Journal of Mathematical Sciences and Informatics Vol.1, No.1 (2006), pp47-52

Spectrum properties of the Koopman and Frobenius-Perron operators

Mohseni Moghadam, M.

Mahani Mathematical Research Center
Kerman University, Kerman, IRAN.
E-mail: mohseni@mail.uk.ac.ir

Abstract. Consider the measure space (X, A, μ). Let $S : X \to X$ be a nonsingular transformation on this space and assume that is the FrobeniusPerron operator associated with s . Also we will assume that U_s is the Koopman operator corresponding to s. Properties of these two operators, namely P_s and U_s have been considered by many authors. For more details see [5], [7], [8] and [9]. In this paper we will present some special aspects concerning the spectrum of P_s. Also it is shown that the spectrum of is a cyclic subset of the unit disk D. In connection with the Koopman operator U_s it is shown that under certain conditions, if μ is a regular measure, then U_s from $L^\infty(X)$ to itself is an isometric.

1 Introduction

Let be a nonsingular transformation on a measure space (X, A, μ). If $f \in L^1(X)$ and $f \geq 0$ then for $A \in A$ define

$$\gamma(A) = \int_{S^{-1}(A)} f(x)d\mu.$$

Clearly γ defines an absolutely continuous measure with respect to μ. As a consequence of the Radon-Nikodym theorem there exists a unique nonnegative L^1 function denoted by P_s such that:

$$\int_A P_s f(X)d\mu = \int_{S^{-1}(A)} f(x)d\mu.$$

In a very standard manner ([4], [3]), the above definition can be extended to a complex- valued function in $L^1(X)$, and the resulting function P_s is the

AMS: 47A36
Keywords: Frobenius Perron Operators, Koopman Operators.

so-called Frobenius Perron Operator. The operator $U_s : L^\infty(X) \to L^\infty(X)$ defined by $U_s(f(x)) = f(s(x))$ is called the Koopman operator. It is known ([2]), that the koopman operator is the adjoint of the FrobeniusPerron Operator. The singularity of s implies that U_s is well-defined. Also U_s is positively bounded and generalized the notion of transition [1]. For a detailed account of the properties of the FrobeniusPerron Operator see ([3], [6] and [10]).

In the remainder of this section, some basic properties of the Frobenius-Perron operator and Koopman operators are presented.

Proposition 1. 1. (Properties of the FrobeniusPerron operator) *P_s is linear, if $f \geq 0$ then $P_s f \geq 0$, $\int_X P_s f(x)d\mu = \int_X f(x)d\mu$ and for any positive n, $P_{s^n} = (P_s)^n$, where S^n and $(P_s)^n$ are respectively the nth iterates of and P_s.*

Proposition 1. 2. (Properties of the Koopman operator) *P_S is linear, if $f \geq 0$ then $U_S f \geq 0$ and if $f \in L^\infty(X)$, then $\|U_S f\|_\infty \geq \|f\|_\infty$ Let (X, A, μ) be a measure space and $S : X \to X$ be a nonsingular transformation. It is well known that if $f \in L^1(X)$ and $g \in L^\infty(X)$, then fg is integrable, and hence we define the scalar product (adjoint) of two functions by*

$$< f, g >= \int_X f(x)g(x)d\mu.$$

By Cauchy-Holder inequality, we have $| < f, g > | \geq \|f\|, \|g\|_\infty$. For operators P_S and U_S we can easily prove the following propositions.

Proposition 1. 3. *For $f \in L^1(X)$ and $g \in L^\infty(X)$, $< P_S, f, g >=< f, U_S, g)$. Moreover if $S : X \to X$ and $T : X \to X$ are both nonsingular, then is nonsingular and*

$$P_{TOS} = P_T P_S.$$

It is easy to prove that $\|P_S\| = \|U_S\| = 1$, and the FrobeniusPerron operator is weakly continuous. The latter means precisely that for every sequence $\{f_n\}$ in $L^1(X)$ if $\{f_n\}$ is weakly converges to f, then $\{P_S f_n\}$ is weakly converges to $P_S f$. The proof of the following Theorem can be found in any standard linear operators book.

Theorem 1. 4. *Suppose P is an operator in a Banach space and suppose $\{n^{-1}P^n\}$ converges to zero in the weak operator topology as n goes to infinity. Then the spectrum of P is a subset of the unit disk in the complex plane. Moreover, any pole λ of P with $|\lambda| = 1$ has order one.*

It follows that if P is quasi-compact, that is $|P^n - K| < 1$ for some positive n and some compact operator K. Then there are at most a finite number of points $\lambda_1, \lambda_2, \cdots, \lambda_k$ of unit modulus in the spectrum of P. Furthermore, each λ_q is an eigenvalue of order one with corresponding finite dimensional eigenspace. For a linear operator P on a Banach space X, $\lambda \in C$ is defined to be an approximate point spectrum of P if there exists a sequence $\{x_n\}$ in X with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(P - \lambda I)x_n\| = 0$ The set of approximate points spectrum of P is denoted by $\sigma_a(P)$. The result of the following theorem (see [1]) will be needed in the following sections.

Theorem 1. 5. *Suppose P is a linear operator on a Banach space X. Then $\sigma_a(P)$ is a nonempty subset of the spectrum of P. namely $\sigma(P)$, and the boundary of $\sigma(P)$ is contained in $\sigma_a(P)$.*

2 Ps With a Bounded Inverse

Through this section we assume that the FrobeniusPerron operator P_S has a bounded inverse. Let $S : X \to X$ be a measurable transformation in a measure space (X, A, μ), μ is said to be regular if $\mu(S^{-1}(A)) > 0$ whenever $\mu(A) > 0$. If $\mu(A) = 0$ implies $S(A) \in \mathbf{A}$ and $\mu(S(A)) = 0$. Then μ is said to be normal.

Lemma 2. 1. *Let (X, A, μ) be a $\sigma-$finite measure space and $S : X \to X$ be nonsingular transformation. Then the Koopman operator associated with S is injective if and only if μ is regular.*

Lemma 2. 2. *Let (X, A, μ) be a $\sigma-$finite measure space and $S : X \to X$ be a nonsingular transformation. If μ is regular, then $D^0 \cap \partial\sigma(U_S) = \emptyset$, where D^0 is the interior of the unit disk and $\partial\sigma(U_S)$ is the boundary of the spectrum of U_S.*

Theorem 2. 3. *Let (X, A, μ) be a $\sigma-$finite measure space and $S : X \to X$ nonsingular transformation. Let P_S, the FrobeniusPerron perator associated with S, have a bounded inverse. If the adjoint of P_S be injective, then $\sigma(P_S) \subset \partial D$.*

Definition 2. 4. Let (X, A, μ) be a measure space and assume that the set $D(X, A, \mu)$ is defined by $D(X, A, \mu) = \{f \in L^{-1}(X, A, \mu) : f \geq 0 \text{ and } \|f\|_1 = 1\}$. Any function $f \in D(X, A, \mu)$ is called a density.

Definition 2. 5. Let (X, A, μ) be a measure space and $P : X \to X$ be a Markov operator, that is for any nonnegative $f \in L^{-1}(X), Pf \geq 0$ and $\|Pf\|_1\|f\|_1$. Any $f \in D(X, A, \mu)$ that satisfies $Pf = f$ is called a stationary density of P.

Remark 2. 6. *By the support of a function f we mean the closure o the set of all such that $f(x) \neq 0$. Since this requires the topological notions which is not used elsewhere, we simply define the support of f to be the set of all x such that $f(x) \neq 0$, that is, Supp $f = \{x : f(x) \neq 0\}$.*

Theorem 2. 7. *Let (X, A, μ) be a $\sigma-$finite measure space, $S : X \to X$ nonsingular transformation, and P_S the FrobeniusPerron operator associated with S. Suppose μ is normal and there is a unique positive stationary density f of P_S. If P_S has a bounded inverse, then $\sigma(P_S) \subset \partial(D)$.*

Definition 2. 8. Let (X, A, μ) be a measure space and let a nonsingular transformation $S : X \to X$ be given. Then S is called ergodic if every invariant set $A \in \mathbf{A}$ is such that $\mu(A) = 0$ or $\mu(X - A) = 0$. That is S is ergodic if all invariant sets are trivial subset of X.

Lemma 2. 9. *Let (X, A, μ) be a measure space and $S : X \to X$ a nonsingular transformation. If S is ergodic, then there exists at most one stationary density for the corresponding FrobeniusPerron operator.*

Proof. Let f_1 and f_2 be two different stationary densities of P. Then it is clear that $Pg^+ = g^+$, $Pg^- = g^-$ where $g = f_1 - f_2$. Definite $A = supp\ g^+$, $B = supp\ g^-$, then both A and B are sets of positive measure and $A \cap B = \emptyset$. Let $\bar{A} = \cup_{n=0}^{\infty} S^{-n}(A)$, and $\bar{B} = \cup_{n=0}^{\infty} S^{-n}(B)$. One can show that $\bar{A} \cup \bar{B} = \emptyset$, $S^{-1}(\bar{A}) = \bar{A}$ and $S^{-1}(\bar{B}) = \bar{B}$. Both $\bar{A}$ and $\bar{B}$ have positive measure and are invariant which contradicts the assumption.

Theorem 2. 10. *Let $S : X \to X$ be a nonsingular transformation in the measure space . Let S be ergodic, μ normal and suppose there is a unique positive stationary density, if P_S has a bounded inverse. Then $\sigma(P_S) \subset \partial D$.*

Proof. Conclusion follows directly from Lemma 9 and Theorem 7.

Theorem 2. 11. *Let (X, A, μ) be a $\sigma-$finite measure space and $S : X \to X$ a nonsingular transformation. The μ is regular if and only if U_S, the Koopman operator associated with S is injective.*

Proof. Let μ be regular measure and $f \in L^{\infty}(X)$ be a nonzero function. For a given positive ε define $A = \{x \in X : |f(x)| \leq \|f\|_{\infty} - \varepsilon\}$. Since $\mu(A) > 0$,

$$\begin{aligned}\{x \in X : \|U_S P(x)\| \geq \|f\|_{\infty} - \varepsilon\} &= \{x : \|f(S(x))| > \|f\|_{\infty} - \varepsilon\} \\ &= \|y \in S^{-1}(A) : \|f(y)| > \|f\|_{\infty} - \varepsilon\} \\ &\subseteq S^{-1}(A)\end{aligned}$$

and μ is regular, we see that

$$\mu\{x \in X : |U_S P(x)| > \|f\|_{\infty} - \varepsilon\} > 0.$$

Hence $\|U_S f\| \geq |f\|_\infty - \varepsilon$ for any $\varepsilon > 0$.

Now Proposition 2 implies U_S is an isometric, that is U_S is injective. Let $U)_S$ be injective and $\mu(A) > 0$. Then $\mathcal{X}_A \in L^\infty(X)$ is a nonzero function. Since U_S is injective $\mathcal{X}_{S^{-1}(A)} = U_S \mathcal{A}_A$ is a nonzero function. Hence $\mu(S^{-1}(A)) > 0$.

Definition 2. 12. A transformation $S : X \to X$ is called measure-preserving in the space (X, A, μ) if and only if for any $A \in \mathbf{A}$,$\mu(A) = \mu(S^{-1}(A))$.,

The concepts of ergodicity with the aid of two types of transformation (namely mixing and exactness), classify various degrees of irregular behaviors in terms of the behaviors of sequences of sets. These three types of transformations exhibit strong chaotic properties. It should be noted that, this is not a complete list of possible behaviors, however, these are probably the most important types of transformations, see [4].

Definition 2. 13. Let (X, A, μ) be a normalized measure space and $S : X \to X$ a measure preserving transformation:
a) S is called mixing if for all $A, B \in \mathbf{A}$, $\lim_{n\to\infty} \mu(A \cap \mu^{-n}(B)) = \mu(A)\mu(B)$.
b) S is called exact if $S(A) \in \mathbf{A}$ for each $A \in \mathbf{A}$ and for any $A \in \mathbf{A}$ with $\mu(A) > 0$.

It is clear that if S is measure preserving then μ is regular, and hence U_S is injective. By these observations the proof of the following proposition is clear.

Proposition 2. 14. *Let (X, A, μ) be a normalized measure space, $S : X \to X$ be a nonsingular transformation and the FrobeniusPerron operator P_S has a bounded inverse. Then if S is mixing or exact then $\sigma(P_S) \subset \partial D$.*

3 Ps Without a Bounded Inverse

In the contrary to the previous section, in this section we assume that the FrobeniusPerron operator does not have a bounded inverse. In this case we will show that under similar assumptions, the spectrum of FrobeniusPerron operator equals the closed unit disk in the complex place.

Note that the conclusion of Lemma **??** is applicable in this section, namely under general setting the regularity of the measure μ implies that if $|\lambda| > 1$, then $\lambda \notin \partial(P_S)$.

Theorem 3. 1. *Let (X, A, μ) be a $\sigma-$finite measure space and $S : X \to X$ a nonsingular transformation. Suppose P_S does not have a bounded inverse. If U_S is injective, then $\sigma(P_S) = D$*

Proof. Suppose that $|\lambda| < 1$ is such that $\lambda \notin \sigma(P_S)$. Since P_S does not have a bounded inverse, namely $0 \in \sigma(P_S)$, it follows that there exist $\lambda_1 \in \partial\sigma(P_S)$ such that $|\lambda_1| < 1$which contradicts Lemma 2. Thus $\sigma(P_S) \subset D$. Since $\sigma(P_S)$ is closed in the complex plane, we see that $\sigma(P_S) = D$.

Theorem 3. 2. *Let (X, A, μ) be a $\sigma-$finite measure space, $S : X \to X$ be a nonsingular transformation and P_S the FrobeniusPerron operator associated with S. Suppose μ is normal and there is a unique positive stationary density f of P_S. if $0 \in \sigma(P_S)$ then $\sigma(P_DS) = D$.*

Proof. Similar to Theorem 7 we show that μ is regular. The conclusion then follows from Lemma 1 and Theorem 1.

Now it is not hard to prove the following propositions.

Proposition 3. 3. *Let $S : X \to X$ be ergodic in the $\sigma-$finite measure space (X, A, μ), where μ is normal. If there exists a unique positive stationary density and $0 \in \sigma(P_S)$, then $\sigma(P_S) = D$.*

Proposition 3. 4. *Let (X, A, μ) be a normalized measure space, $S : X \to X$ a nonsingular transformation, and $0 \in \sigma(P_S)$. Then if S is mixing , then $\sigma(P_S) = D$. Also if S is exact, then $\sigma(P_S) = D$.*

References

1. F. Chatelin, Spectral Approximation of Linear Operators, Academic Press, New York, 1983.
2. S.R. Foruel, The Ergodic Theory of Markov Processes, Van Nostrand Reinhold, New York, 1969.
3. E. Hopf, The General Temporally Discrete Markov Processes, Jour. Rational Mech.Anal., 3, 13-45, 1959.
4. A. Lasota and M. Mackey, Probablistic Properties of Deterministic Systems, Cambridge Univ. Press, 1985.
5. T.Y. Li, Finite Approximation for the Frobenius-Perron Operator, A Solution to Ulam's Conjecture, Jour.of Appr. Theory, 17, 177-186, 1976.
6. T.Y. Li and J.A. Yorke, Ergodic Transformations from an Interval into Itself, Trans. Amer. Math. Soc., 235, 183-192, 1978.
7. M.Mohaeni Moghadam and R.Sabeti, On the Approximation of Unbounded Invariant Densities, Jour. of Non Linear Analysis, Vol.31, No.1/2. PP.137-147, 1998.
8. M.Mohaeni Moghadam and M. Panahi, Approximation Invariant Measure for Piecewise - Trans. Of n-Dimensional Cube, Italian Jour. of Pure and Applied Math.

9. M.Mohaeni Moghadam , An Approximation of the Frobenius-Perron Operators, Proceeding of the First Seminar on Applied Math., Univ. of Zanjan and Bakusate Univ., PP.56-66,1995.

10. M.Mohaeni Moghadam, Aspect Concerning Frobenius-Perron Operators, Inter. Jour. of Applied Mathematics,Vol.8, No.4, PP.417-423, 2002.