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Fuzzy reliability optimization models for redundant systems

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Abstract

In this paper, a special class of redundancy optimization problem with fuzzy random variables is presented. In this model, fuzzy random lifetimes are considered as basic parameters and the Er-expected of system lifetime is used as a major type of system performance. Then a redundancy optimization problem is formulated as a binary integer programming model. Furthermore, illustrative numerical examples are also given to clarify the methods discussed in this paper.

Keywords: Reliability; Redundant system; Fuzzy random theory; Integer programming

1. Introduction

In a classical redundancy optimization model [1, 2, 3, 8, 10, 13, 17], the system and element lifetimes are assumed to be random variables and the system performance such as system reliability is evaluated by using the probability theory. Unfortunately, this assumption is not appropriate in a wide range of situations. In many practical cases, the probability distribution function of the system and element lifetimes may be unknown or partially known. In fact, from a practical viewpoint, the fuzziness and randomness of the element life times are often mixed up with each other. A combination of fuzzy sets and probabilities forms the notion of fuzzy random variable [7, 15, 16]. The observation of fuzzy random variables is fuzzy real numbers. The concept of fuzzy random variable was introduced by Kwakernaak [9] and Puri and Ralescu [14]. The occurrence of fuzzy random variable makes the combination of randomness and fuzziness more persuasive, since the probability theory and the fuzzy sets theory can be used to model uncertainty and imprecision respectively. In the model discussed in this paper, fuzziness and randomness of the element lifetimes are required to be considered simultaneously. Since both fuzzy random theory and random fuzzy theory offer powerful tools for describing and analyzing the uncertainty of combining randomness and fuzziness, we apply them in a redundancy optimization problem involving both fuzziness and randomness.

In this paper, we consider a redundancy optimization model in which the lifetimes of components cannot be known precisely. Recently, a new variable, random fuzzy variable, was presented by Liu [11]. We assume that the lifetime of a component is a fuzzy random variable. In Section 2 some basic concepts on fuzzy theory and fuzzy random theory are presented. In Section 3 our model is defined as a redundant system involving fuzzy random life times and the redundant elements are assumed to be in one of the two cases: parallel or standby. Then the proposed problem is converted to a new model by using the concepts of fuzzy random variables and Er-expected value operator [6]. Finally, an algorithm for solving the proposed problem is presented.

2. Basic Definitions

In this section, some basic definitions are intro-

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duced. For more details see [4, 5, 9, 12, 15].

Definition 1. Let $\tilde{a}_1$ be a fuzzy set on $R = (-\infty, +\infty)$. This fuzzy set is called a level 1 fuzzy point if its membership function is given as follows:

$$\mu_{\tilde{a}_1}(x) = \begin{cases} 1 & if\ x = a \\ 0 & otherwise \end{cases}$$

$F_p(1) = \{\tilde{a}_1 \mid \forall a \in R\}$ denotes the family of all level 1 fuzzy points.

Definition 2. Let $\tilde{A}$ be a fuzzy set on R. $\tilde{A}$ is called a fuzzy number if it satisfies by the following conditions:

(i) $\tilde{A}$ is normal, i.e. $\{x \in R \mid \tilde{A}(x) = 1\}$ is non-empty.

(ii) $\tilde{A}$ is fuzzy convex, i.e. $\tilde{A}(\alpha x + (1-\alpha)y) \geq \min\{\tilde{A}(x), \tilde{A}(y)\}$ for any $x, y \in R$, $\alpha \in (0,1]$.

(iii) $\tilde{A}$ is upper semi-continuous.

(iv) The support set of $\tilde{A}$ is compact, i.e. $\{x \in R \mid \tilde{A}(x) > 0\}$ is closed and bounded.

Definition 3. LR fuzzy number $\tilde{A}$ is defined by the following membership function:

$$\tilde{A}(x) = \begin{cases} L(\frac{A^0 - x}{A^-}) & if\ x \leq A^0 \\ R(\frac{x - A^0}{A^+}) & if\ x \geq A^0 \end{cases} \quad (1)$$

where A^0 denotes the center (or mode) and A^-, A^+ represent the left and right spread respectively; $L, R: [0,1] \to [0,1]$ with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ are strictly decreasing, continuous functions. A possible representation of a LR fuzzy number is $\tilde{A} = (A^0, A^-, A^+)_{LR}$.

Let $\tilde{A} = (A^0, A^-, A^+)_{LR}$ be a LR fuzzy number. It is called a triangular fuzzy number and denoted by $\tilde{A} = (A^0, A^-, A^+)$ if $L(x) = R(x) = 1 - x$. Let $F_N = \{(a^0, a^-, a^+) \mid \forall a^0 - a^- < a^0 < a^0 + a^+; a^0 \in R,$ $a^-, a^+ \in R^+\}$ be the family of all triangular fuzzy numbers. The family of all left triangular fuzzy numbers can be denoted by:

$$F_L = \{(a^0, a^-, 0) \mid a^0 - a^- < a^0; a^0 \in R, a^- \in R^+\} \quad (2)$$

$$F_R = \{(a^0, 0, a^+) \mid a^0 < a^0 + a^+; a^0 \in R, a^+ \in R\}. \quad (3)$$

Similarly, $F_R = \{(a^0, 0, a^+) \mid a^0 < a^0 + a^+; a^0 \in R, a^+ \in R\}$ denotes the family of all right triangular fuzzy numbers. Note that $\tilde{A} = (A^0, A^-, A^+) = (A^0, 0, 0) = A^0$ if $A^- = A^+ = 0$. It is clear that $F_P(1)$, F_L and F_R are all special cases of F_N. Therefore we have:

$$F = F_N \cup F_L \cup F_R \cup F_P(1) = \{(a^0, a^-, a^+) \mid a^0 - a^- \leq a^0 \leq a^0 + a^+; a^0 \in R, a^-, a^+ \in R^+\}. \quad (4)$$

Definition 4. Let $\tilde{A} = (a^0, a^-, a^+)$, $\tilde{B} = (b^0, b^-, b^+)$ be two fuzzy numbers then $\lambda\tilde{A}, \lambda \in R$ and $\tilde{A} + \tilde{B}$ are also fuzzy numbers as follows:

$$\lambda\tilde{A} = \begin{cases} (\lambda a^0, \lambda a^-, \lambda a^+) & if\ \lambda > 0 \\ (\lambda a, -\lambda a^+, -\lambda a^-) & if\ \lambda < 0 \end{cases} \quad (5)$$

$$\tilde{A} + \tilde{B} = (a^0 + b^0, a^- + b^-, a^+ + b^+). \quad (6)$$

Furthermore, $\tilde{A} = (a^0, a^-, a^+) \geq 0$ if $a^0 - a^- \geq 0$, $\tilde{A} = (a^0, a^-, a^+) > 0$ if $a^0 - a^- > 0$, $\tilde{A} \leq 0$ if $a^0 + a^+ \leq 0$ and finally $\tilde{A} < 0$ if $a^0 + a^+ < 0$. We will use standard fuzzy arithmetic, from the extension principle, to perform sums, products, etc. of fuzzy numbers [9, 28].

Definition 5. Let (Ω, A, P) be a complete probability space. A Fuzzy Random Variable (FRV) is a Borel measurable function $X: (\Omega, A) \to (F, d)$. If X is a fuzzy random variable, then an α-cut $X_\alpha(\omega) = \{x \in R \mid X(\omega)(x) > \alpha\} = [X_\alpha^-(\omega), X_\alpha^+(\omega)]$ is a random interval for every $\alpha \in (0,1]$ and (R, B) is a Borel measurable iff:

$$X_\alpha^{-1}(B) = \{\omega \in \Omega; X_\alpha(\omega) \cap B \neq \phi\} \in A. \quad (7)$$

Lemma 1. Let $X(\omega)$ is a fuzzy random variable then $X(\omega) = \bigcup_{\alpha \in (0,1]} \alpha X_\alpha(\omega)$.

Proof. If A is a fuzzy number then $A = \bigcup_{\alpha \in (0,1]} \alpha A_\alpha$. Since $(\bigcup_\alpha \alpha A_\alpha)(x) = Sup\{\alpha(A_\alpha)(x) \mid \alpha \in (0,1]\} = Sup\{\alpha \mid x \in A_\alpha\} = A(x)$

for any $x \in R$ then $A = \bigcup_{\alpha \in (0,1]} \alpha A_\alpha$. Since $X(\omega) \in F$ then the proof is completed.

Definition 6. The expected value of a fuzzy random variable X denoted by E(X) is defined as follows:

$$E(X) = \int_{\Omega} X(\omega) p(d\omega) = \bigcup_{\alpha \in (0,1]} \alpha \int_{\Omega} X_\alpha(\omega) p(d\omega)$$

$$= \bigcup_{\alpha \in (0,1]} \alpha \left[\int_{\Omega} X_\alpha^-(\omega) p(d\omega), \int_{\Omega} X_\alpha^+(\omega) p(d\omega) \right]. \quad (8)$$

Therefore, the expectation of a fuzzy random variable is defined as a unique $U \in F$ whose $\alpha-$cut is $U_\alpha = E(X_\alpha) = \left[E(X_\alpha^-), E(X_\alpha^+)\right]$ and $(E(X))_\alpha = E(X_\alpha)$.

Let $X \in FRV(\omega)$ then we define the scalar expected value of X denoted by $Er(X)$ and called it Er-expected value of X as follows:

$$Er(X) = \int_0^1 \int_{\omega \in \Omega} \overline{X}_\alpha(\omega) p(d\omega) d\alpha, \quad (9)$$

where $\overline{X}_\alpha(\omega) = \frac{1}{2}(X_\alpha^-(\omega) + X_\alpha^+(\omega))$ and for any $\omega \in \Omega$, $X_\alpha(\omega) = [X_\alpha^-(\omega), X_\alpha^+(\omega)]$.

Corollary1. Let $X(\omega)$ is a fuzzy random variable then:

$$Er(X) = \frac{1}{2}\int_0^1 [E(X_\alpha^-) + E(X_\alpha^+)] d\alpha, \quad (10)$$

where $E(X_\alpha^-)$ and $E(X_\alpha^+)$ are expected values of $X_\alpha^-(\omega)$ and $X_\alpha^+(\omega)$ respectively.

Corollary 2. Let X, $Y \in FRV(\Omega)$ and $\lambda \in R$ then:

i) $E(\lambda) = \lambda$, (11)

ii) $E(X + \lambda Y) = E(X) + \lambda E(Y)$, (12)

iii) $Er(X + \lambda Y) = Er(X) + \lambda Er(Y)$. (13)

Definition 7. Let $X, Y \in FRV(\Omega)$. Then the relations "$\cong$", "$\lesssim$" and "$\gtrsim$" are defined respectively as follows:

i) $X \cong Y$ iff $Er(X) = Er(Y)$, (14)

ii) $X \lesssim Y$ iff $Er(X) \leq Er(Y)$, (15)

iii) $X \gtrsim Y$ iff $Er(X) \geq Er(Y)$. (16)

3. Redundant system with fuzzy random lifetimes (RSFRL)

Consider a redundant system consisting of n components. For each component i, $i = 1, 2, ..., n$, there is only one type of elements available,. In this model, variable x_i, $i = 1, 2, ..., n$, is used to indicate the numbers of the i^{th} type of redundant elements. The redundant elements are arranged in one of two ways: parallel-series or standby. The standby and parallel-series systems are shown in Fig.1 and Fig.2, respectively. Let $\tilde{\xi}_{ij}$, $\tilde{T}_i(x, \tilde{\xi})$ and $\tilde{T}(x, \tilde{\xi})$ indicate the lifetimes of the jth redundant element in component i, the lifetimes in component i and the system lifetime respectively for $i = 1, 2, ..., n$, $j = 1, 2, ..., x_i$ and $\tilde{\xi} = (\tilde{\xi}_{11}, \tilde{\xi}_{12}, ..., \tilde{\xi}_{1x_1}, \tilde{\xi}_{21}, \tilde{\xi}_{22}, ..., \tilde{\xi}_{2x_2}, ..., \tilde{\xi}_{n1}, \tilde{\xi}_{n2}, ..., \tilde{\xi}_{nx_n})$.

For a standby redundant system we have $T_i(x, \tilde{\xi}) = \sum_{j=1}^{x_i} \tilde{\xi}_{ij}$, while for a parallel-series redundant system we have $\tilde{T}_i(x, \tilde{\xi}) = \max_{1 \leq j \leq x_i} \tilde{\xi}_{ij}$.

Suppose that our redundant system has the following requirement:

1. Lifetime of the element $\tilde{\xi}_{ij}$, $i = 1, 2, ..., n$, $j = 1, 2, .., x_i$, is a random fuzzy random variable.
2. There is no element repair or system repair or preventive maintenance.
3. The switching device of the standby system is assumed to be perfect.
4. The system and all redundant elements are in one of two states: operating (denoted by 1) or no operating (denoted by 0).

The general fuzzy random programming form of the redundancy system problem is as follows:

$$Ma\overline{x} \quad \tilde{T}(x, \tilde{\xi})$$

Subject to:

$$\sum_{i=1}^{n} c_i x_i \le c \qquad (17)$$

$x = (x_1,..,x_n) \ge 1$, integer vector,

where c is the maximum amount of available resources, c_i is the cost of i^{th} type of the redundant element, $i = 1, 2,..., n$, $Ma\bar{x}$ denotes the maximum operator for Fuzzy Random Variable (FRV) and $\tilde{T}(x,\tilde{\xi})$ indicates the system lifetime which is a fuzzy random.

3.1 Integer programming model for redundant system problem

In this section, a zero-one integer programming model is presented for Redundant System with Fuzzy Random Lifetimes (RSFRL).

Let δ_j^i, $i = 1, ..., n, j = 1, ..., k$, be a binary decision variable defined as:

$$\delta_j^i = \begin{cases} 1 & if \ \ jth \ \ redundant \ \ element \ is \ used \ \ in \ \ component \ \ i \\ 0 & Otherwise \end{cases}$$

where k is determined by expert or decision maker (k is the upper bound of x_i, $i = 1, 2,..., n$, where $k \ge \max_{i=1}^{n}\{x_i\}$) and n is predetermined based on properties of systems. It is clear that $x_i = \sum_{j=1}^{k} \delta_j^i$; $i = 1, 2,..., n$.

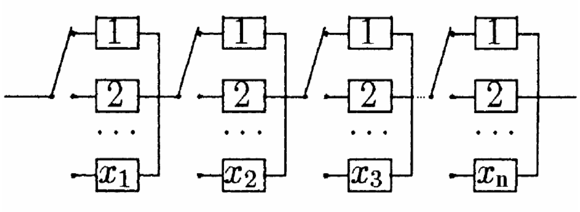

Figure 1. A standby redundant system..

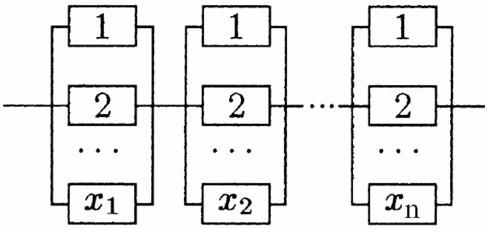

Figure 2. A parallel-series redundant system.

3.1.1. Standby redundant system problem

For a standby system, we have:

$$\tilde{T}_i(x,\xi) = \sum_{j=1}^{x_i} \tilde{\tilde{\xi}}_{ij} = \sum_{j=1}^{k} \delta_j^i \tilde{\tilde{\xi}}_{ij} \, . \qquad (18)$$

Then the integer programming model is formulated as follows:

$$Max \ \tilde{T}(x,\tilde{\xi}) = Min\{\sum_{j=1}^{k} \delta_j^i \tilde{\tilde{\xi}}_{ij} \mid i = 1,..., n\}$$

Subject to:

$$\sum_{i=1}^{n}\sum_{j=1}^{k} c_i \delta_j^i \le c \qquad (19)$$

$$\sum_{j=1}^{k} \delta_j^i \ge 1 \qquad i = 1, 2,..., n$$

$$\delta_j^i \in \{0, 1\}, \ \ i = 1,..., n, \ j = 1,..., k \, .$$

By using the concept of Er-expected value of fuzzy random variables and corollary 2, the above model can be converted to the following zero-one integer programming models:

$$Max \ \ Er(\tilde{T}(x,\tilde{\xi})) = Min\{\sum_{j=1}^{k} \delta_j^i Er(\tilde{\tilde{\xi}}_{ij}) \mid i = 1,..., n\}$$

Subject to:

$$\sum_{i=1}^{n}\sum_{j=1}^{k} c_i \delta_j^i \le c \qquad (20)$$

$$\sum_{j=1}^{k} \delta_j^i \ge 1 \qquad i = 1, 2,..., n$$

$$\delta_j^i \in \{0, 1\}, \ \ i = 1,..., n, \ j = 1,..., k \, .$$

or

$$Max \ \ T(x,\xi) = Min\{\sum_{j=1}^{k} \delta_j^i \xi_{ij} \mid i = 1,..., n\}$$

Subject to:

$$\sum_{i=1}^{n}\sum_{j=1}^{k} c_i \delta_j^i \le c \qquad (21)$$

$$\sum_{j=1}^{k} \delta_j^i \geq 1 \qquad i = 1, 2, \ldots, n$$

$$\delta_j^i \in \{0, 1\}, \quad i = 1, \ldots, n, \; j = 1, \ldots, k \; .$$

In the above model $T(x,\xi)$ and ξ_{ij} are the crisp values of $\tilde{T}(x,\tilde{\xi})$ and $\tilde{\xi}_{ij}$ respectively based on the definition of Er-expected value of a fuzzy random variable. Now let:

$$y = Min\{\sum_{j=1}^{k} \delta_j^i \xi_{ij} \mid i = 1, \ldots, n\} \; . \qquad (22)$$

Then Model (23) can be written as follows:

$Max \;\; y$

Subject to:

$$\sum_{j=1}^{k} \delta_j^i \xi_{ij} \geq y \qquad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} \sum_{j=1}^{k} c_i \delta_j^i \leq c \qquad (23)$$

$$\sum_{j=1}^{k} \delta_j^i \geq 1 \qquad i = 1, 2, \ldots, n$$

$$y \geq 0 \, , \delta_j^i \in \{0, 1\}, \quad i = 1, \ldots, n, \; j = 1, \ldots, k \; .$$

The above model is a zero-one integer programming model and it can be solved by one of the commercial ILP solvers. If we suppose that $\delta_j^{i^*}$ is the optimal solution of Model 25 then an Er-optimal solution of the original problem can be obtained by $x_i^* = \sum_{j=1}^{k} \delta_j^{i^*}$ and $y^* = Er(\tilde{T}^*(x,\tilde{\xi}))$.

3.1.2 Parallel-series redundant system problem

For a parallel-series system, we have:

$$\tilde{T}_i(x,\tilde{\xi}) = \max{}_{1 \leq j \leq x_i} \tilde{\xi}_{ij} = \max{}_{1 \leq j \leq k} \delta_j^i \tilde{\xi}_{ij} \; . \quad (24)$$

Then the corresponding integer programming model is formulated as follows:

$$Max \;\; \tilde{T}(x,\tilde{\xi}) = Min \left\{ \max{}_{1 \leq j \leq k} \delta_j^i \tilde{\xi}_{ij} \mid i = 1, \ldots, n \right\}$$

Subject to:

$$\sum_{i=1}^{n} \sum_{j=1}^{k} c_i \delta_j^i \leq c \qquad (25)$$

$$\sum_{j=1}^{k} \delta_j^i \geq 1 \qquad i = 1, 2, \ldots, n$$

$$\delta_j^i \in \{0, 1\}, \quad i = 1, \ldots, n, \; j = 1, \ldots, k \; .$$

By using the concept of Er-expected value of fuzzy random variables and corollary 2, Model 19 can be converted to the following zero-one integer programming problems:

$$Max \;\; Er(\tilde{T}(x,\tilde{\xi})) = Min \left\{ \max{}_{1 \leq j \leq k} \delta_j^i Er(\tilde{\xi}_{ij}) \mid i = 1, \ldots, n \right\}$$

Subject to:

$$\sum_{i=1}^{n} \sum_{j=1}^{k} c_i \delta_j^i \leq c \qquad (26)$$

$$\sum_{j=1}^{k} \delta_j^i \geq 1 \qquad i = 1, 2, \ldots, n$$

$$\delta_j^i \in \{0, 1\}, \quad i = 1, \ldots, n, \; j = 1, \ldots, k$$

or

$$Max \;\; T(x,\xi) = Min \left\{ \max{}_{1 \leq j \leq k} \delta_j^i \xi_{ij} \mid i = 1, \ldots, n \right\}$$

Subject to:

$$\sum_{i=1}^{n} \sum_{j=1}^{k} c_i \delta_j^i \leq c \qquad (27)$$

$$\sum_{j=1}^{k} \delta_j^i \geq 1 \qquad i = 1, 2, \ldots, n$$

$$\delta_j^i \in \{0, 1\}, \quad i = 1, \ldots, n, \; j = 1, \ldots, k$$

Let $y_i = \max_{1 \leq j \leq k} \delta_j^i \xi_{ij}, i = 1, \ldots, n$ and $y = \min_{1 \leq i \leq n} y_i$.

Model (28) can be written as follows:

$Max \;\; y$

Subject to:

$$y_i \geq y \qquad i = 1, 2, \ldots, n \qquad (28)$$

$$y_i \geq \delta_j^i \xi_{ij} \qquad i = 1, \ldots, n,\ j = 1, \ldots, k$$

$$y_i \leq \delta_j^i \xi_{ij} + M(1 - \alpha_j^i) \qquad i = 1, \ldots, n,\ j = 1, \ldots, k$$

$$\sum_{j=1}^{k} \alpha_j^i = 1 \qquad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} \sum_{j=1}^{k} c_i \delta_j^i \leq c$$

$$\sum_{j=1}^{k} \delta_j^i \geq 1 \qquad i = 1, 2, \ldots, n$$

$$\alpha_j^i, \delta_j^i \in \{0,1\} \qquad i = 1, \ldots, n,\ j = 1, \ldots, k$$

$$y \geq 0$$

$$y_i \geq 0 \qquad i = 1, 2, \ldots, n$$

The above model is a zero-one integer programming model and it can be solved by one of the commercial ILP solvers.

In the following steps, we summarize the necessary steps to solve the redundant system with fuzzy random lifetimes in standby or parallel situation:

Data entry:

Step 0: Define a membership function for each fuzzy random variable in Model (17) and determine the Er-expected values of the fuzzy random variables.

Model structure:

Step 1: Apply the concept of upper bound of numbers of the i^{th} types of elements:

- Convert Model (17) to Model (19) [Standby System],
- Convert Model (17) to Model (25) [Parallel System].

Step 2: Calculate Er-expected values of fuzzy random variables

- Convert Model (19) to Model (20) or Model (4) [Standby System],
- Convert Model (19) to Model (26) or Model (26) [Parallel System].

Step 3: Use the zero-one integer programming model

- Convert Model (20) to Model (23) [Standby System],
- Convert Model (26) to Model (28) [Parallel System].

Solution procedure:

Step 4: Solve Model (23) or Model (28) as a zero-one integer programming model by one of the ILP solvers. Let δ_j^i be its Er-expected solutions. Then an Er-optimal solution of the original problem is obtained by:

$$x_i = \sum_{j=1}^{k} \delta_j^i . \qquad (29)$$

3.2. Numerical examples

In this section, two numerical examples of redundant system problems are given to clarify the model discussed in this section.

Example 1. Consider a standby redundancy system shown in Figure 2. The lifetimes of the 5 types of elements are fuzzy random variables. Suppose that $r_{ij}(\omega)$, $i = 1, \ldots, 5$, $j = 1, \ldots, 4$ are random variables distributed by a normal distribution function $N(\mu_{ij}, \sigma_{ij}^2)$ where μ_{ij} and σ_{ij}^2 are its mean and variance respectively and $\tilde{\xi}_{ij} = (r_{ij}, \beta_{ij}, \gamma_{ij})$, $i = 1, \ldots, 5,\ j = 1, \ldots, 4$, is a fuzzy random lifetime for the jth redundant elements in components i. Suppose that $r \sim N(\mu, \sigma^2)$ is a normal random variable with expectation μ and variance σ^2 on Ω and $X(\omega) = (r(\omega), \beta, \gamma)$. We have the following relations:

$$X_\alpha^-(\omega) = r(\omega) + \beta(\alpha - 1), \qquad (30)$$

$$X_\alpha^+(\omega) = r(\omega) + \gamma(1 - \alpha), \qquad (31)$$

$$X_\alpha(\omega) = [r(\omega) + \beta(\alpha - 1), r(\omega) + \gamma(1 - \alpha)], \qquad (32)$$

$$E(X_\alpha(\omega)) = [\mu + \beta(\alpha - 1), \mu + \gamma(1 - \alpha)], \qquad (33)$$

$$Er(X) = \frac{1}{2}\int_0^1 [2\mu + \alpha(\beta - \gamma) - (\beta - \gamma)] d\alpha = \mu - \frac{1}{4}(\beta - \gamma). \qquad (34)$$

Table 1. Fuzzy random Lifetimes of elements $\tilde{\xi}_{ij}$.

	$\tilde{\xi}_{1j}$, j = 1,..., 4	$\tilde{\xi}_{2j}$, j = 1,..., 4	$\tilde{\xi}_{3j}$, j = 1,..., 4	$\tilde{\xi}_{4j}$, j = 1,..., 4	$\tilde{\xi}_{5j}$, j = 1,..., 4
μ	15	12	9	13	19
β	5, 3.3, 2.5, 2	4, 3, 1, 4	3, 0, 2, 5	1, 0, 4, 2	8, 0, 3, 9
γ	3, 4.3, 6.5, 2	6, 3, 5, 0	1, 2, 6, 0	5, 2, 2, 7	4, 5, 7, 4

In this example, we assume that fuzzy random lifetimes of elements are given in Table 1.

Then the above standby redundancy optimization problem can be converted to a zero-one integer programming as follows by using the concept of Er-expected value of fuzzy random variables:

Max y

Subject to:

$$14.5\delta_1^1 + 15.25\delta_2^1 + 16\delta_3^1 + 15\delta_4^1 \geq y \tag{35}$$

$$12.5\delta_1^2 + 12\delta_2^2 + 13\delta_3^2 + 11\delta_4^2 \geq y$$

$$8.5\delta_1^3 + 9.5\delta_2^3 + 10\delta_3^3 + 7.75\delta_4^3 \geq y$$

$$14\delta_1^4 + 13.5\delta_2^4 + 12.5\delta_3^4 + 14.25\delta_4^4 \geq y$$

$$18\delta_1^5 + 20.25\delta_2^5 + 20\delta_3^5 + 17.75\delta_4^5 \geq y$$

$$89\sum_{j=1}^{4}\delta_j^1 + 102\sum_{j=1}^{4}\delta_j^2 + 109\sum_{j=1}^{4}\delta_j^3 + 95\sum_{j=1}^{4}\delta_j^4 + 113\sum_{j=1}^{4}\delta_j^5 \leq 1200$$

$$\sum_{j=1}^{4}\delta_j^i \geq 1 \qquad i = 1, 2, \ldots, 5$$

$$\delta_j^i \in \{0,1\}, \quad i = 1,\ldots,5, \; j = 1,\ldots,4$$

The above model solved by Lingo which is one of the commercial ILP solvers. Then the Er-optimal solution of the model is obtained as follows:

$$(\delta_1^1, \delta_2^1, \delta_3^1, \delta_4^1)^* = (1, 1, 1, 0),$$

$$(\delta_1^2, \delta_2^2, \delta_3^2, \delta_4^2)^* = (0, 1, 1, 0),$$

$$(\delta_1^3, \delta_2^3, \delta_3^3, \delta_4^3)^* = (1, 1, 0, 0),$$

$$(\delta_1^4, 5\delta_2^4, \delta_3^4, \delta_4^4)^* = (0, 0, 0, 1),$$

$$(\delta_1^5, \delta_2^5, \delta_3^5, \delta_4^5)^* = (1, 1, 1, 0),$$

$$y^* = 14.25,$$

$$x_{Er}^*(SROS) = x^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (3, 2, 2, 1, 3),$$

$$T^* = (Er(\tilde{T}))^* = 14.25.$$

In order to compare the result of our model with the classical redundancy optimization model in which only the element lifetimes are assumed to be random variables, this example was also solved by considering the following assumptions:

$$\tilde{\xi}_{1j} \sim \mathrm{N}(15, \sigma^2), \tag{36}$$

$$\tilde{\xi}_{2j} \sim \mathrm{N}(12, \sigma^2), \tag{37}$$

$$\tilde{\xi}_{3j} \sim \mathrm{N}(9, \sigma^2), \tag{38}$$

$$\tilde{\xi}_{4j} \sim \mathrm{N}(13, \sigma^2), \tag{39}$$

$$\tilde{\xi}_{5j} \sim \mathrm{N}(19, \sigma^2) \text{ for } j = 1, \ldots, 4. \tag{40}$$

The optimal solution in this case when only the randomness of the element lifetimes is important is as follows:

$$(\delta_1^1, \delta_2^1, \delta_3^1, \delta_4^1)_c^* = (0, 1, 0, 0),$$

$$(\delta_1^2, \delta_2^2, \delta_3^2, \delta_4^2)_c^* = (0, 1, 0, 1),$$

$$(\delta_1^3, \delta_2^3, \delta_3^3, \delta_4^3)_c^* = (0, 1, 0, 1),$$

$$(\delta_1^4, 5\delta_2^4, \delta_3^4, \delta_4^4)_c{}^* = (0, 1, 0, 0),$$

$$(\delta_1^5, \delta_2^5, \delta_3^5, \delta_4^5)_c{}^* = (0, 1, 0, 0),$$

$$y_c{}^* = 13.00 ,$$

$$x_c{}^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (1, 2, 2, 1, 1) ,$$

$$T_c{}^* = 13 .$$

As this result shows, the system lifetime in our model has been increased by almost 9% (T^* vs. $T_c{}^*$) and the number of elements used in the system has also been increased by 36% (x^* vs. $x_c{}^*$). Therefore, the combination of randomness and fuzziness in our model generated more reliable and efficient system.

Example 2. Consider a parallel-series redundancy system shown in Figure 1. Furthermore, suppose that all necessary assumptions are the same as Example 1. In this example, we also assume that fuzzy random lifetimes of elements are given in Table 2.

Then the above parallel-series redundancy optimization problem can be converted to zero-one integer programming as follows by using the concept of Er-expected value of fuzzy random variables:

Max y

Subject to:

$$y_1 \geq y \quad (41)$$

$$y_2 \geq y$$

$$y_2 \geq y$$

$$14.5\delta_1^1 \leq y_1$$

$$15.25\delta_2^1 \leq y_1$$

$$16\delta_3^1 \leq y_1$$

$$12.5\delta_1^2 \leq y_2$$

$$12\delta_2^2 \leq y_2$$

$$10\delta_3^3 \leq y_3$$

$$14.5\delta_1^1 + M(1-\alpha_1^1) \geq y_1$$

$$15.25\delta_2^1 + M(1-\alpha_2^1) \geq y_1$$

$$16\delta_3^1 + M(1-\alpha_3^1) \geq y_1$$

$$\alpha_1^1 + \alpha_2^1 + \alpha_3^1 = 1$$

$$12.5\delta_1^2 + M(1-\alpha_1^2) \geq y_2$$

$$12\delta_2^2 + M(1-\alpha_2^2) \geq y_2$$

$$13\delta_3^2 + M(1-\alpha_3^2) \geq y_2$$

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

$$8.5\delta_1^3 + M(1-\alpha_1^3) \geq y_3$$

$$9.5\delta_2^3 + M(1-\alpha_2^3) \geq y_3$$

$$10\delta_3^3 + M(1-\alpha_3^3) \geq y_3$$

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = 1$$

$$\sum_{j=1}^{3} \delta_j^i \geq 1 \qquad i = 1, 2, 3$$

$$89\sum_{j=1}^{3}\delta_j^1 + 102\sum_{j=1}^{3}\delta_j^2 + 109\sum_{j=1}^{3}\delta_j^3 + \leq 700$$

$$\alpha_j^i, \delta_j^i \in \{0, 1\}, \quad i = 1, 2, 3, \ j = 1, 2, 3 .$$

Table 2. Fuzzy random Lifetimes of elements $\tilde{\xi}_{ij}$.

	$\tilde{\xi}_{1j}$, $j = 1,2,3$	$\tilde{\xi}_{2j}$, $j = 1,2,3$	$\tilde{\xi}_{3j}$, $j = 1,2,3$
μ	15	12	9
β	5, 3.3, 2.5, 2	4, 3, 1, 4	3, 0, 2, 5
γ	3, 4.3, 6.5, 2	6, 3, 5, 0	1, 2, 6, 0

Model (41) solved by by Lingo which is one of the commercial ILP solvers. Then the Er-optimal solution of the model is obtained as follows:

$$(\delta_1^1, \delta_2^1, \delta_3^1)^* = (0, 1, 1)\,,$$

$$(\delta_1^2, \delta_2^2, \delta_3^2)^* = (0, 1, 0)\,,$$

$$(\delta_1^3, \delta_2^3, \delta_3^3)^* = (1, 1, 1)\,,$$

$$y^* = 12\,,$$

$$x_{Er}^*(PROS) = x^* = (x_1^*, x_2^*, x_3^*,) = (2, 1, 3)\,,$$

$$T^* = (Er(\tilde{T}))^* = 12\,.$$

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