

Determining a common set of weights in DEA by solving a linear programming

S. Saati*

Assistant Professor, Dep. of Mathematics, Islamic Azad University, Tehran-North Branch, Tehran, Iran

Abstract

In models of Data Envelopment Analysis (DEA), an optimal set of input and output weights is generally assumed to represent the assessed Decision Making Unit (DMU) in the best light in comparison to all the other DMUs. These sets of weights are, typically, different for each of the participating DMUs. Thus, it is important to find a Common Set of Weights (CSW) across the set of DMUs. In this paper, a procedure is suggested to find a CSW in DEA. In the proposed procedure by solving just one linear programming a CSW is achieved. To demonstrate the concept, a numerical example is solved.

Keywords: Data envelopment analysis; Weight restriction; Common set of weights; Linear programming

1. Introduction

Data Envelopment Analysis (DEA), as developed in Charnes et al. [1] (CCR model), does not require any a priori weights for inputs and outputs. DEA is value-free, which is strength and weakness. This strength is sufficient to delineate the DEA technically-efficient DMUs from the DEA technically-inefficient DMUs in multiple-input and multiple-output analysis, without any need for a parametric specification. However, values (prices/costs) must be introduced into the measurement problem to measure technical efficiency. As a means of introducing values into a DEA analysis, one might assume that the production unit faces fixed input and output prices for all inputs and outputs, then, efficiency measures may be defined relative to these exact prices.

Recently, a series of possible approaches for setting bounds on factor weights in DEA have been put forward. The general approaches are presented in Charnes et al. [2], Dyson et al. [3], Jahanshahloo et al. [6], Podinovski [8,9,10], Roll et al. [11,12], Sarrico and Dyson [13] and Thompson et al. [14]. Imposing bounds on factor weights, limits the flexibility of DEA in assigning individual sets of weights to each of the participating DMUs. In the extreme case, when no flexibility is allowed, a Common Set of Weights (CSW) is applied for the assessment of all DMUs. This is the usual approach in all engineering, and

most economic efficiency analyses. Such a common set can serve as a yardstick to which the results of the ordinary ("flexible") DEA outcomes are compared.

Some methods to find CSW are proposed in Franklin and Peng [4], Hosseinzadeh et al. [5], Jahanshahloo et al. [6], Kuosmanen et al. [7] and Roll et al. [11,12]. The main difficulties about these methods are imposing the bounds on variables, and nonlinearity of models.

In this paper, a technique is suggested to find a CSW in DEA. In the proposed procedure first, upper levels of the weights are determined based on the optimal solution of some linear programming problems. These solutions are computed without solving linear programming problems. Then by solving a linear programming problem, a CSW is determined.

The paper is organized as follows: Section 2 provides a short background about standard and bounded CCR models. The suggested method is presented in Section 3. To demonstrate the concept, a numerical example is given in Section 4. Section 5 closes with conclusion.

2. Standard and bounded CCR models

The evaluation of a DMU has long been recognized to be a problem of considerable complexity. This evaluation becomes more difficult when it involves

* Corresponding author. E-mail: ssaatim@yahoo.com

multiple inputs and multiple outputs, in that a set of weights has to be determined to aggregate the outputs and inputs separately to form a ratio as efficiency. To do so, DEA approach is proposed, which allows every DMU to select their most favorable weights while requiring the resulted ratio of the aggregated outputs to the aggregated inputs of all DMUs to be less than or equal to 1.

Consider n DMUs, each consumes varying amounts of m different inputs to produce s different outputs. In the model formulation, x_{ip} ($i=1, \dots, m$) and y_{rp} ($r=1, \dots, s$) denote, respectively, the non-negative input and output values for DMU p , the DMU under consideration. The seminal programming statement for the (input oriented) CCR model is:

$$\text{Max } \sum_{r=1}^s u_r y_{rp}$$

Subject to:

$$\sum_{i=1}^m v_i x_{ip} = 1, \quad (1)$$

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad \forall j,$$

$$u_r \geq \varepsilon, \quad v_i \geq \varepsilon, \quad \forall i, r,$$

where, u_r ($r=1, \dots, s$) and v_i ($i=1, \dots, m$) are the weights associated with output r and input i , respectively and, ε is a non Archimedean infinitesimal.

Apart from the restriction that no weight may be zero, weights on inputs and outputs are only restricted by the requirements that they must not make the efficiency of any DMU more than 1. The advantage of allowing such freeness on the weights is that, a best efficiency rating is associated to each DMU.

However, in this flexibility, some of the weights may be assigned an exceedingly small value. Also, as a different model is run for each DMU, the set of weights will typically be different for each DMU, and it is unacceptable that the same factor has widely different weights. To control the flexibility of weights, the bounded DEA models are proposed.

The CCR model, assuming bounds on factor weights, is as follows:

$$\text{Max } \sum_{r=1}^s u_r y_{rp}$$

Subject to:

$$\sum_{i=1}^m v_i x_{ip} = 1, \quad (2)$$

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad \forall j,$$

$$U_r^l \leq u_r \leq U_r^u \quad \forall r,$$

$$V_i^l \leq v_i \leq V_i^u \quad \forall i,$$

where U_r^l, U_r^u, V_i^l and V_i^u are lower and upper bounds on output and input weights, respectively.

This formulation restricts the flexibility of weights within certain bounds. These bounds are the same for all DMUs and care should be taken in their selection, because unsuitable bounds makes (2) infeasible. The bounded CCR model cannot fully restrict the flexibility of weights, and restricts it in given bounds. When no flexibility is allowed in DEA for assigning the individual set of weights to each of the participating DMUs, a CSW is determined.

3. Proposed model

Bellow, a procedure is suggested to assess a CSW. This is done in two steps. In the first step, by solving a linear programming problem for each factor, an upper bound is determined for each of them. A CSW is determined in the second step by compacting the weight intervals via solving a linear programming problem.

Step 1: Bounds determination. To determine the upper bounds on output weights, the following problems are considered:

$$\text{Max } u_r$$

Subject to:

$$\sum_{i=1}^m v_i x_{ij} \leq 1 \quad \forall j, \quad (3)$$

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad \forall j,$$

$$v_i, u_r \geq 0 \quad \forall i, r.$$

Similar problems are solved to determine the upper bound of input weights.

In (3), the maximum value of each factor weight is determined in such a way that the efficiency of each

DMU does not exceed 1. $\sum_{i=1}^m v_i x_{ij} \leq 1$ is a normaliza-

tion constraint and normalizes the factor weights. The upper bounds of output and input weights are determined by solving $s+m$ linear programming problems.

It is important to prove the feasibility of the problems (3) and, boundedness and positivity of their optimal values, since in the absence of each of these assumptions; the proposed procedure will break down.

Theorem 1. The problems (3) are feasible and their optimal values are bounded and positive.

Proof. Obviously $(\mathbf{U}, \mathbf{V}) = (\mathbf{0}, \mathbf{0})$ is a feasible solution of (3). To prove the boundedness of (3), consider the

constraints $\sum_{i=1}^m v_i x_{ij} \leq 1$ and $\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0$.

It is concluded that $\sum_{r=1}^s u_r y_{rj} \leq 1$.

For each p ($1 \leq p \leq s$) there exists at least one y_{pq} ($1 \leq q \leq n$) such that $y_{pq} \neq 0$. Therefore:

$$u_p \leq (1 - \sum_{\substack{r=1 \\ r \neq p}}^s u_r y_{rq}) / y_{pq}.$$

This yields that the problems for determining the upper bounds of output weights are bounded. In the same way, it can be proved that the upper bounds for the input weights are also bounded. Consider the dual of upper bounds for output weights problem:

$$\text{Min } \sum_{j=1}^n \alpha_j$$

Subject to:

$$\sum_{j=1}^n \alpha_j x_{ij} - \sum_{j=1}^n \beta_j x_{ij} \geq 0 \quad \forall i,$$

$$\sum_{j=1}^n \beta_j y_{rj} \geq 0 \quad \forall r \neq p, \quad (4)$$

$$\sum_{j=1}^n \beta_j y_{pj} \geq 1,$$

$$\alpha_j, \beta_j \geq 0 \quad \forall j.$$

Suppose that the optimal value of upper bounds for output weights problem is zero. Then, by the strong

duality property, $\sum_{j=1}^n \alpha_j^* = 0$ or $\alpha_j^* = 0$ ($j=1, \dots, n$).

Considering the constraints $\sum_{j=1}^n \alpha_j x_{ij} - \sum_{j=1}^n \beta_j x_{ij} \geq 0$

($i=1, \dots, m$), yields that $-\sum_{j=1}^n \beta_j x_{ij} \geq 0$ ($i=1, \dots, m$),

and it is a contradiction. Therefore, the optimal values of (3) are positive. ■

Since (3) has a special structure, so its optimal value may achieved by comparing their constraints.

In (3) it is evident that $\sum_{r=1}^s u_r y_{rj} \leq 1$ ($j=1, \dots, n$).

Therefore, the upper bounds of input and output weights are calculated as follows:

$$u_r^* = 1 / \max_{1 \leq j \leq n} \{y_{rj}\} \quad (r=1, \dots, s).$$

$$v_i^* = 1 / \max_{1 \leq j \leq n} \{x_{ij}\} \quad (i=1, \dots, m). \quad (5)$$

Step 2: Determining a CSW. Starting from bounded model (2), a CSW can be achieved by expressing the deviation from either bound as a fraction of the range between the upper and lower bounds. Assuming the same deviation from bounds across all DMUs, we get:

Max ϕ

Subject to:

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad \forall j, \quad (6)$$

$$U_r^l + \phi(U_r^u - U_r^l) \leq u_r \leq U_r^u - \phi(U_r^u - U_r^l) \quad \forall r,$$

$$V_i^l + \phi(V_i^u - V_i^l) \leq v_i \leq V_i^u - \phi(V_i^u - V_i^l) \quad \forall i.$$

Compacting the weight intervals in (6) is done by a proportion of each interval length, since the upper bounds of factor weights are not the same. Applying (5) and setting the lower bounds of factor weights equal to zero result in the following model:

Max ϕ

Subject to:

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad \forall j, \quad (7)$$

$$\phi U_r \leq u_r \leq (1 - \phi) U_r \quad \forall r,$$

$$\phi V_i \leq v_i \leq (1 - \phi) V_i \quad \forall i,$$

where, U_r ($r=1, \dots, s$) and V_i ($i=1, \dots, m$) are calculated by (5).

Theorem 2. The problem (7) is feasible and its optimal value is bounded and positive.

Proof. The proof is similar to theorem 1. ■

A CSW is obtained by solving (7), and the efficiency of each DMU can be evaluated as follows:

$$e_j = \frac{\sum_{r=1}^s u_r^* y_{rj}}{\sum_{i=1}^m v_i^* x_{ij}} \quad \forall j, \quad (8)$$

where, u_r^* ($r=1, \dots, s$) and v_i^* ($i=1, \dots, m$) are optimal values of (7).

In cases where none of the DMUs is efficient, all output weights can be increased (and / or input weights decreased) by minimal proportion until an efficient DMU is reached. One way to do the task is the following substitutions:

$$M_r = \frac{u_r^*}{e}, \quad N_i = v_i^*, \quad \forall r, i,$$

where, $e = \max_{1 \leq j \leq n} \{e_j\}$. The resulted weights

M_r ($r=1, \dots, s$) and N_i ($i=1, \dots, m$) are the proposed CSW.

After eliciting the CSW, the efficiencies of DMUs are determined by:

$$e_j = \frac{\sum_{r=1}^s M_r y_{rj}}{\sum_{i=1}^m N_i x_{ij}} \quad \forall j. \quad (9)$$

4. Numerical example

For illustrating the suggested approach, consider Table 1. The inputs and outputs of ten DMUs, which each DMU consumes four inputs to produce three outputs, is presented in Table 1.

The results of running CCR model and CSW are summarized in Table 2. The efficiencies of DMUs by CCR model (1) with $\varepsilon = 0.00001$ are shown in "CCR" column of Table 2. As can be seen, 7 out of the 10 DMUs are efficient. There is a large bias in the results. The resulted weights of output and input factors by CCR model are also presented in Table 2. There are large differences in weights accorded to the same factor, with 31 out of the 70 weights being virtually zero. To avoid these differences in weights, the proposed procedure is used to find a CSW.

In the first step of the procedure, the upper bounds of weights are determined by (5). The resulted upper bounds of output and input weights, respectively, are presented in the bottom of Table 2. These bounds are introduced in (7) and the CSW is evaluated. The maximum efficiency by evaluated CSW is 0.924, so none of the DMUs is efficient. The proposed CSW is calculated by dividing each output weight by maximum efficiency. This CSW is presented in the last row of Table 2. After fixing the input and output prices for all inputs and outputs, the efficiencies of DMUs are evaluated by (9). The efficiencies by the CSW are shown in the "CSW" column of Table 2.

5. Conclusion

One of the prominent features of standard DEA is the representation of each of the participating DMUs in the best possible light, relative to the others. Toward this end, factor weights are allowed to vary freely within the general constraints in each run of the model. Weight flexibility in DEA assessments is such that it can lead to some DMUs having all but their most favorable input(s) and output(s) ignored in their assessment. To overcome this problem, a procedure for finding a CSW in CCR model is suggested. In contrast to the other methods, the presented procedure is very useful and applicable.

Table 1. Data for numerical example.

DMU	I_1	I_2	I_3	I_4	O_1	O_2	O_3
D01	12.58	8.67	1.52	1.93	669	1.73	78
D02	7.51	5.58	1.10	2.43	420	2.74	97
D03	8.34	3.83	1.27	2.00	103	1.98	0
D04	6.76	3.83	1.30	2.76	196	1.96	8
D05	6.99	5.58	1.17	1.52	443	2.17	32
D06	5.74	3.83	1.07	2.33	239	2.64	0
D07	6.44	4.50	1.22	1.64	342	2.22	15
D08	10.23	3.58	1.10	2.12	275	2.25	8
D09	5.49	3.58	1.32	1.54	202	1.92	45
D10	4.88	3.83	1.45	1.61	167	1.65	7

Table 2. Results of numerical example.

DMU	Efficiencies		Input Weights				Output Weights		
	CCR	CSW	v_1	v_2	v_3	v_4	u_1	u_2	u_3
D01	1.000	0.713	0.061	0.000	0.000	0.123	0.001	0.000	0.003
D02	1.000	1.000	0.080	0.000	0.000	0.163	0.002	0.000	0.003
D03	0.834	0.356	0.000	0.084	0.000	0.338	0.000	0.421	0.000
D04	0.781	0.417	0.000	0.261	0.000	0.000	0.001	0.303	0.003
D05	1.000	0.766	0.008	0.169	0.000	0.000	0.002	0.076	0.000
D06	1.000	0.584	0.017	0.235	0.000	0.000	0.002	0.168	0.000
D07	1.000	0.658	0.015	0.201	0.000	0.000	0.002	0.144	0.000
D08	1.000	0.524	0.017	0.231	0.000	0.000	0.002	0.165	0.000
D09	1.000	0.698	0.000	0.246	0.000	0.077	0.000	0.425	0.004
D10	0.844	0.423	0.084	0.000	0.000	0.367	0.000	0.506	0.001
Upper Bounds			0.079	0.115	0.658	0.362	0.001	0.365	0.010
CSW			0.040	0.058	0.329	0.181	0.001	0.198	0.006

References

- [1] Charnes, A., Cooper, W. W. and Rhodes, E., 1978, Measuring the efficiency of decision making units. *European Journal of Operations Research*, 2, 429-444.
- [2] Charnes, A., Cooper, W. W., Wei, Q.L. and Huang, Z. M., 1989, Cone-ratio data envelopment analysis and multi-objective programming. *International Journal of System Science*, 20, 1099-1118.
- [3] Dyson, R. G. and Thanassoulis, E., 1988, Reducing weight flexibility in data envelopment analysis. *Journal of Operational Research Society*, 39, 563-576.
- [4] Franklin Liu, F. H. and Peng, H. H., Ranking of units on the DEA frontier with common weights. *Computers & Operations Research*, In press.
- [5] Hosseinzadeh, L., F., Jahanshahloo, G. and Memariani, A., 2000, A method for finding common set of weights by multiple objective programming in data envelopment analysis. *Southwest Journal of Pure and Applied Mathematics*, 1, 44-54.
- [6] Jahanshahloo, G., Alirezaee, M., Saati, M., S. and Mehrabian, S., 1997, The role of bounds on multipliers in DEA; with an empirical study. *Journal of Sciences*, 19(22), 331-347.
- [7] Kuosmanen, T., Cherchye, L. and Sipilainen, T., 2006, The law of one price in data envelopment analysis: Restricting weight flexibility across firms. *European Journal of Operational Research*, 170, 735-757.
- [8] Podinovski, V. V., 2001, Validating absolute weight bounds in data envelopment analysis (DEA) models. *Journal of the Operational Research Society*, 52, 221-225.
- [9] Podinovski, V. V., 2004, Suitability and redundancy of non-homogeneous weight restrictions for measuring the relative efficiency in DEA. *European Journal of Operational Research*, 154, 380-395.
- [10] Podinovski, V. V., 2007, Computation of efficient targets in DEA models with production trade-offs and weight restrictions. *European Journal of Operational Research*, 181, 586-591.
- [11] Roll, Y. and Golany, B., 1993, Alternate methods of treating factor weights in DEA. *Omega*, 21, 99-109.
- [12] Roll, Y., Cook, W. and Golany, B., 1991, Controlling factor weights in data envelopment analysis. *IIE Transactions*, 23, 2-9.
- [13] Sarrico, C. S. and Dyson, R.G., 2004, Restricting virtual weights in data envelopment analysis. *European Journal of Operational Research*, 159, 17-34.
- [14] Thompson, R. G., Langemeier, L.N., Lee, C.T., Lee E. and Thrall, R.M., 1990, The role of multiplier bounds in efficiency analysis with application to Kansas farming. *J. Economet.*, 46, 93-108.