\right)^2 \tag{17}$$

Theorem 2. *Consider the discounted collective risk model described in (7), if $N(t) \sim Poisson(\lambda t)$, then*

$$\operatorname{var}\left[Z_t\right]=\frac{\mu_2 \lambda}{2\delta}\left(1-e^{-2\delta t}\right). \tag{18}$$

Proof. By the law of total variance and equation (17) proof is completed.

Corollary 3. The process $\{B_t\}_{t>0}=\left[\left\{Z_t-\frac{\mu_1\lambda}{\delta}\left(1-e^{-\delta t}\right)\right\}^2-\operatorname{var}\left[Z_t\right]\right]_{t>0}$ is a martingale.

Using the martingale approach, many interesting results can be obtained; refer to Gerber and Shiu (1998) for a thorough discussion. In the next theorem we use similar technique to find the moment generating function of Z_t.

Theorem 3. *Let $M_{Z_t}(u)$ denote the m.g.f of Z_t defined by relation (7) and let $N(t) \sim Poisson(\lambda t)$, then*

$$M_{Z_t}(u)=\exp\left[-\int \lambda\left\{1-M_X\left(ue^{-\delta t}\right)\right\}dt\right] \tag{19}$$

where, $M_X(\cdot)$ is m.g.f of X.

Proof. Consider the process $\{M_t\}_{t>0}=\left\{\frac{e^{uZ_t}}{g(t,u)}\right\}_{t>0}$ where $g(t,u)$ is a function to be determined later and satisfies in the initial condition $g(0,u)=1$. We first seek a value of $g(t,u)$ such that $\{M_t\}_{t>0}$ is a martingale. To do this, we note that based on the properties of martingale, M_t must satisfies in the following relation (for all $h>0$):

$$E\left[\frac{M_{t+h}}{M_t} \mid M_t=m_t\right]=1.$$

In this case, we have:

$$E\left[e^{u\sum_{i=N(t)+1}^{N(t+h)} X_i e^{-\delta W_i}}\right]=\frac{g(t+h,u)}{g(t,u)}.$$

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Now by rule of Iterated expectation it can be shown that

$$E\left[E\left\{e^{u\sum_{i=N(t)+1}^{N(t+h)}X_i e^{-\delta W_i}} \mid N(t+h)-N(t)=k\right\}\right]=\sum_{k=0}^{\infty}E\left[e^{u\sum_{i=N(t)+1}^{N(t+h)}X_i e^{-\delta W_i}} \mid N(t+h)-N(t)=k\right]\Pr\left[N(t+h)-N(t)=k\right]$$

$$=\frac{g(t+h,u)}{g(t,u)} \tag{20}$$

Based on the properties of Poisson process we can rewrite (20) as follows,

$$\sum_{k=0}^{\infty}E\left[e^{u\sum_{i=N(t)+1}^{N(t+h)}X_i e^{-\delta W_i}} \mid N(t+h)-N(t)=k\right]\Pr\left[N(t+h)-N(t)=k\right]$$
$$=(1-\lambda h)+E\left[e^{uX_{N(t+h)}e^{-\delta(t+h)}} \mid N(t+h)-N(t)=1\right]\lambda h+o(h)$$
$$=(1-\lambda h)+M_X(ue^{-\delta(t+h)})\lambda h+o(h)$$
$$=\frac{g(t+h,u)}{g(t,u)}$$

Where $o(h)$ is a generic function that goes to zero faster than h when h goes to zero. By a few simplification we have:

$$-\lambda\left[1-M_X\{ue^{-\delta(t+h)}\}\right]+\frac{o(h)}{h}=\frac{g(t+h,u)-g(t,u)}{h.g(t,u)} \tag{21}$$

Taking limits as $h\to 0$ in above relation, we have:

$$\frac{d}{dt}\ln g(t,u)=-\lambda\left\{1-M_X(ue^{-\delta t})\right\}, \tag{22}$$

Now it is sufficient to show that $M_{Z_t}(u)=g(t,u)$. It follows from the initial condition $g(0,u)=1$ that $M_0=1$. Moreover based on the properties of martingale, we have $E(M_t)=E(M_0)=1$ which complete the proof.

Example 1. In the discounted collective risk model, let claim sizes are exponentially distributed with mean β then the m.g.f of Z_t is given by:

$$M_{Z_t}(u)=E\left[e^{uZ_t}\right]=\left(\frac{1-\beta ue^{-\delta t}}{1-\beta u}\right)^{\frac{\lambda}{\delta}}. \tag{23}$$

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It follows from this example that:

$$\lim_{\delta\to 0} M_{Z_t}(u) = \exp\left(\frac{\lambda t u\beta}{1-u\beta}\right) \tag{24}$$

which is precisely the m.g.f of S_t given in the relation (4). Moreover by limiting when t tend to infinity we have:

$$\lim_{t\to\infty} M_{Z_t}(u) = (1-u\beta)^{-\frac{\lambda}{\delta}} \tag{25}$$

which is coincide to the result that Gerber (1979) has obtained.

An interesting point is to find the distribution; this does not seem an easy task. Nevertheless we can use the Fourier Transform to perform numerical approximation. We know that for given moment generating function, there always exists a unique distribution. The Fast Fourier Transform (FFT) technique gives a fast way to compute a distribution from its moment generating function. This numerical methods can easily be implemented in **R** software and using its **Actuar** package. Figure 1, shows the differences between collective risk model and the discounted collective risk model. As it appear from these figures, the difference between two models increase with t.

4 Conclusion

In studying economic phenomena, the effects of interest and inflation must

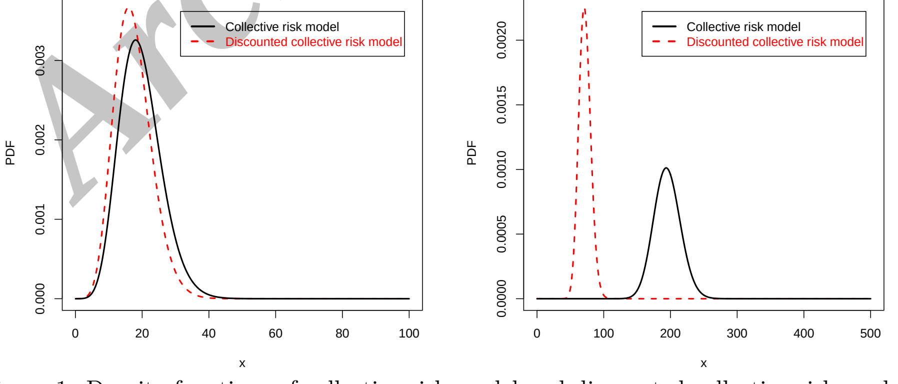

Figure 1. Density functions of collective risk model and discounted collective risk model for λ=4, β=1, δ=0.05, t=5 (left side) and $t = 50$ (right side).

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be taken into account. As it was seen in this paper, Z_t, is a version of S_t that takes into account the interest rate. Knowing the moments of Z_t is useful to approximate the distribution of it. Moreover it is useful for calculating the premium based on the various premium principles. As it said in the text, there are many researches that have studied the distribution and features of total loss collective risk models. In those researches the distribution of Z_t have studied for infinite time horizon. But there are many practical problems that is not true assumption. The interested reader is referred to Gerber (1979) for more applied details.

In this paper we obtain the mean and variance of Z_t for all values of t . It has also been demonstrated that some functions of Z_t are martingale and use this feature to obtain the m.g.f of Z_t for all values of t . Although, we used classic approach to obtain the mean and variance of total loss, it may be use the similar approach with the m.g.f, to get the mean and variance by martingale properties.

The obtained formula for m.g.f of Z_t by Theorem 3 works for every claim size distribution and poisson distribution for the counting process. A formal example is exponential distribution for claim size that the results of it and limiting cases are agree with previous researches. Additionally, we use Fast Fourier Transform to numerically calculate the distribution of discounted collective risk model.

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