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Allocation models for DMUs with negative data

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Abstract

The formulas of cost and allocative efficiencies of decision making units (DMUs) with positive data cannot be used for DMUs with negative data. On the other hand, these formulas are needed to analyze the productivity and performance of DMUs with negative data. To this end, this study introduces the cost and allocative efficiencies of DMUs with negative data and demonstrates that the introduced cost efficiency is equal to the product of allocative and range directional measure efficiencies. The study then intends to extend the definition of the above efficiencies to DMUs with negative data and different unit costs. Finally, two numerical examples are given to illustrate the proposed methods.

JEL classification: C6, D2

Keywords: DEA, Cost efficiency, Negative data, Allocative efficiency, RDM

Introduction

Data envelopment analysis (DEA) is a nonparametric method for computing and assessing the relative efficiency of homogeneous decision making units (DMUs) with multiple inputs and outputs. The traditional DEA models assume that all of the inputs and outputs are nonnegative, while in many situations and applications, the negative values in data might exist, which is a weakness of the traditional DEA models. To overcome the shortcoming, in recent years, different DEA models have been proposed in the literature about DMUs with negative data.

Portela et al. (2004) provided the range directional measure (RDM) approach to measure the efficiency of DMUs with negative data based on a directional distance function without the need to transform the data. The efficiency measurement of DMUs with negative data is also much debated. For example, in order to overcome the shortcomings of the slack-based measure model (Tone 2001) in dealing with negative inputs and outputs, Sharp et al. (2006) presented the modified slack-based measure model. Emrouznejad et al. (2010) proposed the semi-oriented radial measure model for dealing with negative inputs and outputs. Also, recently, a two-phase approach model has been proposed by Kazemi Matin and Azizi (2011) based on a modified version of the additive model to achieve a target with nonnegative components for each DMU with negative inputs and outputs.

One of the most significant types of efficiency is cost efficiency. This type of efficiency is used to identify the different kinds of inefficiencies when information on costs is available. DMUs can achieve the best cost efficiency score with a combination of inputs which allow them to produce the desired outputs at minimum costs.

In many DEA literatures, the economic concepts of cost efficiency have been considered. The primary discussion of cost efficiency can be traced back to Farrell (1957) and Debreu (1951), from whom many of the ideas about DEA are derived. Farrell offered a measure of cost efficiency under fixed and known prices. His method extended to situations with different prices of inputs for DMUs (Tone 2002).

So far, all of the previous studies have explored cost efficiency of DMUs with nonnegative data, and there is no discussion concerning cost efficiency in the presence of negative data. However, in some cases, the inputs of DMUs have negative values with positive costs. This paper defines the cost and allocative efficiencies for DMUs with negative data, then demonstrates that the

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defined cost efficiency with negative data is equal to the product of allocative and RDM efficiencies, and then extends the definition of efficiencies to DMUs with negative data and different unit costs.

The rest of this paper is organized as follows: the Section 'Background models' explains the RDM efficiency and cost efficiency models. 'Cost efficiency in the presence of negative data' briefly introduces cost efficiency under common and different prices in the presence of negative data. The Section 'Illustrative examples' provides two numerical examples, and in the last section, the conclusion is given.

Background models

In productive activities, we assume that there are n homogeneous DMUs. Each DMU produces s different outputs from m different inputs. Input and output vectors for the DMU which is under evaluation are denoted by x_o and y_o, and for DMU$_j$, are denoted by x_j and y_j. The next section explains the RDM and cost efficiency models.

RDM

The RDM model, introduced by Portela et al. (2004), can be used for comparing DMUs when some inputs and/or outputs are negative. Consider a point with maximum outputs and minimum inputs as an ideal point (i.e., the ith (i = 1, …, m) input x_{iI} as min $_j\{x_{ij}\}$ and the rth (r = 1, …, s,) output y_{rI} as max $_j\{y_{rj}\}$). In RDM, a directional vector is considered as $R^+_{ro} = \max_j\left\{y_{rj}\right\} - y_{ro}; r = 1, \ldots, s$ and $R^-_{io} = x_{io} - \max_j\left\{x_{ij}\right\};\ i = 1, \ldots, m$. The RDM model for DMU$_o$ is as follows (Portela et al. 2004):

$$\begin{aligned} &\max \beta \\ \text{s.t.}\ &\sum_{j=1}^{n}\lambda_j x_{ij} \le x_{io} - \beta R^-_{io}, \quad i = 1, \ldots, m \\ &\sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{ro} + \beta R^+_{ro}, \quad r = 1, \ldots, s \\ &\sum_{j=1}^{n}\lambda_j = 1\ ,\ \lambda_j \ge 0, \quad j = 1, \ldots, n. \end{aligned} \tag{1}$$

The optimal value of model (1), β^*, represents the inefficiency measurement for DMU$_o$, while 1 – β^* represents the efficiency measurement for DMU$_o$. Unit invariance and translation invariance are the two important properties of the RDM model.

Cost efficiency

For measuring the cost efficiency of the DMUs with multiple inputs and outputs under common unit input prices, the following linear program is solved (Farrell 1957):

$$\begin{aligned} cx^* = &\min \sum_{i=1}^{m} c_i x_i \\ \text{s.t.}\ &\sum_{j=1}^{n}\lambda_j x_{ij} \le x_i, \quad i = 1, \ldots, m \\ &\sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{ro}, \quad r = 1, \ldots, s \\ &\sum_{j=1}^{n}\lambda_j = 1\ ,\ \lambda_j \ge 0,\ \ j = 1, \ldots, n. \end{aligned} \tag{2}$$

The cost efficiency is obtained as the following ratio:

$$\mathrm{CE} = \frac{cx^*}{cx_o}, \tag{3}$$

where the nominator represents the minimum cost (i.e., the optimal value of model (2)) and the denominator shows the current cost at DMU$_o$.

Cost efficiency in the presence of negative data

In this section, we define cost and allocative efficiencies in the presence of negative data under common and different prices, and then it is shown that cost efficiency is the product of allocative and RDM efficiencies. Finally, the above subjects are extended to new cost, allocative, and RDM efficiencies.

Cost and allocative efficiencies with negative data under common unit prices

Definition 1 Under common unit input prices, we define cost (overall) efficiency in the presence of negative data as follows:

$$\mathrm{CE} = \frac{cx^* - cx_I}{cx_o - cx_I}. \tag{4}$$

In the above mentioned ratio, the nominator represents the difference between cx^* (the optimal value of model (2)) and the cost of the ideal point, i.e., cx_I. In addition, the denominator depicts the difference between the observed cost of DMU$_o$, i.e., cx_o and cx_I. It is clear that the value of CE is equal to or less than 1. Cost efficiency might be less than 1 for one of the following two reasons: excessive input usage in production or production with a wrong input mix in light of input prices or both. In a particular case, when the ideal point is one of the observed DMUs, we define CE = 1.

Figure 1 illustrates the concepts dealing with allocative efficiency, RDM efficiency, and cost efficiency, using the units A, B, D, E, F, H, G, and P in the presence of

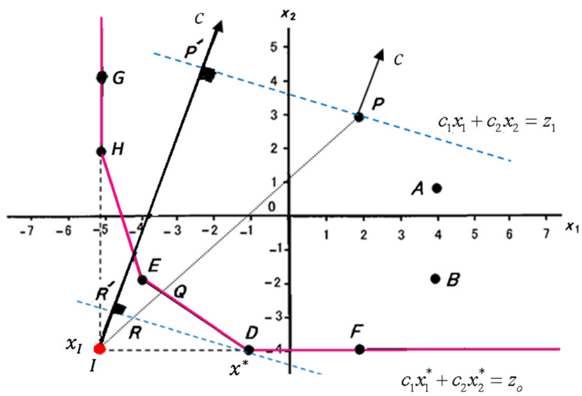

Figure 1 RDM, allocative, and overall efficiencies. The solid lines represent the segments of an isoquant which is composed of a set of all inputs (x_1, x_2) that produce the same amount of a single output. The dashed lines passing through *P* and *D* are the budget (or cost) lines.

negative data. In Figure 1, the RDM inefficiency of DMU_P can be evaluated by

$$\beta = \frac{d(Q,P)}{d(I,P)}. \tag{5}$$

Hence,

$$1-\beta = \frac{d(I,P)-d(Q,P)}{d(I,P)} = \frac{d(I,Q)}{d(I,P)} \tag{6}$$

Equation 6 represents the RDM efficiency of DMU_P, which is between 0 and 1. $d(I, P)$ and $d(I, Q)$ denote the distance from the ideal point I to P and the distance from the ideal point I to Q, respectively, and $d(Q, P)$ is the distance from Q to P.

In order to illustrate allocative efficiency in Figure 1, we consider the budget (cost) line $c_1x_1 + c_2x_2 = z_1$ passing through the point P. We move this cost line in parallel form until it crosses the isoquant at D.

By moving the budget line in parallel form, cost can be reduced. The lowest cost is associated with the budget (cost) line $c_1x_1^* + c_2x_2^* = z_o$, where $z_o < z_1$ and z_o can be obtained by substituting the coordinates of DMU_D in the budget (cost) line $c_1x_1 + c_2x_2 = z_1$. The best point D is achieved as the optimal solution x^* of the linear program (2) (Farrell 1957).

Now, we define $\frac{d(I,R)}{d(I,Q)}$ as a measure of allocative efficiency, where $d(I, R)$ and $d(I, Q)$ denote the distance from the ideal point I to R and the distance from the ideal point I to Q, respectively. It shows that the minimum cost is not reached since we have failed to make the replacements which are involved in moving from point Q to D along the efficiency frontier.

According to Definition 1, we have

$$\mathrm{CE} = \frac{cx^*-cx_I}{cx_o-cx_I}$$

$$\mathrm{CE} = \frac{cx^*-cx_I}{cx_o-cx_I} = \frac{|c||x^*-x_I|\cos\theta}{|c||x_o-x_I|\cos\phi} = \frac{|x^*-x_I|\cos\theta}{|x_o-x_I|\cos\phi} \tag{7}$$

where $|.|$ in Equation 7 represents the norm function of a vector, and θ and ϕ are the angles between the vector c and vectors $(x^* - x_I)$ and $(x_o - x_I)$, respectively. The numerator in Equation 7 represents the projection of vector $(x^* - x_I)$ on vector c which is equal to $d(I, R')$, and the denominator in Equation 7 demonstrates the projection of vector $(x_o - x_I)$ on vector c which is equal to $d(I, P')$. We can represent the defined cost efficiency in the presence of negative data by means of the following ratio:

$$\mathrm{CE} = \frac{d(I,R')}{d(I,P')} \tag{8}$$

where $d(I, R')$ and $d(I, P')$ denote the distance from the ideal point I to R' and P', respectively. According to Thales theorem, we have:

$$\mathrm{CE} = \frac{d(I,R')}{d(I,P')} = \frac{d(I,R)}{d(I,P)}. \tag{9}$$

Therefore, the three above mentioned efficiencies in the presence of negative data have the following relationship:

$$\frac{d(I,R)}{d(I,Q)} \times \frac{d(I,Q)}{d(I,P)} = \frac{d(I,R)}{d(I,P)}. \tag{10}$$

That is, the product of allocative and RDM efficiencies is equal to cost efficiency.

Cost efficiency with negative data under different prices

In some situations, the unit prices of input are not the same among DMUs. Therefore, the above definitions of cost and allocative efficiencies have shortcomings, and they are not applicable to these cases (Farrell 1957). In this case, we define new cost and allocative efficiencies, which are extensions of the prior definitions and are applicable to situations in which the input prices are not the same among DMUs.

In order to discuss cost efficiency in the presence of different unit input costs, we consider the cost-based production possibility set, P_C, as follows (Tone 2002):

$$P_C = \left\{ (\bar{x}, y) | \bar{x} \geq \sum_{j=1}^{n} \lambda_j \bar{x}_j, \right. \\ \left. y \leq \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \ldots, n \right\} \tag{11}$$

where $\bar{x}_j = (\bar{x}_{1j}, \ldots, \bar{x}_{mj})^T = (c_{1j}x_{1j}, \ldots, c_{mj}x_{mj})^T$, in which $c_j = (c_{1j}, \ldots, c_{mj})^T$ is the positive cost vector of DMU_j.

Definition 2 The ideal point, by using the new production possibility set, is a point with maximum outputs and minimum inputs, i.e., $\bar{x}_I = (\bar{x}_{1I}, \ldots, \bar{x}_{mI})^T$ where $\bar{x}_{iI} = \min_j \bar{x}_{ij}$, $i = 1, \ldots, m$, and $y_I = (y_{1I}, \ldots, y_{sI})^T$ where $y_{rI} = \max_j y_{rj}$, $r = 1, \ldots, s$. In the new production possibility set, P_C, the new RDM inefficiency, $\bar{\beta}^*$, is obtained by solving the following linear program:

$$\begin{aligned} &[\text{NRDM}] \quad \bar{\beta}^* = \max \bar{\beta} \\ &\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \bar{x}_{ij} \leq \bar{x}_{io} - \bar{\beta} \bar{R}_{io}^-, \quad i = 1, \ldots, m \\ &\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \bar{\beta} R_{ro}^+, \quad r = 1, \ldots, s \\ &\qquad \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0 \quad , \quad j = 1, \ldots, n, \end{aligned} \tag{12}$$

where $\bar{R}_{io}^- = \bar{x}_{io} - \min_j \{\bar{x}_{ij}\}$; $i = 1, \ldots, m$ and $R_{ro}^+ = \max_j \{y_{rj}\} - y_{ro}$; $r = 1, \ldots, s$. To interpret $\bar{\beta}^*$ (the optimal value of model (12)) and $1-\bar{\beta}^*$, it can be said that $\bar{\beta}^*$ demonstrates the new RDM inefficiency, and $1-\bar{\beta}^*$ demonstrates the new RDM efficiency of DMU_o.

Definition 3 The new cost efficiency in the presence of negative data under different unit prices is defined as follows:

$$\bar{\text{CE}} = \frac{e\bar{x}^* - e\bar{x}_I}{e\bar{x}_o - e\bar{x}_I} \tag{13}$$

where e is a vector in R_m with each component equal to 1, and $\bar{x}^*$ is an optimal solution of the following model (Tone 2002):

$$\begin{aligned} &[\text{NCost}] \quad e\bar{x}^* = \min e\bar{x} \\ &\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \bar{x}_j \leq \bar{x} \\ &\qquad \sum_{j=1}^{n} \lambda_j y_j \geq y_o \\ &\qquad \sum_{j=1}^{n} \lambda_j = 1, \\ &\qquad \lambda_j \geq 0, \quad j = 1, \ldots, n. \end{aligned} \tag{14}$$

It is obvious that $0 \leq \bar{\text{CE}} \leq 1$. The following theorem states a relationship between new RDM efficiency and new cost efficiency scores.

Theorem 1 $\bar{\text{CE}} \leq 1 - \bar{\beta}^*$.

Proof Let $(\bar{\beta}^*, \lambda^*)$ be an optimal solution of model (12); then $(\bar{x}_o - \bar{\beta}^* \bar{R}_o^-, \lambda^*)$ is a feasible solution for model (14). This shows that $e\bar{x}^* \leq e\bar{x}_o - e\bar{\beta}^* \bar{R}_o^-$. Using Equation 13 we have:

$$\begin{aligned} \bar{\text{CE}} &= \frac{e\bar{x}^* - e\bar{x}_I}{e\bar{x}_o - e\bar{x}_I} \leq \frac{e\bar{x}_o - e\bar{\beta}^* \bar{R}_o^- - e\bar{x}_I}{e\bar{x}_o - e\bar{x}_I} \leq \frac{e\bar{x}_o - e\bar{\beta}^* (\bar{x}_o - \bar{x}_I) - e\bar{x}_I}{e\bar{x}_o - e\bar{x}_I} \\ &= \frac{e\bar{x}_o (1 - \bar{\beta}^*) - e\bar{x}_I (1 - \bar{\beta}^*)}{e\bar{x}_o - e\bar{x}_I} = (1 - \bar{\beta}^*). \end{aligned} \tag{15}$$

This fact completes the proof.

Definition 4 The new allocative efficiency under the different unit prices in the presence of negative data is defined as follows:

$$\bar{\text{AE}} = \frac{\bar{\text{CE}}}{1 - \bar{\beta}^*}. \tag{16}$$

It is obvious that $0 \leq \bar{\text{AE}} \leq 1$.

Illustrative examples

In this section, two numerical examples are used to illustrate the concepts of what was mentioned earlier.

Table 1 The inputs, outputs, and costs for eight DMUs

DMUs	x_1	c_1	x_2	c_2	y
A	4	1	1	2	1
B	4	1	−2	2	1
D	−1	1	−4	2	1
E	−4	1	−2	2	1
F	2	1	−4	2	1
G	−5	1	4	2	1
H	−5	1	2	2	1
P	2	1	3	2	1

Table 2 RDM, cost, and allocative efficiencies

DMUs	β^*	$1-\beta^*$	cx^*	cx_o	cx_I	CE	AL
A	0.76	0.24	−9	6	−13	0.21	0.87
B	0.67	0.33	−9	0	−13	0.30	0.90
D	0	1	−9	−9	−13	1	1
E	0	1	−9	−8	−13	0.8	0.8
F	0.43	0.57	−9	−6	−13	0.57	1
G	0.25	0.75	−9	3	−13	0.25	0.33
H	0	1	−9	−1	−13	0.33	0.33
P	0.77	0.23	−9	8	−13	0.19	0.82

Example 1 In Table 1, we have eight DMUs with two inputs and one output. Figure 1 depicts the production possibility set composed of these input and output data. The values of x_1 and x_2 and their relative unit costs are exhibited in the columns of Table 1.

Table 2 reports the obtained results for the data of Table 1. To compare the results of Table 2 with Equations 5, 6, 9, and 10 as an example, we select DMU_P. As it can be seen in Figure 1, the coordinates of Q are $\left(\frac{-17}{5}, \frac{-12}{5}\right)$, which are obtained from the intersection of the line passing through E and D and the line passing through I and P.

The RDM efficiency for P is obtained as follows:

$$1-\beta^* = \frac{d(I,Q)}{d(I,P)} = \frac{\sqrt{\left(-5+\frac{17}{5}\right)^2+\left(-4+\frac{12}{5}\right)^2}}{\sqrt{(-5-2)^2+(-4-3)^2}} = \frac{8}{35} \simeq 0.23$$

where (−5,−4) are the coordinates of the ideal point. The coordinates of R are $\left(\frac{-11}{3}, \frac{-8}{3}\right)$, which were achieved from the intersection of the line passing through D and R' and the line passing through I and P. Hence, the cost and the allocative efficiencies of DMU_P are obtained as follows:

Table 4 New data set

DMUs	$\bar{x}_1$	$\bar{x}_2$	y
A	12	2	1
B	4	−4	1
D	−2	−8	1
E	−4	−6	1
F	6	−4	1
G	−20	20	1
H	−15	8	1
P	6	6	1
Ideal	−20	−8	1

$$\text{CE} = \frac{d(I,R)}{d(I,P)} = \frac{\sqrt{\left(-5+\frac{11}{3}\right)^2+\left(-4+\frac{8}{3}\right)^2}}{\sqrt{(-5-2)^2+(-4-3)^2}} = \frac{4}{21} \simeq 0.19$$

$$\text{AL} = \frac{d(I,R)}{d(I,Q)} = \frac{\sqrt{\left(-5+\frac{11}{3}\right)^2+\left(-4+\frac{8}{3}\right)^2}}{\sqrt{\left(-5+\frac{17}{5}\right)^2+\left(-4+\frac{12}{5}\right)^2}} = \frac{5}{6} \simeq 0.82.$$

Hence,

$$\frac{d(I,R)}{d(I,Q)} \times \frac{d(I,Q)}{d(I,P)} = \frac{d(I,R)}{d(I,P)}.$$

The above results are the same as the results which are exhibited in Table 2.

Example 2 Table 3 represents the data set of Table 1 under the different unit costs. The coordinates of inputs and output in the PPS P_C and the ideal point are represented in Table 4. These data were obtained by multiplying the relevant unit costs of x_1 and x_2 by the values of x_1 and x_2.

Table 5 shows the new RDM inefficiency, new RDM, cost, and allocative efficiencies under the different unit prices in the presence of the negative data of Table 4. The results of Table 5 indicate that the DMUs D and E have

Table 3 Data for eight DMUs

DMUs	x_1	c_1	x_2	c_2	y
A	4	3	1	2	1
B	4	1	−2	2	1
D	−1	2	−4	2	1
E	−4	1	−2	3	1
F	2	3	−4	1	1
G	−5	4	4	5	1
H	−5	3	2	4	1
P	2	3	3	2	1

Table 5 New RDM, new cost, and new allocative efficiencies

DMU	$\bar{\beta}^*$	$1-\bar{\beta}^*$	$e\bar{x}^*$	$e\bar{x}_o$	$e\bar{x}_I$	$\bar{\text{CE}}$	$\bar{\text{AL}}$
A	0.56	0.44	−10	14	−13	0.11	0.25
B	0.35	0.65	−10	0	−13	0.23	0.35
D	0	1	−10	−10	−13	1	1
E	0	1	−10	−10	−13	1	1
F	0.45	0.55	−10	2	−13	0.2	0.36
G	0	1	−10	0	−13	0.23	0.23
H	0	1	−10	−7	−13	0.5	0.5
P	0.53	0.47	−10	12	−13	0.12	0.25

the best performance. The DMUs G and H are new RDM efficient, in spite of the fact that these DMUs fell short in their new cost and new allocative efficiency scores.

Conclusions

This paper introduced cost and allocative efficiencies under common and different unit prices in the presence of negative data. It was shown that under common and different unit prices, the defined cost efficiency is the product of allocative efficiency and RDM efficiency. Finally, to illustrate the mentioned concepts, two numerical examples were used.

Competing interests

Both authors declare that they have no competing interests.

Authors' contributions

GT proposed the definition of cost efficiency of DMUs in the presence of negative data. Then MKH and GT extended this definition to DMUs with negative data and different unit costs and then demonstrated that the introduced cost efficiency is equal to the product of allocative and range directional measure efficiencies. Both authors read and approved the final manuscript.

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References

Debreu G (1951) The coefficient of resource utilization. Econometrica 19:273–292
Emrouznejad A, Anouze AL, Thanassoulis E (2010) A semi-oriented radial measure for measuring the efficiency of decision making units with negative data, using DEA. Eur J Oper Res 200:297–304
Farrell MJ (1957) The measurement of productive efficiency. J Roy Stat Soc 120:253–281
Kazemi Matin R, Azizi R (2011) A two-phase approach for setting targets in DEA with negative data. Appl Math Model 35(12):5794–5803
Portela MCAS, Thanassoulis E, Simpson GPM (2004) Negative data in DEA: a directional distance approach applied to bank branches. J Oper Res Soc 55:1111–1121
Sharp JA, Liu WB, Meng WA (2006) Modified slacks-based measure model for data envelopment analysis with 'natural' negative outputs and inputs. J Oper Res Soc 57:1–6
Tone K (2001) Slacks-based measure of efficiency in data envelopment analysis. Eur J Oper Res 130:498–509
Tone K (2002) A strange case of the cost and allocative efficiencies in DEA. J Oper Res Soc 53:1225–1231

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