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Bayesian Prediction Intervals under Bivariate Truncated Generalized Cauchy Distribution

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Abstract. Ateya and Madhagi (2011) introduced a multivariate form of truncated generalized Cauchy distribution (TGCD), which introduced by Ateya and Al-Hussaini (2007). The multivariate version of (TGCD) is denoted by (MVTGCD). Among the features of this form are that subvectors and conditional subvectors of random vectors, distributed according to this distribution, have the same form of distribution (MVTGCD). They also introduced the joint density function, conditional density function, moment generating function and mixed moments. Also, they estimated all parameters of the distribution using the maximum likelihood and Bayes methods. In this paper, we used the point of view, introduced by Al-Hussaini and Ateya (2010), to obtain the Highest Posterior Density (HPD) prediction intervals of future observations from bivariate truncated generalized Cauchy distribution (BVTGCD).

Keywords. Bayesian prediction intervals; highest posterior density (HPD) prediction intervals; generalized Cauchy distribution; moment generating function; mixed moments.

MSC 2010: 62F10, 62F15.

1 Introduction

The Cauchy distribution is a well known symmetric distribution which is often used in outlier analysis. It is also well-known that the Cauchy distribution

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can arise as the ratio of two independent normal variates. The probability density function (*pdf*) with location parameter μ (representing the population median) and scale parameter γ (representing the semi-quartile range) is given by

$$f_X(x) = \frac{1}{\pi\gamma}\left\{1+\left(\frac{x-\mu}{\gamma}\right)^2\right\}^{-1}, \quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \gamma > 0. \tag{1}$$

The bivariate Cauchy distribution has received applications in many areas, including biological analysis, clinical trials, stochastic modelling of decreasing failure rate life components, queueing theory, and reliability (see, for example, Nayak (1987) and Lee and Gross (1991)). For data from these areas, there is no reason to believe that empirical moments of any order should be infinite. Thus, the choice of the bivariate Cauchy distribution as a model is unrealistic since its mixed moments of all orders are not finite. The introduced bivariate truncated generalized Cauchy distribution can be a more appropriate model for the kind of data mentioned.

The books by Johnson et al(1994) and Kotz et al. (2000) cover the Cauchy distribution and many univariate and multivariate distributions in many of their aspects starting from the history, properties, developments and applications up to the most recent research done in the subject matter, to the date of the books' publication.

A random variable X is said to have a truncated generalized Cauchy distribution (TGCD) according to Ateya and AL-Hussaini (2007), if its *pdf*, takes the form

$$f_X(x) = \frac{2}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\gamma\Gamma(\alpha)}\left\{1+\left(\frac{x-\mu}{\gamma}\right)^2\right\}^{-\alpha-\frac{1}{2}}, \quad x \geqslant \mu, (\mu,\gamma,\alpha>0). \tag{2}$$

Among the features of (TGCD) are that while the moment generating function (mgf) of the Cauchy *pdf* (1) (and the moments of any order) do not exist, the (mgf) of the TGCD and moments of order b do exist if $b < 2\alpha$. Ateya and AL-Hussaini (2007), studied the properties of TGCD and used the maximum likelihood and Bayes methods to estimate the parameters μ, γ and α. We shall write $X \sim \text{TGCD}(\mu,\gamma,\alpha)$ to denote that the random variable X has *pdf* (2).

The positive value μ is very important in industry because it represents the minimum time μ before which no failure occurs (guarantee time). Another use for μ is in epidemiological or biomedical applications where μ may

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represent the latent period of some disease (the time elapsed between first exposure to carcinogen and the appearance of the disease).

Ateya and Madhagi (2011) introduced a multivariate version of TGCD and gave it the notation MVTGCD by compounding $L(\theta;\boldsymbol{x}) = \prod_{i=1}^{k} f_{X_i|\Theta}(x_i|\theta)$ and $g_\Theta(\theta)$ as follows

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \int_0^\infty L(\theta;\boldsymbol{x})\, g_\Theta(\theta) d\theta,$$

where $\boldsymbol{X} = (X_1, X_2, \ldots, X_k)$ is a random vector such that $X_i|\Theta$ is distributed as half normal with location parameter μ_i and scale parameter $1/\sqrt{\theta}$ with the following *pdf*

$$f_{X_i|\Theta}(x_i|\theta) = \frac{2}{\sqrt{2\pi}} \theta^{\frac{1}{2}} \exp\left\{-\frac{\theta}{2}(x_i-\mu_i)^2\right\} \quad x_i \geqslant \mu_i,\ i = 1, 2, \ldots, k, \tag{3}$$

and Θ is a positive random variable follows the Gamma $(\alpha, \frac{\gamma_i^2}{2})$ distribution with the following *pdf* $(g_\Theta(\theta))$

$$g_\Theta(\theta) = \frac{\gamma_i^{2\alpha}}{2^\alpha\,\Gamma(\alpha)}\, \theta^{\alpha-1} \exp\left\{-\frac{\gamma_i^2}{2}\,\theta\right\}, \quad \theta > 0,\ (\alpha, \gamma_i > 0). \tag{4}$$

Then the pdf $f_{\boldsymbol{X}}(\boldsymbol{x})$ takes the form

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \left(\frac{4}{\pi}\right)^{\frac{k}{2}} \frac{\Gamma\left(\alpha + \frac{k}{2}\right)}{\left(\prod_{i=1}^{k}\gamma_i\right)\Gamma(\alpha)} \left\{1 + \sum_{i=1}^{k}\left(\frac{x_i-\mu_i}{\gamma_i}\right)^2\right\}^{-\alpha-\frac{k}{2}}, \quad x_i \geqslant \mu_i,\ \ (\mu_i,\, \gamma_i,\, \alpha > 0), i = 1, 2, \ldots, k. \tag{5}$$

and also the corresponding (mgf) and mixed moments are in the forms

$$M_{X_1^{b_1}, X_2^{b_2}, \ldots, X_k^{b_k}}(t_1, t_2, \ldots, t_k) = \left(\frac{4}{\pi}\right)^{\frac{k}{2}} \frac{\Gamma\left(\alpha+\frac{k}{2}\right)}{\Gamma(\alpha)} \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \exp\left[\sum_{i=1}^{k} t_i \left(\mu_i + \gamma_i \tan\theta_i \prod_{j=i+1}^{k} \sec\theta_j\right)^{b_i}\right] \prod_{s=1}^{k} (\cos\theta_s)^{2\alpha+k-(s+1)} d\theta_1 d\theta_2 \ldots d\theta_k. \tag{6}$$

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$$\mu'_{b_1, b_2, \ldots, b_k} = \left(\frac{1}{\pi}\right)^{\frac{k}{2}} \frac{\Gamma\left(\alpha+\frac{k}{2}\right)}{\Gamma(\alpha)} \sum_{j_1=0}^{b_1} \sum_{j_2=0}^{b_2} \cdots \sum_{j_k=0}^{b_k} \binom{b_1}{j_1}\binom{b_2}{j_2} \cdots \binom{b_k}{j_k}$$
$$\left[\gamma_1^{j_1}\,\gamma_2^{j_2} \ldots \gamma_k^{j_k}\right]\left[\mu_1^{b_1-j_1}\,\mu_2^{b_2-j_2} \ldots \mu_k^{b_k-j_k}\right] \prod_{s=1}^{k} B\left(\frac{j_s+1}{2}, \alpha+\frac{k}{2}-\frac{\sum_{l=1}^{s} j_l}{2}\right.$$
$$\left.-\frac{s}{2}\right), \qquad \sum_{i=1}^{k} b_i < 2\alpha, \tag{7}$$

where $B(a, b)$ is the standard beta function.

1.1 Univariate, Bivariate and Trivariate Cases

In this subsection we write the *pdf*, (mgf) and the mixed moments in case of $n = 1$ (univariate case), $n = 2$ (bivariate case) and n=3 (trivariate case) as follows

Univariate Case ($n = 1$)

For positive parameters μ, γ, α and non-negative int integer b we have
pdf

$$f_X(x) = \left(\frac{4}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\gamma\Gamma(\alpha)} \left\{1+\left(\frac{x-\mu}{\gamma}\right)^2\right\}^{-\alpha-\frac{1}{2}}, \quad x \geqslant \mu,\ (\mu,\ \gamma,\ \alpha > 0),$$

mgf

$$M_{X^b}(t) = \left(\frac{4}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)} \int_0^{\pi/2} \exp\left\{t\Big(\mu+\gamma\tan\theta\Big)^b\right\} (\cos\theta)^{2\alpha-1} d\theta,$$

Moments

$$E[X^b] = \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)} \sum_{j=0}^{b} \binom{b}{j} \gamma^j\, \mu^{b-j} B\left(\frac{j+1}{2}, \alpha-\frac{j}{2}\right), \quad b < 2\alpha.$$

Bivariate Case ($n = 2$)

For positive parameters $\mu_1, \mu_2, \gamma_1, \gamma_2, \alpha$ and non-negative integers b_1, b_2 we

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have
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$$f_{X_1,X_2}(x_1,x_2) = \left(\frac{4}{\pi}\right)\frac{\Gamma(\alpha+1)}{\gamma_1\,\gamma_2\,\Gamma(\alpha)}\left\{1+\left(\frac{x_1-\mu_1}{\gamma_1}\right)^2+\left(\frac{x_2-\mu_2}{\gamma_2}\right)^2\right\}^{-\alpha-1},$$
$$x_i \geqslant \mu_i,\ \ (\mu_i,\,\gamma_i,\,\alpha>0),\quad i=1,2,$$

mgf

$$M_{X_1^{b_1},X_2^{b_2}}(t_1,t_2) = \left(\frac{4}{\pi}\right)\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\exp\left\{\sum_{i=1}^{2}t_i\Big(\mu_i + \gamma_i\tan\theta_i\prod_{j=i+1}^{2}\sec\theta_j\Big)^{b_i}\right\}\prod_{s=1}^{2}(\cos\theta_s)^{2\alpha+2-(s+1)}d\theta_1 d\theta_2,$$

Moments

$$E[X_1^{b_1}X_2^{b_2}] = \left(\frac{1}{\pi}\right)\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\sum_{j_1=0}^{b_1}\sum_{j_2=0}^{b_2}\binom{b_1}{j_1}\binom{b_2}{j_2}\gamma_1^{j_1}\gamma_2^{j_2}\,\mu_1^{b_1-j_1}\mu_2^{b_2-j_2}$$
$$\prod_{s=1}^{2}B\left(\frac{j_s+1}{2},\alpha+1-\frac{\sum_{l=1}^{s}j_l}{2}-\frac{s}{2}\right),\quad b_1+b_2<2\alpha,$$

Trivariate Case $(n=3)$
For positive parameters $\mu_1,\mu_2,\mu_3,\gamma_1,\gamma_2,\gamma_3,\alpha$ and non-negative integers $b_1,b_2,$ b_3 we have

pdf

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \left(\frac{4}{\pi}\right)^{\frac{3}{2}}\frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\gamma_1\,\gamma_2\,\gamma_3\,\Gamma(\alpha)}\left\{1+\left(\frac{x_1-\mu_1}{\gamma_1}\right)^2+\left(\frac{x_2-\mu_2}{\gamma_2}\right)^2 + \left(\frac{x_3-\mu_3}{\gamma_3}\right)^2\right\}^{-\alpha-\frac{3}{2}},\ x_i\geqslant\mu_i,\,(\mu_i,\,\gamma_i,\,\alpha>0),\quad i=1,2,3,$$

mgf

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$$M_{X_1^{b_1},X_2^{b_2},X_3^{b_3}}(t_1,t_2,t_3) = \left(\frac{4}{\pi}\right)^{\frac{3}{2}} \frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha)} \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \exp\left\{\sum_{i=1}^{3} t_i\Big(\mu_i + \gamma_i \tan\theta_i \prod_{j=i+1}^{3} \sec\theta_j\Big)^{b_i}\right\} \prod_{s=1}^{3} (\cos\theta_s)^{2\alpha+3-(s+1)} d\theta_1 d\theta_2 d\theta_3,$$

Moments

$$E[X_1^{b_1}X_2^{b_2}X_3^{b_3}] = \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{\Gamma(\alpha+3/2)}{\Gamma(\alpha)} \sum_{j_1=0}^{b_1}\sum_{j_2=0}^{b_2}\sum_{j_3=0}^{b_3} \binom{b_1}{j_1}\binom{b_2}{j_2}\binom{b_3}{j_3} \times \gamma_1^{j_1}\gamma_2^{j_2}\gamma_3^{j_3}\,\mu_1^{b_1-j_1}\mu_2^{b_2-j_2}\mu_3^{b_3-j_3} \prod_{s=1}^{3} B\Big(\frac{j_s+1}{2}, \alpha+\frac{3}{2} - \frac{\sum_{l=1}^{s} j_l}{2} - \frac{s}{2}\Big), \qquad b_1+b_2+b_3 < 2\alpha.$$

where $B(a,b)$ is the standard beta function.

1.2 Generation of a Multivariate Random Sample of Size n From MVTGCD

In this subsection, we generate a multivariate random sample of size n from MTGCD by the following steps:

1. Using the parameters α, μ_i and $\gamma_i, i = 1, 2, \ldots, k$ and for fixed i generate θ from $Gamma(\alpha\,, \frac{2}{\gamma_i^2})$.
2. Generate $z_j = \frac{x_{ij}-\mu_i}{\frac{1}{\sqrt{\theta}}}$ from N$(0,1)$.
3. $x_{ij} = \frac{z_j}{\sqrt{\theta}} + \mu_i$ from N$(\mu_i, \frac{1}{\sqrt{\theta}})$.
4. If $x_{ij} \geqslant \mu_i$ then accept x_{ij}, else go to step 2.
5. Repeat steps $2, 3$ and 4 n times to obtain the sample $x_{i1}\,, x_{i2}, \ldots, x_{in}$.
6. Repeat steps $1, 2, 3$ and 4 for $i = 1, 2, 3, \ldots, k$ to get the multivariate sample $\boldsymbol{x} = (\boldsymbol{x_1}\,, \boldsymbol{x_2}, \ldots, \boldsymbol{x_j}, \ldots, \boldsymbol{x_n}), \boldsymbol{x_j} = (x_{1j}, \ldots, x_{kj}), j = 1, 2, \ldots, n.$

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1.3 The Relations Between the Proposed Univariate and Bivariate Truncated Generalized Cauchy Distribution and the Truncated Univariate and Bivariate Cauchy

Univariate Case

A random variable X is said to have Cauchy distribution with location parameter μ and scale parameter γ if its *pdf*, takes the form

$$f_X(x) = \frac{1}{\pi}\left\{ 1+\left(\frac{x-\mu}{\gamma}\right)^2 \right\}^{-1}, -\infty < x < \infty,\ -\infty < \mu < \infty,\ \gamma > 0. \quad (8)$$

The doubly truncated version of (8) can be easily derived in the form

$$f_X(x) = \frac{\left\{1+\left(\frac{x-\mu}{\gamma}\right)^2\right\}^{-1}}{\gamma\left\{\arctan\left(\frac{b-\mu}{\gamma}\right)+\arctan\left(\frac{a-\mu}{\gamma}\right)\right\}}, \quad a \leqslant x \leqslant b,\ \ a \leqslant \mu \leqslant b,\ \ -\infty < a, b < \infty,\ \ \gamma > 0. \quad (9)$$

If the random variable X represents a failure time and the parameter μ represents a guarantee time ($\mu \geqslant 0$), then $\mu = a, b = \infty$ and the *pdf* in (9) will be in the form

$$f_X(x) = \frac{2}{\pi\gamma\left\{ 1+\left(\frac{x-\mu}{\gamma}\right)^2 \right\}}, \qquad x \geq \mu,\ (\mu,\ \gamma > 0). \quad (10)$$

It is clear that (10) is a special case of the proposed *pdf* in (2) in case of $\alpha = \frac{1}{2}$.

Bivariate Case

It is well-known that the bivariate version of (8) takes the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi\,\gamma_1\,\gamma_2}\left\{ 1+\left(\frac{x-\mu_1}{\gamma_1}\right)^2 + \left(\frac{y-\mu_2}{\gamma_2}\right)^2 \right\}^{-\frac{3}{2}}, \quad -\infty < x, y < \infty,\ \ -\infty < \mu_1, \mu_2 < \infty,\ \ \gamma_1, \gamma_2 > 0, \quad (11)$$

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and the corresponding cumulative distribution function can be written in the form (see Nadarajah and Kotz (2007))

$$F_{X,Y}(x,y) = \frac{1}{4} + \frac{1}{2\pi}\left[\arctan\left(\frac{x-\mu_1}{\gamma_1}\right) + \arctan\left(\frac{y-\mu_2}{\gamma_2}\right) + \arctan\left\{\frac{\left(\frac{x-\mu_1}{\gamma_1}\right)\left(\frac{y-\mu_2}{\gamma_2}\right)}{\sqrt{1+\left(\frac{x-\mu_1}{\gamma_1}\right)^2+\left(\frac{y-\mu_2}{\gamma_2}\right)^2}}\right\}\right],$$

$$-\infty < x, y < \infty, \quad -\infty < \mu_1, \mu_2 < \infty, \quad \gamma_1, \gamma_2 > 0, \tag{12}$$

and the truncated version of (11) can be written in the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi\,\gamma_1\,\gamma_2\Omega}\left\{1+\left(\frac{x-\mu_1}{\gamma_1}\right)^2+\left(\frac{y-\mu_2}{\gamma_2}\right)^2\right\}^{-\frac{3}{2}},$$

$$-\infty < a \leqslant x \leqslant b < \infty, \quad -\infty < c \leqslant y \leqslant d < \infty,$$

$$-\infty < a \leqslant \mu_1 \leqslant b < \infty, -\infty < c \leqslant \mu_2 \leqslant d < \infty, \quad \gamma_1, \gamma_2 > 0, \tag{13}$$

where

$$\Omega = F_{X,Y}(a,c) + F_{X,Y}(b,d) - F_{X,Y}(a,d) - F_{X,Y}(b,c). \tag{14}$$

If the random variables X and Y represent failure times and the parameters μ_1 and μ_2 represent a guarantee times($\mu_1 \geqslant 0$, $\mu_2 \geqslant 0$), then $\mu_1 = a, b = \infty$ and $\mu_2 = c, d = \infty$, then $\Omega = \frac{1}{4}$ and the *pdf* in (13) will be in the form

$$f_{X,Y}(x,y) = \left(\frac{2}{\pi\,\gamma_1\,\gamma_2}\right)\left\{1+\left(\frac{x-\mu_1}{\gamma_1}\right)^2+\left(\frac{y-\mu_2}{\gamma_2}\right)^2\right\}^{-\frac{3}{2}},$$

$$x \geqslant \mu_1, \quad y \geqslant \mu_2, \ (\mu_1,\, \mu_2,\, \gamma_1,\, \gamma_2,\, \alpha > 0), i = 1, 2, \tag{15}$$

which is a special case of the proposed bivariate truncated generalized Cauchy distribution in case of $\alpha = \frac{1}{2}$.

2 BPI's of Future Observation Coming from Bivariate Distribution

The main goal in this section is to introduce a point of view in studying the one-sample and two-sample prediction problems in case of bivariate informative observations.

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While ordering a set of univariate random variables is a clear and straightforward matter as it can be done by simply ordering the set of random variables, such ordering is not as clear if we are dealing with a set of random vectors.

Barnett (1976) classified the principles used for ordering multivariate data into four principles: marginal, reduced (aggregate), partial and conditional (sequential) ordering. An interesting detailed discussion of such principles with illustrative examples are given in Barnett's paper.

In our paper, we wish to predict bivariate random vectors. Our point of view can be summarized as, the first components of the predicted random vectors are based on the ordered first components of the informative sample, as is done in the univariate case. To predict the second components, we compute the norms of each vector of the informative sample, order the norms and then predict the future norms as is done in the univariate case. The relation between the components of vectors and norms enables us to obtain the second components of the predicted vectors. In other words, we obtain the second component of a predicted vector from the knowledge of the values of the first component and the norm of the vector.
This point of view can be shown in the following subsections

2.1 One-sample Prediction

Let $(X_1, Y_1), \ldots, (X_r, Y_r)$ be the first r bivariate informative observations from a random sample of size n of bivariate observations. Suppose that the first components of such informative vectors are ordered, that is $X_1 < X_2 < \cdots < X_r$ and that their norms are given by $Z_1, Z_2, \ldots, Z_r$.

To obtain BPI's for the remaining future vectors, denoted by $(X_1^*, Y_1^*), \ldots, (X_{n-r}^*, Y_{n-r}^*)$, where $X_1^* < X_2^* < \cdots < X_{n-r}^*$ and norms $Z_1^* < Z_2^* < \cdots < Z_{n-r}^*$ we apply the following steps:

1. Based on ordered $Z_1, Z_2, \ldots, Z_r$, compute the BPI's for $Z_s^*, s = 1, 2, \ldots, (n-r)$, say (L_{1s}, U_{1s}),

2. Based on $X_1 < X_2 < \cdots < X_r$ compute the BPI's for $X_s^*, s = 1, 2, \ldots, (n-r)$, say (L_{2s}, U_{2s}),

3. From (1) and (2), compute the BPI's for $Y_s^*, s = 1, 2, \ldots, (n-r)$ which are $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$. This is true, since $z_s^* = (x_s^{*\,2} + y_s^{*\,2})^{1/2}$,

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4. From (2) and (3), the BPI's for $(X_s^*, Y_s^*), s = 1, 2, \ldots, (n-r)$ is $([L_{1s}^2 - L_{2s}^2]^{1/2}, L_{2s})$, $([U_{1s}^2 - U_{2s}^2]^{1/2}, U_{2s})$.

2.2 Two-sample Prediction

In this case the first r bivariate informative observations $(X_1, Y_1), \ldots, (X_r, Y_r)$ from a random sample of size n is such that $X_1 < X_2 < \cdots < X_r$ with norms $Z_1, Z_2, \ldots, Z_r$. An independent future sample of size m is $(X_1^*, Y_1^*), \ldots, (X_m^*, Y_m^*)$, where $X_1^* < X_2^* < \cdots < X_m^*$ and norms $Z_1^* < Z_2^* < \cdots < Z_m^*$. To obtain the BPI's of the future sample, we apply the following steps:

1. Based on ordered $Z_1, Z_2, \ldots, Z_r$, compute the BPI's for $Z_s^*, s = 1, 2, \ldots, m$, say (L_{1s}, U_{1s}) ,

2. Based on $X_1 < X_2 < \cdots < X_r$ compute the BPI's for $X_s^*, s = 1, 2, \ldots, m$, say (L_{2s}, U_{2s}),

3. From (1) and (2), compute the BPI's for $Y_s^*, s = 1, 2, \ldots, m$ which are $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$,

4. From (2) and (3), the $BPI's$ for $(X_s^*, Y_s^*), s = 1, 2, \ldots, m$ is $([L_{1s}^2 - L_{2s}^2]^{1/2}, L_{2s}), ([U_{1s}^2 - U_{2s}^2]^{1/2}, U_{2s})$.

3 One-sample Prediction in Case of (BVTGCD)

If, in (5), $k = 2, \gamma_1 = \gamma_2 = 1, \mu_1 = 0$ and $\mu_2 = 0$, and let $X_1 \equiv X$ and $X_2 \equiv Y$, then (X, Y) has a BTGCD *pdf*, given by

$$f_{X,Y}(x,y) = \left(\frac{4\,\alpha}{\pi}\right)\left\{1 + x^2 + y^2\right\}^{-\alpha-1}, \qquad x \geqslant 0, y \geqslant 0, (\alpha > 0), \quad (16)$$

The marginal pdf's of the random variables X and Y are given, respectively, by

$$f_X(x) = \left(\frac{4}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma\left(\alpha + \frac{1}{2}\right)}{\Gamma(\alpha)} \left\{1 + x^2\right\}^{-\alpha-\frac{1}{2}}, \quad x \geqslant 0, \ (\alpha > 0), \qquad (17)$$

$$f_Y(y) = \left(\frac{4}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma\left(\alpha + \frac{1}{2}\right)}{\Gamma(\alpha)} \left\{1 + y^2\right\}^{-\alpha-\frac{1}{2}}, y \geqslant 0, \ (\alpha > 0), \qquad (18)$$

In this section we apply the steps given in Subsection 2.1.

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Step 1

The norm Z of the vector (X, Y) is given by $Z = (X^2 + Y^2)^{1/2}$. In Appendix A the *pdf* and hence *cdf* and rf are derived. Such functions are given in the forms

$$f_Z(z) = 2\,\alpha\, z\left\{1 + z^2\right\}^{-\alpha-1}, \qquad z \geqslant 0, (\alpha > 0), \tag{19}$$

$$F_Z(z) = 1 - [1 + z^2]^{-\alpha}, \qquad z \geqslant 0,\ (\alpha > 0), \tag{20}$$

$$R_Z(z) = [1 + z^2]^{-\alpha}, \qquad z \geqslant 0,\ (\alpha > 0). \tag{21}$$

From (19) and (21), the conditional density of Z_s^* given α is obtained (see Appendix B), as

$$\begin{aligned} g_1(z_s^*|\alpha) \propto \alpha\, z_s^* \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} [1 + z_s^{*\,2}]^{-\alpha(n-r-s+i+1)-1} \\ \times [1 + z_r^2]^{-\alpha(s-i-n+r-1)}, \quad z_s^* > z_r. \end{aligned} \tag{22}$$

Suppose that the prior belief of the experimenter is given by the *pdf*,$\pi(\alpha) \propto \alpha^{c_1-1}\, e^{-c_2\alpha}$.
The likelihood function of α given $Z_1, \ldots, Z_r$ is given by

$$L(\alpha|z_1, \ldots, z_r) \propto [R_Z(z_r)]^{n-r} \prod_{i=1}^{r} f_Z(z_i) \propto \alpha^r [1 + z_r^2]^{-\alpha(n-r)} \left\{ \prod_{i=1}^{r} [1 + z_i^2] \right\}^{-\alpha-1} \tag{23}$$

Since the posterior density $\pi^*(\alpha|z_1, \ldots, z_r) \propto \pi(\alpha)\, L(\alpha|z_1, \ldots, z_r)$, can be written in the form

$$\pi^*(\alpha|z_1, \ldots, z_r) \propto \alpha^{r+c_1-1} [1 + z_r^2]^{-\alpha(n-r)} \exp\left[-\alpha \left\{ c_2 + \sum_{i=1}^{r} \ln(1 + z_i^2) \right\} \right]. \tag{24}$$

From (22) and (24), the predictive density function of Z_s^* is given by

$$g_1^*(z_s^*|\, z_1, \ldots, z_r) = \int_0^\infty g_1(z_s^*|\,\alpha) \pi^*(\alpha|\, z_1, \ldots, z_r) d\alpha. \tag{25}$$

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Which can be rewritten in the form

$$g_1^*(z_s^*|\, z_1,\ldots,z_r) = A_1 \int_0^\infty z_s^* \alpha^{r+c_1} \exp\left[-\alpha\left\{c_2 + \sum_{j=1}^{r} \ln(1+z_j^2)\right\}\right] \times \sum_{i=0}^{s-1}(-1)^i \binom{s-1}{i}[1+z_s^{*\,2}]^{-\alpha(n-r-s+i+1)-1} \times [1+z_r^2]^{-\alpha(s-i-1)}\, d\alpha,\ z_s^* > z_r. \tag{26}$$

By choosing

$$A_1^{-1} = \int_{z_r}^\infty \int_0^\infty z_s^* \alpha^{r+c_1} \exp\left[-\alpha\left\{c_2 + \sum_{j=1}^{r} \ln[1+z_j^2)\right\}\right] \times \sum_{i=0}^{s-1}(-1)^i \binom{s-1}{i}[1+z_s^{*\,2}]^{-\alpha(n-r-s+i+1)-1} \times [1+z_r^2]^{-\alpha(s-i-1)}\, d\alpha dz_s^*, \tag{27}$$

then, $g_1^*(z_s^*|\, z_1,\ldots,z_r)$, will be a *pdf*.

It's clear that the predictive density function in (26) is not symmetric, so that we will use the HPD Prediction method to obtain $(1-\tau)$ % HPD Prediction interval (L_{1s}, U_{1s}) by solving the following two nonlinear equations

$$P(L_{1s} < Z_s^* < U_{1s}|\, z_1,\ldots,z_r) = \int_{L_{1s}}^{U_{1s}} g_1^*(z_s^*|\, z_1,\ldots,z_r) dz_s^* = 1-\tau, \tag{28}$$

$$g_1^*(L_{1s}|\, z_1,\ldots,z_r) = g_1^*(U_{1s}|\, z_1,\ldots,z_r). \tag{29}$$

Step 2

By using the *pdf*(17), the rf can be written in the form (after some simple mathematical steps),

$$R_X(x) = \int_x^\infty \left(\frac{4}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)} \left\{1+u^2\right\}^{-\alpha-\frac{1}{2}} du = \frac{1}{B\left(\alpha,\frac{1}{2}\right)} \int_x^\infty \left\{1+u^2\right\}^{-\alpha-\frac{1}{2}} du = \zeta_x(\alpha, 1/2), \tag{30}$$

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where $B(\alpha, 1/2)$ is the complete beta function.

The predictive density function of X_s^* can be written as follows

$$g_2^*(x_s^*|\, x_1, \ldots, x_r) = \int_0^\infty g_2(x_s^*|\alpha)\pi^*(\alpha|\, x_1, \ldots, x_r)d\alpha, \tag{31}$$

where

$$\begin{aligned} g_2(x_s^*|\alpha)\pi^*(\alpha|\, x_1, \ldots, x_r) = {} & A_2\alpha^{c_1-1}\exp[-c_2\alpha] \\ & \sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}\left\{\zeta_{x_s^*}\left(\alpha, \frac{1}{2}\right)\right\}^{n-r-s+i} \\ & \left\{\zeta_{x_r}\left(\alpha, \frac{1}{2}\right)\right\}^{s-i-1}\left\{\frac{1}{B\left(\alpha, \frac{1}{2}\right)}\right\}^{r+1} \\ & \left[1 + x_s^{*\,2}\right]^{\left(-\alpha-\frac{1}{2}\right)}\prod_{i=1}^{r}\left[1 + x_i^2\right]^{\left(-\alpha-\frac{1}{2}\right)}, \quad x_s^* > x_r \end{aligned} \tag{32}$$

where A_2 is a normalizing constant which can be computed from the relation

$$\begin{aligned} A_2^{-1} = {} & \int_{x_r}^\infty\int_0^\infty \alpha^{c_1-1}\exp[-c_2\alpha]\sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}\left\{\zeta_{x_s^*}\left(\alpha, \frac{1}{2}\right)\right\}^{n-r-s+i} \\ & \times\left\{\zeta_{x_r}\left(\alpha, \frac{1}{2}\right)\right\}^{s-i-1}\left\{\frac{1}{B\left(\alpha, \frac{1}{2}\right)}\right\}^{r+1}\left[1 + x_s^{*\,2}\right]^{\left(-\alpha-\frac{1}{2}\right)} \\ & \prod_{i=1}^{r}\left[1 + x_i^2\right]^{\left(-\alpha-\frac{1}{2}\right)} d\alpha\, dx_s^*. \end{aligned} \tag{33}$$

The $(1-\tau)$ % HPD Prediction interval (L_{2s}, U_{2s}) is obtained by solving the following two nonlinear equations

$$P(L_{1s} < X_s^* < U_{1s}|\, x_1, \ldots, x_r) = \int_{L_{1s}}^{U_{1s}} g_2^*(x_s^*|\, x_1, \ldots, x_r)dx_s^* = 1 - \tau, \tag{34}$$

$$g_2^*(L_{2s}|\, x_1, \ldots, x_r) = g_1^*(U_{2s}|\, x_1, \ldots, z_r). \tag{35}$$

Step 3

From steps 2 and 3, a $(1-\tau)$ % HPD Prediction interval for Y_s^* is $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

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4 Two-sample Prediction in Case of (BVTGCD)

In this case we apply the steps in Subsection 2.2 as follows

Step 1

Substituting from (19) and (21) in (14), we can write

$$g_3(z_s^*|\alpha) \propto \alpha z_s^* \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \left[1 + {z_s^*}^2\right]^{-\alpha(m+i-s+1)-1}, \qquad z_s^* > 0. \quad (36)$$

By using the posterior (24), we get

$$\begin{aligned} g_3(z_s^*|\alpha)\pi^*(\alpha|\, z_1, \ldots, z_r) = A_3 z_s^* \alpha^{r+c_1} [1+z_r^2]^{-\alpha(n-r)} \\ \exp\left[-\alpha\left\{c_2 + \sum_{i=1}^{r} \ln\left(1+z_i^2\right)\right\}\right] \\ \times \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \left[1 + {z_s^*}^2\right]^{-\alpha(m+i-s+1)-1}, z_s^* > 0. \end{aligned} \quad (37)$$

and A_3 is a normalizing constant.
It then follows that the predictive density function of Z_s^* is given by

$$\begin{aligned} g_3^*(z_s^*|\, z_1, \ldots, z_r) &= \int_0^\infty g_3(z_s^*|\alpha)\pi^*(\alpha|\, z_1, \ldots, z_r) d\alpha \\ &= A_3 z_s^* \int_0^\infty \alpha^{r+c_1} [1+z_r^2]^{-\alpha(n-r)} \\ &\quad \exp\left[-\alpha\left\{c_2 + \sum_{i=1}^{r} \ln\left(1+z_i^2\right)\right\}\right] \\ &\quad \times \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \left[1 + {z_s^*}^2\right]^{-\alpha(m+i-s+1)-1} d\alpha, \qquad z_s^* > 0. \end{aligned} \quad (38)$$

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By choosing

$$A_3^{-1} = \int_0^\infty \int_0^\infty z_s^* \alpha^{r+c_1} [1 + z_r^2]^{-\alpha(n-r)} \exp\left[-\alpha \left\{c_2 + \sum_{i=1}^{r} \ln\left(1 + z_i^2\right)\right\}\right]$$
$$\times \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \left[1 + {z_s^*}^2\right]^{-\alpha(m+i-s+1)-1} d\alpha dz_s^*, \quad (39)$$

the function $g_3^*(z_s^*|\, z_1, ..., z_r)$ will be a *pdf*.

The $(1-\tau)$ % HPD Prediction interval (L_{1s}, U_{1s}) is obtained by solving the following two nonlinear equations

$$P(L_{1s} < Z_s^* < U_{1s}|\, z_1, \ldots, z_r) = \int_{L_{1s}}^{U_{1s}} g_3^*(z_s^*|\, z_1, \ldots, z_r) dz_s^* = 1 - \tau, \quad (40)$$

$$g_3^*(L_{1s}|\, z_1, \ldots, z_r) = g_3^*(U_{1s}|\, z_1, \ldots, z_r). \quad (41)$$

Step 2

Using the *pdf* (17), its *cdf* and the same posterior as in (24) the predictive density function of X_s^* is given by

$$g_4^*(x_s^*|\, x_1, \ldots, x_r) = \int_0^\infty g_4(x_s^*|\alpha) \pi^*(\alpha|\, x_1, \ldots, x_r) d\alpha, \qquad x_s^* > 0, \quad (42)$$

where

$$g_4(x_s^*|\alpha)\pi^*(\alpha|\, x_1, \ldots, x_r) = A_4 \alpha^{c_1 - 1} \exp[-c_2 \alpha] \left\{\zeta_{x_r}\left(\alpha, \frac{1}{2}\right)\right\}^{n-r}$$
$$\sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \left\{\zeta_{x_s^*}\left(\alpha, \frac{1}{2}\right)\right\}^{m+i-s} \left\{\frac{1}{B\left(\alpha, \frac{1}{2}\right)}\right\}^{r+1}$$
$$\left[1 + {x_s^*}^2\right]^{\left(-\alpha - \frac{1}{2}\right)} \prod_{i=0}^{r} \left[1 + x_i^2\right]^{\left(-\alpha - \frac{1}{2}\right)}, \quad x_s^* > 0 \quad (43)$$

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By choosing A_4 such that

$$A_4^{-1} = \int_0^\infty \int_0^\infty \alpha^{c_1-1} \exp[-c_2\alpha] \left\{\zeta_{x_r}\left(\alpha, \frac{1}{2}\right)\right\}^{n-r} \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \times \left\{\zeta_{x_s^*}\left(\alpha, \frac{1}{2}\right)\right\}^{m+i-s} \left\{\frac{1}{B\left(\alpha, \frac{1}{2}\right)}\right\}^{r+1} \left[1 + x_s^{*\,2}\right]^{\left(-\alpha-\frac{1}{2}\right)} \prod_{i=0}^{r} \left[1 + x_i^2\right]^{\left(-\alpha-\frac{1}{2}\right)} d\alpha dx_s^* \quad (44)$$

The $(1-\tau)$ % HPD Prediction interval (L_{2s}, U_{2s}) is obtained by solving the following two nonlinear equations

$$P(L_{2s} < X_s^* < U_{2s}|\, x_1, \ldots, x_r) = \int_{L_{2s}}^{U_{2s}} g_4^*(x_s^*|\, x_1, \ldots, x_r) dx_s^* = 1 - \tau, \quad (45)$$

$$g_4^*(L_{2s}|\, x_1, \ldots, x_r) = g_4^*(U_{2s}|\, x_1, \ldots, z_r). \quad (46)$$

Step 3

From steps 2 and 3, a $(1-\tau)$ % HPD Prediction interval for Y_s^* is $([L_{1s}^2 - L_{2s}^2]^{1/2}, [U_{1s}^2 - U_{2s}^2]^{1/2})$.

5 Numerical Example

In this section we follow the steps

1. Given the set of prior parameters (c_1, c_2), generate the parameter α,
2. Using the generated population parameter, generate a bivariate random sample of size n, say $(X_1, Y_1), \ldots, (X_n, Y_n)$ as shown in Subsection 1.2
3. Follow steps in Sections 3 and 4.

In Tables 1 and 2 95% HPD Prediction intervals are computed in case of the one- and two-sample predictions, respectively, using informative samples of different sizes, r.

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Table 1. One-sample prediction- 95 % HPD Prediction intervals for Z_s^*, Y_s^* and X_s^*, $s = 1, 2, 3$.

1 Number of samples which cover the HPD Prediction intervals from 10000 samples.

2 HPD Prediction intervals for z_s^*, x_s^*, y_s^*.

3 Length of the HPD Prediction intervals.

r		$z_1{}^*$	z_2^*	z_3^*
	1	9621	9707	9807
10	2	(1.9201,3.2213)	(3.5115,5.6536)	(5.8433,8.1477)
	3	1.3012	2.1421	2.3044
	1	9587	9652	9738
20	2	(2.3104,3.5119)	(4.1005,6.1860)	(6.0115,8.1181)
	3	1.2015	2.0855	2.1066
	1	9517	9594	9672
45	2	(2.8920,3.9109)	(4.1702,5.6754)	(5.7003,7.6017)
	3	1.0189	1.5052	1.9014
r		x_1^*	x_2^*	x_3^*
	1	9692	9865	9893
10	2	(1.9105,2.3196)	(2.5205,3.2955)	(3.5048,4.4609)
	3	0.4091	0.7750	0.9561
	1	9605	9717	9799
20	2	(2.0720,2.7231)	(2.8309,3.5415)	(3.9004,4.7104)
	3	06511	0.7106	0.8100
	1	9588	9676	9717
45	2	(2.7713,3.0256)	(3.1615,3.7017)	(3.9114,4.5717)
	3	0.2543	0.5402	0.6603
r		y_1^*	y_2^*	y_3^*
	1	9703	9805	9897
10	2	(0.1918,2.2352)	(2.4449,4.5938)	(4.6755,6.8180)
	3	2.0434	2.1488	2.2425
	1	9687	9704	9759
20	2	(1.0221,2.2177)	(2.9665,5.0719)	(4.5744,6.6118)
	3	1.1955	2.1054	2.1374
	1	9672	9694	9724
45	2	(0.8268,2.4781)	(2.7195,4.3020)	(4.1466,6.0733)
	3	1.1513	1.5825	1.9267

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Table 2. One-sample prediction- 95 % HPD Prediction intervals for Z_s^*, Y_s^* and X_s^*, $s = 1, 2, 3$.

1 Number of samples which cover the HPD Prediction intervals from 10000 samples.

2 HPD Prediction intervals for z_s^*, x_s^*, y_s^*.

3 Length of the HPD Prediction intervals.

r		$z_1{}^*$	z_2^*		z_3^*
	1	9788	9806		9873
10	2	(1.1503,1.9608)	(2.1037,3.1205)		(3.2207,4.6278)
	3	0.8105	1.0168		1.4071
	1	9704		9754	9805
20	2	(1.0506,1.8426)	(2.0308,2.9388)		(3.2316,4.4335)
	3	0.7920		0.9080	1.2019
	1	9628	9704		9729
45	2	(1.6915,2.2021)	(2.4014,3.0107)		(3.2105,4.1179)
	3	0.5106	0.6093		0.9074
r		x_1^*	x_2^*		x_3^*
	1	9726	9794		9878
10	2	(0.6508,0.9925)	(1.3047,1.7706)		(1.9148,2.6454)
	3	0.3417	0.4509		0.7306
	1	9701		9750	9805
20	2	(0.5704,0.9634)	(1.1508,1.6167)		(1.7309,2.4310)
	3	0.3930	0.4659		0.7001
	1	9663	9713		9788
45	2	(0.9327,1.2133)	(1.3215,1.6757)		(1.9115,2.3345)
	3	0.2806	0.3542		0.4230
r		y_1^*	y_2^*		y_3^*
	1	9804	9894		9927
10	2	(0.9485,1.6911)	(1.6502,2.5695)		(2.5897,3.7972)
	3	0.7426	0.9193		1.2075
	1	9737	9806		9897
20	2	(0.8822,1.5707)	(1.6733,2.4541)		(2.7289,3.7076)
	3	0.6885	0.7808		0.9787
	1	9616	9756		9801
45	2	(1.4111,1.8877)	(2.0050,2.5012)		(2.5794,3.3922)
	3	0.4266	0.4963		0.8128

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Concluding Remarks

In Tables 1 and 2 we take different sizes for the informative sample, 10, 20 and 45 and predict the first three future observations .
In these tables, we observe that

1. The length of the HPD Prediction intervals and the number of samples which cover these intervals increase by increasing s and decrease by increasing the informative sample size,

2. tTe results become better as the informative sample size r gets larger.

3. In all cases, the simulated percentage coverages are at least 95%.

4. If the hyperparameters are unknown, they can be estimated by using the empirical Bayes method, see Maritz and Lwin (1989), or the hierarchical method, see Bernardo and Smith (1994).

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Appendix A

Proof of equations (19)-(21) From the joint density function of the random variables X and Y which is given by (15) and using the transforms $X = Z\cos\Theta$ and $Y = Z\sin\Theta$ we get the joint density function of the random variables Z and Θ in the form

$$f_{Z,\Theta}(z,\theta) = \left(\frac{4\alpha z}{\pi}\right)\left[1+z^2\right]^{-\alpha-1}, \quad z \geqslant 0, \ \ 0 \leqslant \theta \leqslant \frac{\pi}{2}, \ \ (\alpha > 0). \quad (47)$$

Integrating (47) with respect to θ, we get the density function of Z in the form

$$f_Z(z) = 2\,\alpha\, z\left[1+z^2\right]^{-\alpha-1}, \quad z \geqslant 0, \ \ (\alpha > 0). \quad (48)$$

The (cdf) of the random variable Z is given by

$$F_Z(z) = 2\,\alpha\int_0^z u\,[1+u^2]^{-\alpha-1}du = 1-[1+z^2]^{-\alpha}, \quad z \geqslant 0, \ \ (\alpha > 0) \quad (49)$$

The rf) of z is given by

$$R_Z(z) = 1 - F_Z(z) = [1+z^2]^{-\alpha}, \quad z \geqslant 0, \ \ (\alpha > 0) \quad (50)$$

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Appendix B

Proof of equation (22)

From (8), (19) and (21) we have

$$g_1(z_s^*|c,\alpha) \propto [R_Z(z_r) - R_Z(z_s^*)]^{(s-1)}[R_Z(z_s^*)]^{n-r-s}[R_Z(z_r^*)]^{-(n-r)} f_Z(z_s^*)$$
$$= \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} [R_Z(z_s^*)]^{n-r-s+i} \, [R_Z(z_r)]^{s-i-n+r-1} \, f_Z(z_s^*), \tag{51}$$

where the reliability function $R_Z(z)$, given by (21) yields

$$g_1(z_s^*|\alpha) \propto \alpha \; z_s^* \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} [1 + z_s^{*\,2}]^{-\alpha(n-r-s+i+1)-1}$$
$$\times \, [1 + z_r^2]^{-\alpha(s-i-n+r-1)} \tag{52}$$

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