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On the Distribution Functions of the Range and Quasi-range for the Extended Type I Generalized Logistic Distribution

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Abstract. In this paper, we obtain the distribution functions of the range and the quasi-range of the random variables arising from the extended type I generalized logistic distribution.

Keywords. Extended type I generalized logistic distribution; order statistics; Quasi-range and range.

1 Introduction

The probability density function (pdf) of the standard logistic distribution is

$$f_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \qquad -\infty < x < \infty,$$

and hence its cumulative distribution function (cdf) is

$$F_X(x) = (1+e^{-x})^{-1}, \qquad -\infty < x < \infty.$$

The shape of this distribution is similar to that of normal distribution has made it to be preferred to normal distribution by some researchers like Berkson (1944, 1950, 1953), Berkson and Hodges (1960), etc. Ojo (1989) used the logistic model to analyze some social data sets. Researchers have been working on order statistics from the logistic distribution for a long time. It was considered in Plackett (1958), Birnbaum and Dudman (1963), Tarter and Clark (1965), Shah (1966, 1970). Gupta et al. (1967) obtained best unbiased estimators for the parameters of the logistic distribution using order statistics. It is well known that the range and quasi-range are important statistics which are defined based on order statistics. Gupta and Shah (1965) obtained the distribution of the range from the logistic distribution while Malik (1980) obtained the distribution function of the quasi-range from the logistic distribution. In recent times, researchers have focused more on generalizing pdfs with the aim of making the functions to be more robust and applicable to model different types of data. The logistic distribution has enjoyed the practice of generalization in many forms as could be seen in George and Ojo (1980), Balakrishnan and Leung (1988a), Wu et al. (2000), Olapade (2004, 2005, 2006). Though, many works have been done on the order statistics from the logistics distribution, many of its various generalizations of the distribution have not enjoyed such privilege. Balakrishnan and Leung (1988a) studied order statistics from the type I generalized logistic distribution while Balakrishnan and Leung (1988b) obtained the means, variances and covariances of the order statistics, BLUE's for the type I generalized logistic distribution. In this paper, we shall obtain the distributions of the range and quasi-range of the extended type I generalized logistic distribution with pdf

$$f_X(x;\lambda,p) = \frac{p\lambda^p e^{-x}}{(\lambda+e^{-x})^{p+1}}, \qquad -\infty < x < \infty,\ p>0,\ \lambda>0 \tag{1}$$

and cdf

$$F_X(x;\lambda,p) = \frac{\lambda^p}{(\lambda+e^{-x})^{p}}, \qquad -\infty < x < \infty,\ p>0,\ \lambda>0. \tag{2}$$

Some properties and application of this distribution were presented in Olapade (2009) who obtained the distribution of the rth order statistics and established the pdfs of the maximum and minimum order statistics in a random sample.

2 Distribution of the Range

Given a set of random variables $X_1, X_2, \ldots, X_n$ of size n coming from the extended type I generalized logistic distribution, let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ be the corresponding order statistics. Let $F_{X_{r:n}}(x)$ and $f_{X_{r:n}}(x)$, $r = 1, 2, \ldots, n$ be the cdf and pdf of the rth order statistics $X_{r:n}$ respectively.

David (1970) obtained the pdf of $X_{r:n}$ as

$$f_{X_{r:n}}(x) = \frac{1}{B(r, n-r+1)}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x).$$

Let us define the sample range W_n by $W_n = X_{n:n} - X_{1:n}$. The cdf of W_n can be written as (Gupta and Shah, 1965)

$$Pr(W_n \leqslant w) = n\int_{-\infty}^{+\infty}\{F(x+w) - F(x)\}^{n-1}f(x)dx.$$

By expanding $\{F(x+w) - F(x)\}^{n-1}$, we have

$$Pr(W_n \leqslant w) = n\sum_{k=0}^{n-1}\binom{n-1}{k}\int_{-\infty}^{+\infty}\{F(x+w)\}^{n-1-k}\{-F(x)\}^k f(x)dx. \quad (3)$$

Substituting (1) and (2) in (3) we have

$$\begin{aligned}
Pr(W_n \leqslant w) &= n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\int_{-\infty}^{+\infty}\left\{\frac{\lambda}{\lambda+e^{-w-x}}\right\}^{p(n-1-k)} \\
&\quad \times\left\{\frac{\lambda}{\lambda+e^{-x}}\right\}^{pk}\frac{p\lambda^p e^{-x}}{(\lambda+e^{-x})^{p+1}}dx \\
&= np\lambda^{np}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k} \\
&\quad \int_{-\infty}^{+\infty}\frac{e^{-x}}{(\lambda+e^{-w-x})^{p(n-1-k)}(\lambda+e^{-x})^{pk+p+1}}dx.
\end{aligned}$$

Let $t = (\lambda + ae^{-x})^{-1}$, where $a = e^{-w}$, then

$$\begin{aligned}
Pr(W_n \leqslant w) &= np\lambda^{np}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}a^{pk+p} \\
&\quad \int_0^{\frac{1}{\lambda}}\frac{t^{np-1}}{\{1+\lambda(a-1)t\}^{pk+p+1}}dt \\
&= np\lambda^{np}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}a^{p(k+1)}\int_0^{\frac{1}{\lambda}}\frac{t^{np-1}}{(1+bt)^{pk+p+1}}dt \\
&= np\lambda^{np}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}a^{p(k+1)}A(k,p,n,\lambda), \quad (4)
\end{aligned}$$

where $b = \lambda(a-1)$, $v = 1 + bt$ and

$$
\begin{aligned}
A(k,p,n,\lambda) &= \int_0^{\frac{1}{\lambda}} \frac{t^{np-1}}{(1+bt)^{pk+p+1}} dt \\
&= \frac{-1}{(-b)^{np}} \int_1^{\frac{1+b}{\lambda}} v^{-pk-p-1}(1-v)^{np-1} dv, \\
&= \frac{-1}{(-b)^{np}} \sum_{j=0}^{np-1} (-1)^j \binom{np-1}{j} \int_1^{\frac{1+b}{\lambda}} v^{j-pk-p-1} dv \\
&= \frac{-1}{(-b)^{np}} \Bigg\{ (-1)^{pk+p} \binom{np-1}{pk+p} \ln\left(1+\frac{b}{\lambda}\right) \\
&\quad + \sum_{j=0, j\neq pk+p}^{np-1} (-1)^j \binom{np-1}{j} \frac{\left(\frac{1+b}{\lambda}\right)^{j-pk-p} - 1}{j-pk-p} \Bigg\}
\end{aligned}
$$

Substitute for $A(k,p,n,\lambda)$ in (4), we have

$$
\begin{aligned}
Pr(W_n \leqslant w) &= np \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n-1}{k} \\
&\quad a^{p(k+1)} \frac{1}{(1-a)^{np}} \Bigg\{ (-1)^{pk+p} \binom{np-1}{pk+p} \ln a \\
&\quad + \sum_{j=0, j\neq pk+p}^{np-1} (-1)^j \binom{np-1}{j} \frac{a^{j-pk-p} - 1}{j-pk-p} \Bigg\} \\
&= \frac{np}{(1-e^{-w})^{np}} \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n-1}{k} \\
&\quad \Bigg\{ (-1)^{pk+p-1} \binom{np-1}{pk+p} w e^{-wp(k+1)} \\
&\quad + \sum_{j=0, j\neq pk+p}^{np-1} (-1)^j \binom{np-1}{j} \frac{e^{-wj} - e^{-wp(k+1)}}{j-pk-p} \Bigg\}.
\end{aligned}
$$

By differentiating the distribution function of the sample range in equation (2.14) with respect to w, we derive the pdf of W_n as

$$
\begin{aligned}
p(w) = \frac{n^2p^2e^{-w}}{(1-e^{-w})^{np+1}} \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k} \\
\Big\{(-1)^{pk+p-1}\binom{np-1}{pk+p} we^{-wp(k+1)} \\
+ \sum_{j=0, j\neq pk+p}^{np-1} (-1)^j \binom{np-1}{j} \frac{e^{-wj} - e^{-wp(k+1)}}{j-pk-p}\Big\} \\
+ \frac{np}{(1-e^{-w})^{np}} \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k} \\
\Big[(-1)^{pk+p-1}\binom{np-1}{pk+p}\{1-wp(k+1)\}e^{-wp(k+1)} \\
+ \sum_{j=0, j\neq pk+p}^{np-1} (-1)^j \binom{np-1}{j} \frac{p(k+1)e^{-wp(k+1)} - je^{-wj}}{j-pk-p}\Big].
\end{aligned}
$$

It could be noted that the $F_W(w)$ and $f_W(w)$ of the extended type I generalized logistic distribution are free of λ.

3 Distribution of the Quasi-range

The sample rth quasi-range denoted by W, is defined as

$$W = X_{n-r:n} - X_{r+1:n}, \qquad r = 0, 1, \ldots, \frac{n-1}{2},$$

where n is odd. Thus the joint pdf of $X_{r+1:n}$ and $X_{n-r:n}$ is

$$
\begin{aligned}
f(x_{r+1:n}, x_{n-r:n}) &= \frac{n!}{r!(n-2r-2)!r!}\{F(x_{r+1:n})\}^r \\
&\times \{F(x_{n-r:n}) - F(x_{r+1:n})\}^{n-2r-2}\{1-F(x_{n-r:n})\}^r \\
&\times f(x_{n-r:n})f(x_{r+1:n}), \quad -\infty < X_{r+1:n} < X_{n-r:n} < \infty.
\end{aligned}
$$

Since $X_{n-r:n} = X_{r+1:n} + W$, we have

$$\begin{aligned}
Pr(W \leqslant w) &= \int_{-\infty}^{+\infty} \int_{x_{r+1:n}}^{x_{r+1:n}+w} f(x_{r+1:n}, x_{n-r:n}) dx_{n-r:n} dx_{r+1:n} \\
&= \frac{n!}{r!(n-2r-2)!r!} \int_{-\infty}^{+\infty} \{F(x)\}^r f(x) \\
&\quad \times \left[\int_0^{x+w} \{1-F(u)\}^r \{F(u)-F(x)\}^{n-2r-2} f(u) du \right] dx \\
&= \frac{n!}{r!(n-2r-2)!r!} \int_{-\infty}^{+\infty} \{F(x)\}^r f(x) \\
&\quad \left[\int_{F(x)}^{F(x+w)} (1-y)^r \{y-F(x)\}^{n-2r-2} dy \right] dx.
\end{aligned}$$

Integrating the expression in braces r times by parts, we have

$$\begin{aligned}
Pr(W \leqslant w) &= \sum_{k=0}^{r} \prod_{i=0}^{2r-k} \frac{(n-i)}{r!(r-k)!} \int_{-\infty}^{+\infty} \{F(x)\}^r \{1-F(x+w)\}^{r-k} \\
&\quad \{F(x+w)-F(x)\}^{n-2r+k-1} f(x) dx \\
&= \sum_{k=0}^{r} \prod_{i=0}^{2r-k} \frac{(n-i)}{r!(r-k)!} \sum_{j=0}^{n-2r+k-1} (-1)^j \\
&\quad \binom{n-2r+k-1}{j} \sum_{l=0}^{r-k} (-1)^l \binom{r-k}{l} \\
&\quad \times \int_{-\infty}^{\infty} \{F(x)\}^{r+j} \{F(x+w)\}^{n-2r+k-j+l-1} f(x) dx.
\end{aligned}$$

Let $\Lambda = \int_{-\infty}^{\infty} \{F(x)\}^{r+j} \{F(x+w)\}^{n-2r+k-j+l-1} f(x) dx$, using $f(x; \lambda, p)$ and $F(x; \lambda, p)$ shown in equations (1) and (2) respectively we have

$$\begin{aligned}
\Lambda &= \int_{-\infty}^{+\infty} \left(\frac{\lambda}{\lambda + e^{-x}} \right)^{p(r+j)} \left(\frac{\lambda}{\lambda + e^{-x-w}} \right)^{p(n-2r+k-j+l-1)} \\
&\quad \times \frac{p\lambda^p e^{-x}}{(\lambda + e^{-x})^{p+1}} \, dx \\
&= \int_{-\infty}^{+\infty} \frac{p\lambda^{p(n-r+k+l)} e^{-x}}{(\lambda + a e^{-x})^{p(n-2r+k-j+l-1)} (\lambda + e^{-x})^{pr+pj+p+1}} \, dx,
\end{aligned}$$

Let $t = (\lambda + ae^{-x})^{-1}$, where $a = e^{-w}$, then

$$\Lambda = pa^{p(r+j+1)}\lambda^{p(n-r+k+l)} \int_0^{\frac{1}{\lambda}} \frac{t^{pn-pr+pk+pl-1}}{\{1+t\lambda(a-1)\}^{pr+pj+p+1}}\ dt.$$

Taking $\lambda(a-1) = b$ and $1 + bt = u$, we have

$$\begin{aligned}
\Lambda &= \frac{-pa^{p(r+j+1)}}{(1-e^{-w})^{p(n-r+k+l)}} \\
&\quad \times \int_1^{1+\frac{b}{\lambda}} (1-u)^{pn-pr+pk+pl-1} u^{-(pr+pj+p+1)} du. \\
&= \frac{-pa^{p(r+j+1)}}{(1-e^{-w})^{p(n-r+k+l)}} \\
&\quad \times \int_1^{1+\frac{b}{\lambda}} \sum_{m=0}^{pn-pr+pk+pl-1} (-1)^m \binom{pn-pr+pk+pl-1}{m} u^{m-pr-pj-p-1} du \\
&= \frac{-pa^{p(r+j+1)}}{(1-e^{-w})^{p(n-r+k+l)}} \\
&\quad \times \int_1^{1+\frac{b}{\lambda}} \Bigg\{ (-1)^{p(r+j+1)} \binom{pn-pr+pk+pl-1}{pr+pj-p} u^{-1} \\
&\quad + \sum_{m=0, m\neq p(r+j+1)}^{pn-pr+pk+pl-1} (-1)^m \binom{pn-pr+pk+pl-1}{m} u^{m-pr-pj-p-1} \Bigg\} du \\
&= \frac{pe^{-wp(r+j+1)}}{(1-e^{-w})^{p(n-r+k+l)}} \Bigg\{ (-1)^{p(r+j+1)} \binom{pn-pr+pk+pl-1}{pr+pj-p} w \\
&\quad + \sum_{m=0, m\neq p(r+j+1)}^{pn-pr+pk+pl-1} (-1)^{m+1} \binom{pn-pr+pk+pl-1}{m} \frac{e^{-w(m-pr-pj-p)}-1}{m-pr-pj-p} \Bigg\}.
\end{aligned}$$

Finally,

$$\begin{aligned}
Pr(W \leqslant w) = \sum_{k=0}^{r} \prod_{i=0}^{2r-k} \frac{(n-i)}{r!(r-k)!} \sum_{j=0}^{n-2r+k-1} (-1)^j \binom{n-2r+k-1}{j} \\
\times \sum_{l=0}^{r-k} (-1)^l \binom{r-k}{l} \cdot \frac{pe^{-wp(r+j+1)}}{(1-e^{-w})^{p(n-r+k+l)}}
\end{aligned}$$

$$\times \left\{ (-1)^{p(r+j+1)} \binom{pn-pr+pk+pl-1}{pr+pj-p} w \right.$$
$$\left. + \sum_{m=0, m\neq p(r+j+1)}^{pn-pr+pk+pl-1} (-1)^{m+1} \binom{pn-pr+pk+pl-1}{m} \frac{e^{-w(m-pr-pj-p)}-1}{m-pr-pj-p} \right\}.$$

It should be noted also that the distribution function of quasi-range of the extended type I generalized logistic distribution is free of the parameter λ. When $p = 1$, the result obtained agrees with Malik (1980) for the standard logistic distribution.

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