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A Review On Hyper K-algebras

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Abstract. In this note we review briefly some topics on hyper K-algebras.

1 Introduction And Preliminaries

The notion of BCK-algebra was formulated first in 1966 by K.Iseki, Japanese Mathematician. This notion is originated from two different ways.

One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations introduced by L. Kantorovic and E. Livenson to make a new set from old sets. These fundamental operations are union, intersection and the set difference. Then, as a generalization of those three operations and properties, we have the notion of Boolean algebra. If we take both of the union and the intersection, then as a general algebra, the notion of distributive lattice is obtained. Moreover, if we consider the notion of union or intersection, we have the notion of an upper semilattice or a lower semilattice. But the notion of set difference was not considered systematically before K. Iseki.

Another Motivation is taken from classical and non-classical propositional calcului. There are some systems which contain the only implication functor among the logical functors. These examples are the systems of positive implicational calculus, weak positive implicational calculus by A.Church, and BCI, BCK-systems by C.AS.Meredith.

We know the following simple relations in set theory:

$$(A - B) - (A - C) \subset C - B$$

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$$A - (A - B) \subset B$$

In propositional calcului, these relations are denoted by

$$(p \longrightarrow q) \longrightarrow ((q \longrightarrow r) \longrightarrow (p \longrightarrow r))$$

$$p \longrightarrow ((p \longrightarrow q) \longrightarrow q)$$

From these relationships, K.Iseki introduced a new notion called a BCK-algebra.

Definition 1. 1. [33]. Let X be a set with a binary operation " $*$ " and a constant "0". Then $(X, *, 0)$ is called a BCK-algebra if it satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$.
for all $x, y, z \in X$.

For brevity we also call X a BCK-algebra. In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. Then $(X, *, 0)$ is a BCK-algebra if and only if it satisfies that:

For all $x, y \in X$;
(i) $(x * y) * (x * z) \leq z * y$,
(ii) $x * (x * y) \leq y$,
(iii) $x \leq x$,
(iv) $0 \leq x$,
(v) $x \leq y$ and $y \leq x$ imply $x = y$.

Theorem 1. 2. [33]. *In a BCK-algebra $(X, *, 0)$, we have the following properties:*

For all $x, y, z \in H$;
(i) $x \leq y$ implies $z * y \leq z * x$,
(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$,
(iii) $(x * y) * z = (x * z) * y$,
(iv) $x * y \leq z$ implies $x * z \leq y$,
(v) $(x * z) * (y * z) \leq x * y$,
(vi) $x \leq y$ implies $x * z \leq y * z$,
(vii) $x * y \leq x$,
(vii) $x * 0 = x$.

Definition 1. 3. [33]. If there is an element 1 of a BCK-algebra X satisfying $x \leq 1$ for all $x \in X$, then the element 1 is called unit of X. A BCK-algebra with unit is called to be bounded.

Theorem 1.4. [33]. *Let $(X,*,0)$ be a BCK-algebra and $1 \notin X$. We define the operation " $*'$ " on $\overline{X} = X \cup \{1\}$ as follows*

$$x *' y = \begin{cases} x * y & \text{if } x, y \in X \\ \{0\} & \text{if } x \in X \text{ and } y = 1 \\ 1 & \text{if } x = 1 \text{ and } y \in X \\ 0 & \text{if } x = y = 1, \end{cases}$$

Then $(\overline{X}, *', 0)$ is a bounded BCK-algebra with unit 1.

Definition 1.5. [33]. Let $(X_1, *_1, 0)$ and $(X_2, *_2, 0)$ be two BCK-algebras and $X_1 \cap X_2 = \{0\}$. Suppose $X = X_1 \cup X_2$, define $*$ on X as follows:

$$x * y = \begin{cases} x *_1 y & \text{if } x \text{ and } y \text{ belong to } X_1 \\ x *_2 y & \text{if } x \text{ and } y \text{ belong to } X_2 \\ x & \text{if } x \text{ and } y \text{ do not belong to the same algebra} \end{cases}$$

Next we will verify that $(X,*,0)$ is a BCK-algebra, this algebra is called to be the *union* of $(X_1, *_1, 0)$ and $(X_2, *_2, 0)$, written $X_1 \bigoplus X_2$.

Theorem 1.6. [33]. *Let $(X_i, *_i, 0_i), (i \in I)$ be an indexed family of BCK-algebras and let $\prod_{i \in I} X_i$ be the set of all mappings $f : I \longrightarrow \cup_{i \in I} X_i$, where $f(i) \in X_i$ for all $i \in I$. For $f, g \in \prod_{i \in I} X_i$, we define $f * g$ by $(f * g)(i) = f(i) * g(i)$ for every $i \in I$, and by 0 we mean $0(i) = 0_i, \forall i \in I$. Then $(\prod_{i \in I} X_i, *, 0)$ is a BCK-algebra, which is called the direct product of $X_i (i \in I)$.*

Definition 1.7. [33]. Let $(X,*,0)$ be a BCK-algebra and S be a non-empty subset of X. Then S is called to be a *subalgebra* of X if, for any $x, y \in S, x*y \in S$, i.e., S is closed under the binary operation $*$ of X.

Definition 1.8. [33]. A non-empty subset I of a BCK-algebra X is called an *ideal* of X if for all $x, y \in X$:
(i) $0 \in I$
(ii) $x * y \in I$ and $y \in I$ imply that $x \in I$.

Definition 1.9. [33]. Give a BCK-algebra $(X,*,0)$, a non-empty subset I of X is said to be a *positive implicative ideal* if it satisfies, for all x, y, z in X,
(i) $0 \in I$,
(ii) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$.

Definition 1.10. [33]. A BCK-algebra $(X,*,0)$ is called to be positive implicative if it satisfies for all x, y and z in X,

$$(x * z) * (y * z) = (x * y) * z.$$

Definition 1.11. [33]. Give a BCK-algebra $(X, *, 0)$ and given elements a, b of X, we define:

$$A(a, b) = \{x \in X : x * a \leq b\}$$

Obviously,0, a and b are in $A(a, b)$. If for all $x, y \in X$, $A(x, y)$ has a greatest element, written $x + y$, then the BCK-algebra is called to be with *condition* (S).

Theorem 1.12. [33]. *Any positive implicative ideal must be an ideal, but the inverse is not true.*

Definition 1.13. [33]. A non-empty subset I of a BCK-algebra X is said to be a *varlet ideal* if , for all $x, y \in X$,

(i) $x \in I$ and $y \leq x$ imply $y \in I$,

(ii) $x \in I$ and $y \in I$ imply that there exists $z \in I$ such that $x \leq z$ and $y \leq z$.

Definition 1.14. [33]. Let X be a BCK-algebra with condition (S) and I be a non-empty subset of X. Then I is called to be an *additive ideal* if , for all $x, y \in X$,

(i) $x \in I$ and $y \leq x$ imply $y \in I$,

(ii) $x \in I$ and $y \in I$ imply $x + y \in I$.

Definition 1.15. [41] A non-empty subset I of a BCK-algebra X is called a *commutative ideal* of X if for all $x, y \in X$:

(i) $0 \in I$,

(ii) $(x * y) * z \in I$ and $z \in I$ imply that $x * (y * (y * x)) \in I$.

Theorem 1.16. *Any positive implicative ideal (commutative ideal) must be an ideal, but the inverse is not true.*

The theory of hypercompositional Structure has been introduced by F. Marty in 1934 during the 8^{th} congress of Scandinavian Mathematiciens, where he presented his work [32]. Several references have also been made to H.S.Wall who has presented his paper [41] in 1937. Unfortunately Marty himself did not present more than 3 or 4 papers, because he died, very young during the world war II. So F.Marty introduced the notion of the hypergroup. Today the research in the field of hypercompositional structures is very vivid. In many universities in the world are working teams on this theory.

Definition 1.17. Let H be a non-empty set and "$\circ$" be a function from $H \times H$ to $P^*(H) = P(H) - \{\emptyset\}$. Then "$\circ$" is called a hyperoperation on H.

Definition 1.18. For two non-empty subsets A and B of H, denote by $A \circ B$ the set

$$\bigcup_{a \in A, b \in B} a \circ b.$$

Notation 1.19. *Let "$\circ$" be a hyperopration on H and $a \in H, A \in P^*(H)$. Then by $a \circ A$ and $A \circ a$ we mean $\{a\} \circ A$ and $A \circ \{a\}$ respectively.*

2 On Hyper K-algebras

Definition 2.1. By a *hyper$K$-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y$ and $y < x$ imply $x = y$,
(HK5) $0 < x$ for all $x \in H$

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$ and $\exists b \in B$ such that $a < b$.

Example 2.2. (1) Let $(X, *, 0)$ be a BCK-algebra and define a hyperoperation "$\circ$" on X by $x \circ y = \{x * y\}$ for all $x, y \in X$. Then $(X, \circ, 0)$ is a hyper$K$-algebra.

(2) Let $n \in \mathcal{N} \cup \{0\}$. Define a hyperoperation "$\circ$" on $H_n = [n, \infty)$ by

$$x \circ y := \begin{cases} [n, x] & \text{if } \; x \leq y \\ (n, y] & \text{if } \; x > y \neq n \\ \{x\} & \text{if } \; y = n \end{cases}$$

for all $x, y \in H$. Then $(H_n, \circ, n)$ is a hyper$K$-algebra.

(3) Let $H = \{0, x, y\}$. Consider the following table:

$\circ$	0	x	y
0	$\{0\}$	$\{0, x, y\}$	$\{0, x, y\}$
x	$\{x\}$	$\{0, x, y\}$	$\{0, x, y\}$
y	$\{y\}$	$\{x, y\}$	$\{0, x, y\}$

Then $(H, \circ, 0)$ is a hyper$K$-algebra which is not a hyper BCK-algebra, since $x \circ y \not\ll \{x\}$.

(4) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
2	$\{2\}$	$\{2\}$	$\{0, 1, 2\}$

Then $(H, \circ, 0)$ is a hyper$K$-algebra.

Proposition 2. 3. *Let $(H, \circ, 0)$ be a hyperK-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H the following hold:*

(i) $(A \circ B) \circ C = (A \circ C) \circ B$,
(ii) $x \circ (x \circ y) < y$,
(iii) $x \circ y < z \Leftrightarrow x \circ z < y$,
(iv) $A \circ B < C \Leftrightarrow A \circ C < B$,
(v) $(x \circ z) \circ (x \circ y) < y \circ z$,
(vi) $(A \circ C) \circ (B \circ C) < A \circ B$,
(vii) $(A \circ C) \circ (A \circ B) < B \circ C$,
(viii) $A \circ (A \circ B) < B$,
(ix) $A < A$,
(x) $A \subseteq B$ *implies* $A < B$.
(xi) $x \circ y < x$,
(xii) $A \circ B < A$,
(xiii) $A \circ A < A$,
(xiv) $0 \in x \circ (x \circ 0)$,
(xv) $x \in x \circ 0$,
(xvi) $0 \in x \circ y \Rightarrow 0 \in (x \circ y) \circ 0$

Theorem 2. 4. *Let $(H_1, \circ_1, 0)$ and $(H_2, \circ_2, 0)$ be hyperK-algebras such that $H_1 \cap H_2 = \{0\}$ and $H = H_1 \cup H_2$. Then $(H, \circ, 0)$ is a hyperK-algebra , where the hyperoperation "$\circ$" on H is defined as follows:*

$$x \circ y := \begin{cases} x \circ_1 y \text{ if } x, y \in H_1, \\ x \circ_2 y \text{ if } x, y \in H_2, \\ \{x\} \quad \text{otherwise}, \end{cases}$$

for all $x, y \in H$.

Notation 2. 5. *We use the notation $H_1 \oplus H_2$ for the union of two hyperK-algebras H_1 and H_2.*

Theorem 2. 6. *Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be hyperK-algebras and $H = H_1 \times H_2$. We define a hyperopration "$\circ$" on H as follows,*

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 \circ_1 a_2, b_1 \circ_2 b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in H$, where for $A \subseteq H_1$ and $B \subseteq H_2$ by (A, B) we mean

$$(A, B) = \{(a, b) : a \in A, b \in B\}, \quad 0 = (0_1, 0_2)$$

and

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow a_1 < a_2, b_1 < b_2$$

Then $(H, \circ, 0)$ is a hyperK-algebra , and it is called the hyperK-product of H_1 and H_2.

Definition 2. 7. Let I be a non-empty subset of a hyper$K$-algebra $(H, \circ, 0)$. Then I is called a *weak hyperK-ideal* of H if

(H1) $0 \in I$,

(WHK) $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$.

Definition 2. 8. Let I be a non-empty subset of a hyper$K$-algebra $(H, \circ, 0)$. Then I is said to be a *hyperK-ideal* of H if

(H1) $0 \in I$,

(HK) $x \circ y < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Proposition 2. 9. *Let $(H, \circ, 0)$ be a hyperK-algebra and let I be a hyperK-ideal of H. Then I is a weak hyperK-ideal of H.*

Note that the converse of proposition 2.8 may not be true. To see this, consider the following example:

Example 2. 10. Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

Then $(H, \circ, 0)$ is a hyper K-algebra.

Now it is easy to see that $I := \{0, 2\}$ is a weak hyper$K$-ideal which is not a hyper$K$-ideal.

Lemma 2. 11. *Let H be a hyperK-algebra and I a hyperK-ideal of H. If $x < y$ and $y \in I$, then $x \in I$.*

Proof. Since $x < y$, we have $0 \in x \circ y$. It follows from $0 \in I$ and (HK3) that $x \circ y < I$ so from (HK) that $x \in I$.

Theorem 2. 12. *Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be hyperK-algebras and consider the hyperK-algebra $(H_1 \times H_2, \circ, (0_1 \circ 0_2))$. Then*

(i) If I_1 and I_2 are hyperK-ideals of H_1 and H_2 respectively, then $I_1 \times I_2$ is a hyperK-ideal of $H_1 \times H_2$.

(ii) If I is a hyperK-ideal of $H_1 \times H_2$, then there are unique hyperK-ideals I_1 and I_2 of H_1 and H_2 respectively such that $I = I_1 \times I_2$.

Definition 2. 13. Let $(H, \circ, 0)$ be a hyper$K$-algebra and let S be a subset of H containing 0. If S is a hyper$K$-algebra with respect to the hyperoperation "$\circ$" on H, we say that S is a *hyperK-subalgebra* of H.

Theorem 2. 14. *Let S be a non-empty subset of a hyperK-algebra $(H, \circ, 0)$. Then S is a hyperK-subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Assume that $x \circ y \subseteq S$ for all $x, y \in S$ and let $a \in S$. Since $a \circ a < \{a\}$, we have $0 \in (a \circ a) \circ a = \bigcup_{x \in a \circ a} x \circ a \subseteq S$. Now for any $x, y, z \in S$, we have $x \circ z \subseteq S$, $y \circ z \subseteq S$ and $x \circ y \subseteq S$. Hence

$$(x \circ z) \circ (y \circ z) = \bigcup_{\substack{a \in x \circ z \\ b \in y \circ z}} a \circ b \subseteq S$$

and so (HK1) holds in S. Similarly we can prove that the axioms (HK2), (HK3), (HK4) and (HK5) are hold in S. Therefore S is a hyper$K$-subalgebra of H.

Example 2. 15. (1) Let $(X, \circ, 0)$ be the hyper$K$-algebra in Example 3.1.2(1) and let S be a subalgebra of a BCK-algebra $(X, *, 0)$. Then S is a hyper$K$-subalgebra of $(X, \circ, 0)$.

(2) Let $(H_n, \circ, n)$ be the hyper$K$-algebra in Example 2.2(2) and let $S = [n, a]$ for every $a \in [n, \infty)$. Then S is a hyper$K$-subalgebra of $(H_n, \circ, n)$, which is not a hyper BCK-ideal.

(3) Let $(H, \circ, 0)$ be the hyper$K$-algebra in Example 2.9 and let $S = \{0, 1\}$. Then S is a hyper$K$-subalgebra and a weak hyper$K$-ideal of H, but S is not a hyper$K$-ideal of H.

(4) Let H_1 be the hyper$K$-algebra in Example 2.2(3) and let $H = H_1 \times H_1$. Assume $S = H_1 \times \{0\}$. Then S is a nontrivial hyper$K$-subalgebra of H.

Definition 2. 16. Let $(H, \circ, 0)$ be a hyper$K$-algebra. If there exists an element $e \in H$ such that $x < e$ for all $x \in H$, then H is called a *bounded hyperK-algebra* and e is said to be the *unit* of H.

Note that (HK4) implies that the unit of H is unique.

Example 2. 17. (i) Let $(X, *, 0)$ be a bounded BCK-algebra. Define the hyper operation "$\circ$" on X as follows:

$$x \circ y = \{x * y\}, \quad \forall x, y \in X.$$

Then $(X, \circ, 0)$ is a bounded hyper$K$-algebra.

(ii) The hyper$K$-algebra $(H_n, \circ, n)$ in Example 2.2(2) is not bounded, because if $a \in H_n$ is unit, then $(a + 1) \circ a = (n, a]$. Thus $n \notin (a + 1) \circ a$, i.e., $a + 1 \not< a$. (iii) In Example 2.9, H is bounded and $2 \in H$ is unit.

(iv) The hyper$K$-algebra $(H, \circ, 0)$ in Example 2.2(4) is bounded and $2 \in H$ is unit.

Proposition 2. 18. *Let H_1 and H_2 be two bounded hyperK-algebras. Then the hyperK-product $H_1 \times H_2$ of H_1 and H_2 is also bounded.*

Definition 2. 19. Let H be a hyper$K$-algebra. If $0 \circ x = \{0\}$ for all $x \in H$, then we say that H satisfies the *zero condition.*

Example 2. 20. Let H be a hyper$K$-algebra as in Example 3.1.2(2). Then H satisfies the zero condition.

Theorem 2. 21. *Let $(H_1, \circ_1, 0)$ be a hyperK-algebra, which satisfies the zero condition. Then $(H_1, \circ_1, 0)$ can be extended to a bounded hyperK-algebra.*

Proof. Let $e \notin H_1$ and $H = H_1 \cup \{e\}$. Define the hyper operation "$\circ$" on H as follows:

$$x \circ y = \begin{cases} \{e\} & \text{if } x = e,\ y \in H_1 \\ \{0\} & \text{if } x = e,\ y = e \\ \{0, x\} & \text{if } x \in H_1,\ y = e \\ x \circ_1 y & \text{if } x, y \in H_1, \end{cases}$$

for all $x, y \in H$. Now we can show that $(H, \circ, 0)$ is a bounded hyper$K$-algebra and e is its unit.

Definition 2. 22. (i) A non-empty subset I of H is called s-reflexive, if for any $x, y \in H$ which $(x \circ y) \bigcap I \neq \emptyset$, implies $x \circ y \subseteq I$.

(ii) A non empty subset I of H is called reflexive, if for any $x \in H$, $x \circ x \subseteq I$.

(iii) If I is hyper K-ideal of H and it is s-reflexive (reflexive), then I is called s-reflexive (reflexive) hyper K-ideal of H.

Example 2. 23. Let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper K-algebra structure on H

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{2\}$	$\{1\}$	$\{0, 1\}$

Then $I = \{0, 1\}$ is s-reflexive hyper K-ideal of H.

3 Homomorphism of Hyper$K$-algebras

Definition 3. 1. Let H_1 and H_2 be two hyper$K$-algebras. A mapping $f : H_1 \to H_2$ is said to be a *homomorphism* if

(i) $f(0) = 0$

(ii) $f(x \circ y) = f(x) \circ f(y), \quad \forall x, y \in H_1$.

If f is 1-1 (or onto) we say that f is a *monomorphism* (or *epimorphism*). And if f is both 1-1 and onto, we say that f is an *isomorphism.*

Example 3. 2. Let H_0 be as in Example 2.2(2) and $t \in \mathbb{R}^+$ be constant. Define

$$f : H_0 \to H_0, \quad f(x) = tx, \quad \forall x \in H_0.$$

Then f is an isomorphism of hyper$K$-algebras. To do this, let $x, y \in H_0$ and $x \leq y$. Then $tx \leq ty$ and thus $f(x \circ y) = f([0, x]) = [0, tx] = tx \circ ty = f(x) \circ f(y)$. If $x > y \neq 0$, then $tx > ty$ and so

$$f(x \circ y) = f((0, y]) = (0, ty] = tx \circ ty = f(x) \circ f(y).$$

If $y = 0$, then

$$f(x \circ 0) = f(\{x\}) = tx = tx \circ t0 = f(x) \circ f(0).$$

Also $f(0) = 0$, consequently f is a homomorphism. Clearly f is onto and 1-1. Thus f is an isomorphism.

Theorem 3. 3. *Let $f : H_1 \to H_2$ be a homomorphism of hyper$K$-algebras. Then*

(i) If S is a hyper$K$-subalgebra of H_1, then $f(S)$ is a hyper$K$-subalgebra of H_2,

(ii) $f(H_1)$ is a hyper$K$-subalgebra of H_2,

(iii) If H_1 satisfies the zero condition, then so is $f(H_1)$,

(iv) If S is a hyper$K$-subalgebra of H_2, then $f^{-1}(S)$ is a hyper$K$-subalgebra of H_1,

(v) If I is a (weak) hyper$K$-ideal of H_2, then $f^{-1}(I)$ is a (weak) hyper$K$-ideal of H_1,

(vi) $\mathrm{Ker}f := \{x \in H_1 \mid f(x) = 0\}$ *is a hyper$K$-ideal and hence a weak hyper$K$-ideal of H_1,*

(vii) If f is onto and I is a hyper$K$-ideal of H_1 which contains $\mathrm{Ker}f$, *then $f(I)$ is a hyper$K$-ideal of H_2.*

Theorem 3. 4. *Let $f : H_1 \to H_2$ be an epimorphism of hyper$K$-algebras. Then there is a one to one correspondence between the set of all hyper$K$-ideals of H_1 containing* $\mathrm{Ker}f$ *and the set of all hyper$K$-ideals of H_2.*

Lemma 3. 5. *Let $f : H_1 \to H_2$ be a homomorphism of hyper$K$-algebras. If $x < y$ in H_1, then $f(x) < f(y)$ in H_2.*

Theorem 3. 6. *Let $f : H_1 \to H_2$ be an epimorphism of hyper$K$-algebras. If H_1 is bounded, then H_2 is also bounded.*

Proof. Let e be the unit of H_1 and $y \in H_2$ be an arbitrary element. Then there exists $x \in H_1$ such that $f(x) = y$. Since $x < e$, by Lemma 3.2.5, we have $y = f(x) < f(e)$. Thus $f(e)$ is the unit of H_2 and H_2 is bounded.

Theorem 3. 7. *Let $f : H_1 \to H_2$ and $g : H_1 \to H_3$ be two homomorphisms of hyperK-algebras such that f is onto and* $\mathrm{Ker} f \subseteq \mathrm{Ker} g$. *Then there exists a homomorphism $h : H_2 \to H_3$ such that $h \circ f = g$.*

Proof. Let $y \in H_2$ be arbitrary. Since f is onto, there exists $x \in H_1$ such that $y = f(x)$. Define $h : H_2 \to H_3$ by $h(y) = g(x)$, $\forall y \in H_2$. Now we show that h is well-defined. Let $y_1, y_2 \in H_2$ and $y_1 = y_2$. Since f is onto, there are $x_1, x_2 \in H_1$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Therefore $f(x_1) = f(x_2)$ and thus $0 \in f(x_1) \circ f(x_2) = f(x_1 \circ x_2)$. It follows that there exists $t \in x_1 \circ x_2$ such that $f(t) = 0$. Thus $t \in \mathrm{Ker} f \subseteq \mathrm{Ker} g$ and so $g(t) = 0$. Since $t \in x_1 \circ x_2$ we conclude that

$$0 = g(t) \in g(x_1 \circ x_2) = g(x_1) \circ g(x_2)$$

which implies that $g(x_1) < g(x_2)$. On the other hand since $0 \in f(x_2) \circ f(x_1) = f(x_2 \circ x_1)$, similarly we can conclude that $0 \in g(x_2) \circ g(x_1)$, i.e., $g(x_2) < g(x_1)$. Thus $g(x_1) = g(x_2)$, which shows that h is well-defined. Clearly $h \circ f = g$. Finally we show that h is a homomorphism. Let $y_1, y_2 \in H_2$ be arbitrary. Since f is onto there are $x_1, x_2 \in H_1$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then

$$\begin{aligned} h(y_1 \circ y_2) &= h(f(x_1) \circ f(x_2)) = h(f(x_1 \circ x_2)) = (h \circ f)(x_1 \circ x_2) \\ &= g(x_1 \circ x_2) = g(x_1) \circ g(x_2) = (h \circ f)(x_1) \circ (h \circ f)(x_2) \\ &= h(f(x_1)) \circ h(f(x_2)) = h(y_1) \circ h(y_2). \end{aligned}$$

Moreover since $f(0) = 0$ and $g(0) = 0$, we conclude that

$$h(0) = h(f(0)) = (h \circ f)(0) = g(0) = 0.$$

Thus h is a homomorphism, ending the proof.

Theorem 3. 8. *Let $f : H_1 \to H_2$ be a monomorphism of hyperK-algebras. If H_2 is bounded with unit element e and $e \in \mathrm{Im} f$, then H_1 is also bounded and $f^{-1}(e)$ is its unit.*

4 Positive Implicative Hyper K-ideals

Definition 4. 1. Let I be a nonempty subset of H such that $0 \in I$. Then I is said to be a *positive implicative hyperK-ideal* of
(i) type 1, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ implies that $x \circ z \subseteq I$,
(ii) type 2, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ implies that $x \circ z \subseteq I$,

(iii) type 3, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ implies that $x \circ z \subseteq I$,
(iv) type 4, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ implies that $x \circ z < I$.

Example 4. 2. (i) Let H be the hyper$K$-algebra of Example 3.1.2(1). If I is a positive implicative ideal of BCK-algebra $(H, *, 0)$, then I is a positive implicative hyper$K$-ideal of type 1,2,3 and 4 of hyper$K$-algebra H.

(ii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

Clearly $I_2 = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 1. But $I_1 = \{0, 1\}$ is not, since $(2 \circ 1) \circ 0 = \{0, 1\} \subseteq I_1$, $1 \circ 0 = \{1\} \subseteq I_1$ and $2 \circ 0 = \{2\} \not\subseteq I_1$

(iii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

It can be checked that $I_1 = \{0, 1\}$ is a positive implicative hyper$K$-ideal of type 2. But $I_2 = \{0, 2\}$ is not, since $(1 \circ 2) \circ 0 < I_2, 2 \circ 0 \subseteq I_2$ and $1 \circ 0 \not\subseteq I_2$.

(iv) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

Now we can check that $I_2 = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 3. But $I_1 = \{0, 1\}$ is not, since $(2 \circ 1) \circ 0 < I_1, 1 \circ 0 < I_1$ and $2 \circ 0 \not\subseteq I_1$.

(v) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

It is easy to see that $I_2 = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 4. But $I_1 = \{0, 1\}$ is not, since $(2 \circ 1) \circ 0 = \{1, 2\} < I_1$, $1 \circ 0 = \{1\} < I_1$ and $2 \circ 0 = \{2\} \not< I_1$.

Theorem 4. 3. *Let $0 \in H$ be a right scalar element. If I is a positive implicative hyper$K$-ideal of type 1, then I is a weak hyper$K$-ideal of H.*

Example 4. 4. (i) Let $H = \{0,1,2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

We see that the $0 \in H$ is not a right scalar element and $I = \{0,2\}$ is a positive implicative hyper$K$-ideal of type 1. But I is not a weak hyper$K$-ideal, since $1 \circ 2 \subseteq I, 2 \in I$ and $1 \notin I$.

(ii) Let $H = \{0,1,2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

Clearly that $0 \in H$ is a right scalar element. Moreover $I = \{0,2\}$ is a weak hyper$K$-ideal of H, but it is not a positive implicative hyper$K$-ideal of type 1.

Definition 4. 5. H is called to be a positive implicative hyper$K$-algebra, if it satisfies the following condition,

$$(x \circ z) \circ (y \circ z) = (x \circ y) \circ z$$

for all $x, y, z \in H$.

Example 4. 6. (i) Let $H = \{0,1,2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

Then $(H, \circ, 0)$ is a positive implicative hyper$K$-algebra.

(ii) Consider Example 4.4(i). Since,

$$(2 \circ 0) \circ (1 \circ 0) = \{0,1,2\} \neq \{1,2\} = (2 \circ 1) \circ 0$$

then H is not a positive implicative hyper$K$-algebra.

Theorem 4. 7. *Let H be a positive implicative hyper$K$-algebra. Then any weak hyper$K$-ideal of H is a positive implicative hyper$K$-ideal of type 1.*

Proof. Let I be a weak hyper$K$-ideal of H and let $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ for $x, y, z \in H$. Since $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z \subseteq I, y \circ z \subseteq I$ and I is a weak hyper$K$-ideal of H, then we get that $x \circ z \subseteq I$. Therefore I is a positive implicative hyper$K$-ideal of type 1.

Corollary 4. 8. *Let H be a positive implicative hyper$K$-algebra, such that $0 \in H$ is a right scalar element and I is a nonempty subset of H. Then I is a positive implicative hyper$K$-ideal of type 1 iff I is a weak hyper$K$- ideal of H.*

Theorem 4. 9. *Let I be a nonempty subset of H. Then the following statements hold:*

(i) If I is a positive implicative hyper$K$-ideal of type 2, then I is a positive implicative hyper$K$-ideal of type 1.

(ii) If I is a positive implicative hyper$K$-ideal of type 3, then I is a positive implicative hyper$K$-ideal of type 2 and 4.

Proof. (i) Let $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ for $x, y, z \in I$. Then $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$. So by hypothesis we get that $x \circ z \subseteq I$. Therefore I is a positive implicative hyper$K$-ideal of type 1.

(ii) The proof is similar to the proof of (i).

Example 4. 10. (i) Consider Example 4.4(i). Then $I = \{0, 1\}$ is a positive implicative hyper$K$-ideal of type 1. But it is not of type 2, since $(1 \circ 0) \circ 0 = \{1, 2\} < \{0, 1\} = I$, $0 \circ 0 = \{0, 1\} \subseteq I$ and $1 \circ 0 = \{1, 2\} \not\subseteq I$.

(ii) The following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0, 1\}$	$\{0, 1\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 1\}$	$\{1, 2\}$
2	$\{1, 2\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$

Then $I = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 2. But it is not a positive implicative hyper$K$-ideal of type 3, since $(2 \circ 2) \circ 0 = \{0, 1, 2\} < I$, $2 \circ 0 = \{1, 2\} < I$ and $2 \circ 0 = \{1, 2\} \not\subseteq I$.

(iii) Consider Example 4.2(v). Then $I = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 4, but it is not a positive implicative hyper$K$-ideal of type 3, since $(2 \circ 1) \circ 1 < I$, $1 \circ 1 < I$ and $2 \circ 1 \not\subseteq I$

Theorem 4. 11. *Let I be a nonempty subset of H and $0 \in H$ is a right scalar element. If I is a positive implicative hyper$K$-ideal of type 2 or 3, then I is a hyper$K$-ideal of H.*

Example 4. 12. (i) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0, 1, 2\}$

We see that $0 \in H$ is a right scalar element and $I = \{0, 2\}$ is a hyper$K$-ideal of H. But I is not a positive implicative hyper$K$-ideal of type 2, since $(2 \circ 0) \circ 2 = \{0, 1, 2\} < I$, $0 \circ 2 = \{0\} \subseteq I$ and $2 \circ 2 = \{0, 1, 2\} \nsubseteq I$.

(ii) The following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0, 1\}$	$\{0, 1\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 1\}$	$\{1, 2\}$
2	$\{1, 2\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$

Clearly $0 \in H$ is not a right scalar element and $I = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 2. But I is not a hyper$K$-ideal of H, since $1 \circ 2 = \{1, 2\} < I$, $2 \in I$ and $1 \notin I$.

(iii) Consider the following table which shows that a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 1, 2\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0, 1, 2\}$

Then $0 \in H$ is a right scalar element and $I = \{0, 1\}$ is a hyper$K$-ideal of H. But it is not a positive implicative hyper$K$-ideal of type 3.

(iv) Consider the following table which shows that a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0, 1\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 1\}$	$\{1, 2\}$
2	$\{1, 2\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$

Then $0 \in H$ is not a right scalar element and $I = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 2. But it is not a hyper$K$-ideal, since $1 \circ 2 < I$, $2 \in I$ and $1 \notin I$.

Definition 4. 13. Let I be a nonempty subset of H. Then we say that I satisfies the *additive condition*, if $x < y$ and $y \in I$ implies that $x \in I$, for all $x, y \in H$.

Example 4.14. Consider Example 4.2(iii). Then $I_1 = \{0, 1\}$ satisfies the additive condition. But $I_2 = \{0, 2\}$ does not satisfy the additive condition, since $1 < 2$, $2 \in I_2$ and $1 \notin I_2$.

Theorem 4.15. *Let I be a positive implicative hyper$K$-ideal of type 4 and satisfies the additive condition. Then I is a hyper$K$-ideal of H.*

Proof. Let $x \circ y < I$ and $y \in I$ for $x, y \in H$. By Theorem 2.3(xv), $(x \circ y) \circ 0 < I$ and $y \circ 0 < I$. Since I is a positive implicative hyper$K$-ideal of type 4, then $x \circ 0 < I$. Thus there is $b \in I$ such that $x \circ 0 < b$. By Theorem 3.1.3(iii), $x \circ b < 0$ and so there is $a \in x \circ b$ such that $a < 0$. By (HK5) and (HK4) we have $a = 0$. Therefore $0 \in x \circ b$ and hence $x < b$. Since I satisfies the additive condition and $b \in I$, we get that $x \in I$. So I is a hyper$K$-ideal of H.

Example 4.16. (i) Consider Example 4.2(v). Then $I = \{0\}$ is a hyper$K$-ideal of H and it satisfies the additive condition. But I is not a positive implicative hyper$K$-ideal of type 4, since $(2 \circ 1) \circ 1 < \{0\}, 1 \circ 1 < \{0\}$ and $2 \circ 1 \not< \{0\}$.

(ii) Consider Example 4.6(i). Then $I = \{0, 1\}$ is a positive implicative hyper$K$-ideal of type 4 and dose not satisfies the additive condition and it is not a hyper$K$-ideal of H, since $1 \circ 2 = \{0\} < I$, $2 \in I$ and $1 \notin I$. Therefore the additive condition in theorem 4.15 is a necessary.

Corollary 4.17. *If H be a positive implicative hyper$K$-algebra, which $0 \in H$ is a right scalar element. If I is a positive implicative hyper$K$-ideal of type 4 such that it satisfies the additive condition then, I is a positive implicative hyper$K$-ideal of type 1.*

Proof. By Theorem 4.15, I is a hyper$K$-ideal of H. By Theorem 2.8, I is a weak hyper$K$-ideal of H. Thus by Theorem 4.7, I is positive implicative hyper$K$-ideal of type 1.

Theorem 4.18. *Let $f : H_1 \to H_2$ be a homomorphism of hyper$K$-algebras. Then*

(i) If J is a positive implicative hyper$K$-ideal of type 1(resp 2,3,4) of H_2, then $f^{-1}(J)$ is also a positive implicative hyper$K$-ideal of type 1(resp 2,3,4) of H_1.

(ii) Let f be onto and $ker f \subseteq I$. Then

(a) If I is a positive implicative hyper$K$-ideal of type 1 of H_1 and I be a hyper$K$-ideal of H_1, then $f(I)$ is a positive implicative hyper$K$-ideal of type 1 of H_2.

(b) If $0 \in H_1$ is a right scalar element and I is a positive implicative hyper$K$-ideal of type 2(type 3) of H_1, then $f(I)$ is a positive implicative hyper$K$-ideal of type 2(type 3) of H_2.

(c) If I is a positive implicative hyper$K$-ideal of type 4 of H_1 and I satisfies the additive condition, then $f(I)$ is a positive implicative hyper$K$-ideal of type 4 of H_2.

Example 4.19. (i) Let $H_1 = \{0,1,2\}$ and $H_2 = H_3 = \{0,1,2,3\}$. Consider the following tables:

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

$\circ_2$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{0,1\}$	$\{3\}$	$\{0,1,3\}$

$\circ_3$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

Then $(H_1, \circ_1, 0)$, $(H_2, \circ_2, 0)$ and $(H_3, \circ_3, 0)$ are hyper$K$-algebras. Let $f_1 : H_1 \to H_2$ and $f_2 : H_1 \to H_3$ are defined as follows:

$$f_1(x) = \begin{cases} 1 & \text{if } x = 2 \\ 2 & \text{if } x = 1 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f_2(x) = x \quad , \quad \forall x \in H_1$$

Then f_1 and f_2 are homomorphism, but are not onto. Moreover, $I = \{0,1\}$ is a positive implicative hyper$K$-ideal of type 1,2,3,4 of H_1 and $kerf_1 = kerf_2 \subseteq I$. But $f_1(I)$ is not a positive implicative hyper$K$-ideal of type 1,2,3 and $f_2(I)$ is not a positive implicative hyper$K$-ideal of type 4.

Theorem 4.20. *Let I be a nonempty subset of H. Then*

(i) I is a positive implicative hyper$K$-ideal of type 1 if and only if, for all $a \in H, I_a = \{x \in H : x \circ a \subseteq I\}$ is a weak hyper$K$-ideal of H.

(ii) Let I be a positive implicative hyper$K$-ideal of type 2, then for all $a \in H, I_a = \{x \in H : x \circ a \subseteq I\}$ is a hyper$K$-ideal of H.

Proof. (i) Let $x, y, a \in H, x \circ y \subseteq I_a$ and $y \in I_a$. Thus $(x \circ y) \circ a \subseteq I$ and $y \circ a \subseteq I$. Since I is of type 1, then $x \circ a \subseteq I$ and so $x \in I_a$. Therefore I_a is a weak hyper$K$-ideal of H.

Conversely, let $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ for $x, y, z \in H$. Then $x \circ y \subseteq I_z$ and

$y \in I_z$. Since I_z is a weak hyper$K$-ideal of H, then $x \in I_z$ and so $x \circ z \subseteq I$. Thus I is a positive implicative hyper$K$-ideal of type 1.

(ii) Let $x, y, z \in H, x \circ y < I_a$ and $y \in I_a$. Then, there are $z \in x \circ y$ and $w \in I_a$ such that $0 \in z \circ w$. Since $w \circ a \subseteq I$, then

$$0 \in 0 \circ a \subseteq (z \circ w) \circ a \subseteq ((x \circ y) \circ w) \circ a$$

This implies that $((x \circ y) \circ w) \circ a < I$. Thus there is $d \in x \circ y$ such that $(d \circ w) \circ a < I$. Since $w \circ a \subseteq I$ and I is a positive implicative hyper$K$-ideal of type 2, then $d \circ a \subseteq I$. Thus $(x \circ y) \circ a < I$. Now since $y \circ a \subseteq I$ we get that $x \circ a \subseteq I$ and so $x \in I_a$. Therefore I_a is a hyper$K$-ideal of H.

Theorem 4. 21. *Let I be a nonempty subset of H. Then I is a positive implicative hyperK-ideal of type 4 if and only if, for all $a \in H$, $I_a^< = \{x \in H : x \circ a < I\}$ is a least hyperK-ideal of H containing $I \cup \{a\}$, that is $I_a^< = < I \cup \{a\} >$.*

Definition 4. 22. Let $a \in H$. We define the subset $(a]$ of H as follows:

$$(a] = \{x \in H \ : \ x < a\}$$

Note that it is clear that $\{0, a\} \subseteq (a]$.

Theorem 4. 23. *The following conditions on H are equivalent:*

(i) $\{0\}$ is a positive implicative hyperK-ideal of type 4,

(ii) $(a]$ is a hyperK-ideal of H, for all $a \in H$.

Proof. $(i) \Rightarrow (ii)$ Let $\{0\}$ be a positive implicative hyper$K$-ideal of type 4. Then by Theorem 3.3.22, for all $a \in H$, $\{0\}_a^<$ is a hyper$K$-ideal of H . But,

$$\{0\}_a^< = \{x \ : \ x \circ a < \{0\}\} = \{x \ : \ x < a\} = (a] \quad , \quad (1)$$

Therefore for all $a \in H, (a]$ is a hyper$K$-ideal of H.

$(ii) \Rightarrow (i)$ Let for all $a \in H, (a]$ is a hyper$K$-ideal of H. By (1), $\{0\}_a^< = (a]$. Then for all $a \in H, \{0\}_a^<$ is a hyper$K$-ideal of H contain $\{a\}$. So by the proof of $(\Leftarrow)$ in Theorem 4.21, $\{0\}$ is a positive implicative hyper$K$-ideal of type 4.

Theorem 4. 24. *Let A be a nonempty subset of H and let $x \in H$ be such that $(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n < \{0\}$, for some $a_1, a_2, \cdots, a_n \in A$. Then $x \in < A >$, i.e.,*

$$< A > \supseteq \{x \in H : (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n < \{0\} \text{ for some } a_1, a_2, \cdots, a_n \in A\}.$$

In particular, if $\{0\}$ is a positive implicative hyperK-ideal of type 4 and $a \in H$, then

$$< a >= \{x \in H : \underbrace{(\cdots((x \circ a) \circ a) \circ \cdots) \circ a}_{n\ times} < \{0\}\ for\ some\ \ n \in \mathcal{N}\}.$$

5 Commutative Hyper K-ideals And Quasi-commutative Hyper K-algebra

Definition 5. 1. Let I be a nonempty subset of a hyper K-algebra H such that $0 \in I$. Then I is called a commutative hyper K-ideal of
(i) *type 1*, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $z \in I$ imply that $x \circ (y \circ (y \circ x)) \subseteq I$,
(ii) *type 2*, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $z \in I$ imply that $x \circ (y \circ (y \circ x)) < I$,
(iii) *type 3*, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $z \in I$ imply that $x \circ (y \circ (y \circ x)) \subseteq I$,
(iv) *type 4*, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $z \in I$ imply that $x \circ (y \circ (y \circ x)) < I$.

Example 5. 2. (i) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

We can check that $I = \{0, 2\}$ is a commutative hyper K-ideal of any types of 1,2,3 and 4. Also $J = \{0, 1\}$ is a commutative hyper K-ideal of any types of 1,2 and 4 while it is not of type 3, since $(1 \circ 1) \circ 0 = \{0, 2\} < J$ and $0 \in J$ but $1 \circ (1 \circ (1 \circ 1)) = \{0, 2\} \not\subseteq J$.
(ii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

We see that $I = \{0\}$ is a commutative hyper K-ideal of type 1, but $J = \{0, 1\}$ is not, since $(2 \circ 1) \circ 0 = \{0, 1\} \subseteq J$ and $0 \in J$, while $2 \circ (1 \circ (1 \circ 2)) = \{2\} \not\subseteq J$.

(iii) Let $H = \{0, 1, 2\}$. Consider the following hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

Then $I = \{0, 1\}$ is a commutative hyper K-ideal of type 2(4), but $J = \{0\}$ is not, since $(2 \circ 1) \circ 0 = \{0\} \subseteq J$ and $0 \in J$, while $2 \circ (1 \circ (1 \circ 2)) = \{2\} \not< J$.

Theorem 5. 3. *Let I be a nonempty subset of H. Then the following statements hold:*
(i) if I is a commutative hyper K-ideal of type 1, then I is a commutative hyper K-ideal of type 2,
(ii) if I is a commutative hyper K-ideal of type 3, then I is a commutative hyper K-ideal of any types of 1,2 and 4,
(iii) if I is a commutative hyper K-ideal of type 4, then I is a commutative hyper K-ideal of type 2.

Example 5. 4. (i) In Example 5.2(ii), $I = \{0, 1\}$ is a commutative hyper K-ideal of type 2, while it is not of type 1.
(ii) In Example 5.2(i), $I = \{0, 1\}$ is a commutative hyper K-ideal of type 1(2,4), while it is not of type 3.
(iii) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

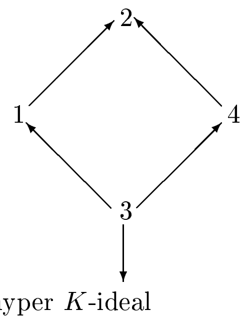
Then $I = \{0\}$ is a commutative hyper K-ideal of type 2, but it is not of type 4, since $(2 \circ 1) \circ 0 = \{0, 2\} < I$, while $2 \circ (1 \circ (1 \circ 2)) = \{2\} \not< I$.

Theorem 5. 5. *If I is a commutative hyper K-ideal of type 3, then I is a hyper K-ideal.*

Example 5. 6. The following table shows a hyper K-algebra structure on $\{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

Then $I = \{0,1\}$ is a hyper K-ideal, but it is not a commutative hyper K-ideal of type 3, since $(1 \circ 2) \circ 0 = \{1\} < I$, while $1 \circ (2 \circ (2 \circ 1)) = \{0,1,2\} \not\subseteq I$.
The following diagram shows the relationships between commutative hyper K-ideals of types 1,2,3 and 4 and hyper K-ideals.



Theorem 5. 7. *Let $0 \in H$ be a left scalar and I be closed. If I is a commutative hyper K-ideal of type 4, then I is a hyper K-ideal.*

The following example shows that each of the hypotheses of the above theorem is necessary.

Example 5. 8. (i) Let H be the hyper K-algebra of Example 5.2(i). Then $0 \in H$ is a left scalar. Also $I = \{0,1\}$ is a commutative hyper K-ideal of type 4, while it is not a hyper K-ideal and I is not closed.
(ii) Consider the following hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then $I = \{0,1\}$ is a closed commutative hyper K-ideal of type 4, while $0 \in H$ is not a left scalar and I is not a hyper K-ideal, because $2 \circ 1 < I$, $1 \in I$ and $2 \notin I$.
(iii) Let $H = \{0,1,2\}$. Then the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0,2\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

We can check that $I = \{0, 1\}$ is a commutative hyper K-ideal of type 4, while it is not a hyper K-ideal. Here $0 \in H$ is not a left scalar and I is not closed. The following example shows that the converse of Theorem 5.7 is not true in general.

Example 5. 9. The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

Then $I = \{0\}$ is a closed hyper K-ideal and $0 \in H$ is a left scalar, while it is not a commutative hyper K-ideal of type 4, because $(2 \circ 1) \circ 0 = \{0\} < I$, while $2 \circ (1 \circ (1 \circ 2)) = 2 \not< I$.

Theorem 5. 10. *Let $0 \in H$ be a scalar and I be closed. If I is a commutative hyper K-ideal of type 2, then I is a weak hyper K-ideal.*

The following example shows that each of the hypotheses of the above theorem is necessary.

Example 5. 11. (i) Let H be the hyper K-algebra of Example 5.2(ii). Then $I = \{0, 1\}$ is a commutative hyper K-ideal of type 2 and 0 is a scalar, while I is not a weak hyper K-ideal and is not closed too.
(ii) The following table shows a hyper K-algebra structure on $\{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

Then $I = \{0, 2\}$ is a closed commutative hyper K-ideal of type 2, while it is not a weak hyper K-ideal and $0 \in H$ is not a scalar.

Theorem 5. 12. *Let $0 \in H$ be a right scalar. If I is a commutative hyper K-ideal of type 1, then I is a weak hyper K-ideal.*

Example 5. 13. The following table shows a hyper K-algebra structure on $\{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then $0 \in H$ is a right scalar and $I = \{0, 1\}$ is a weak hyper K-ideal, while it is not a commutative hyper K-ideal of type 1, because $(1 \circ 0) \circ 0 = \{1\} \subseteq I$, $0 \in I$ and $1 \circ (0 \circ (0 \circ 1)) = \{0, 1, 2\} \not\subseteq I$.

Theorem 5. 14. *Let I be a weak hyper K-ideal. Then the following statements hold:*
(i) if $0 \in H$ is a right scalar and I is a commutative hyper K-ideal of type 1, then for all $x, y \in H$, $x \circ y \subseteq I$ implies that $x \circ (y \circ (y \circ x)) \subseteq I$,
(ii) if $x \circ y \subseteq I$ implies that $x \circ (y \circ (y \circ x)) \subseteq I$, where $x, y \in H$, then I is a commutative hyper K-ideal of type 1,
(iii) if $0 \in H$ is a right scalar and I is a commutative hyper K-ideal of type 2, then for all $x, y \in H$, $x \circ y \subseteq I$ implies that $x \circ (y \circ (y \circ x)) < I$,
(iv) if $x \circ y \subseteq I$ implies that $x \circ (y \circ (y \circ x)) < I$, where $x, y \in H$, then I is a commutative hyper K-ideal of type 2.

Theorem 5. 15. *Let I be a hyper K-ideal. Then the following statements hold:*
(i) I is a commutative hyper K-ideal of type 3 if and only if for all $x, y \in H$, $x \circ y < I$ implies that $x \circ (y \circ (y \circ x)) \subseteq I$,
(ii) I is a commutative hyper K-ideal of type 4 if and only if for all $x, y \in H$, $x \circ y < I$ implies that $x \circ (y \circ (y \circ x)) < I$.

Proof. (i) Let $x \circ y < I$. Then $(x \circ y) \circ 0 < I$. Since $0 \in I$ and I is a commutative hyper K-ideal of type 3 we conclude that $x \circ (y \circ (y \circ x)) \subseteq I$. Conversely, let $(x \circ y) \circ z < I$ and $z \in I$. Then there exists $t \in x \circ y$ such that $t \circ z < I$. Since I is a hyper K-ideal we get that $t \in I$, so $x \circ y < I$. Therefore by hypothesis we conclude that $x \circ (y \circ (y \circ x)) \subseteq I$. Thus I is a commutative hyper K-ideal of type 3.
The proof of (ii) is the same as (i).

Example 5. 16. The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0, 2\}$

Then $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$ are commutative hyper K-ideals of type 2(4), but $I_1 \bigcap I_2 = \{0\}$ is not a commutative hyper K-ideal of type 2(4), because $(2 \circ 1) \circ 0 = \{0\}$ and $2 \circ (1 \circ (1 \circ 2)) = \{2\} \not< \{0\}$.

Definition 5. 17. A hyper K-algebra H is said to be quasi-commutative, if for all $x, y \in H$, $x < y$ implies that $x \in y \circ (y \circ x)$.

Example 5. 18. Let H be the hyper K-algebra of Example 5.11(ii). Then H is a quasi-commutative hyper K-algebra.

Theorem 5. 19. *If H is a quasi-commutative hyper K-algebra, then the hyper K-ideal $\{0\}$ is a commutative hyper K-ideal of type 4(2).*

Example 5. 20. Let H be the hyper K-algebra of Example 5.2(ii). Then $\{0\}$ is a commutative hyper K-ideal of types 2 and 4, but H is not a quasi-commutative, because $2 < 1$ and $2 \notin 1 \circ (1 \circ 2) = \{0\}$. Thus the converse of the above theorem is not true in general.

Theorem 5. 21. *If $\{0\}$ is a commutative hyper K-ideal of type 3, then H is a quasi-commutative hyper K-algebra.*

Proof. Let $x < y$. Then $0 \in x \circ y \subseteq (x \circ y) \circ 0$, so $(x \circ y) \circ 0 < \{0\}$. Since $\{0\}$ is a commutative hyper K-ideal of type 3, then $x \circ (y \circ (y \circ x)) \subseteq \{0\}$. By Theorem 1.2.3(iii) we have $y \circ (y \circ x) < x$, thus there exists $t \in y \circ (y \circ x)$ such that $t < x$. Therefore we get that $x \circ t \subseteq x \circ (y \circ (y \circ x)) \subseteq \{0\}$. Hence $x \circ t = 0$, i.e. $x < t$, so $x = t$. Therefore $x \in y \circ (y \circ x)$.

Example 5. 22. Let H be the hyper K-algebra of Example 5.2(i). Then H is a quasi-commutative, but $\{0\}$ is not a commutative hyper K-ideal of type 3, because $(1 \circ 1) \circ 0 = \{0, 2\} < \{0\}$ and $1 \circ (1 \circ (1 \circ 1)) = \{0, 2\} \nsubseteq \{0\}$. Therefore the converse of the above theorem is not true in general.

The following theorem shows that if we restrict ourselves to a hyper K-algebra of order 3, then the condition "$0 \in H$ be a right scalar" is superfluous in Theorem 5.12.

Theorem 5. 23. *Let H be a hyper K-algebra of order 3. Then any commutative hyper K-ideal of type 1 is a weak hyper K-ideal.*

Proof. We consider three cases: (i) $2 \circ 0 = \{1, 2\}$ (ii) $1 \circ 0 = \{1, 2\}$, (iii) $2 \circ 0 = \{2\}$ and $1 \circ 0 = \{1\}$.
(i) Let $2 \circ 0 = \{1, 2\}$. We prove that if $I_1 = \{0, 1\}(I_2 = \{0, 2\})$ is a commutative hyper K-ideal of type 1, then $I_1(I_2)$ is a weak hyper K-ideal. On the contrary, let I_1 be a commutative hyper K-ideal of type 1, while it is not a weak hyper K-ideal. Thus we must have $2 \circ 1 \subseteq \{0, 1\}$. Also $2 \circ 0 = \{1, 2\}$ implies that $1 \circ 0 \neq \{1, 2\}$, because if $1 \circ 0 = \{1, 2\}$, then by (HK2) we have $0 \in 2 \circ 2 \subseteq (1 \circ 0) \circ 2 = (1 \circ 2) \circ 0$, so $0 \in 1 \circ 2$ i.e. $1 < 2$ and similarly, $2 \circ 0 = \{1, 2\}$ implies that $0 \in 2 \circ 1$, hence $2 < 1$. So $1 = 2$, which is a contradiction. Thus we have

$$1\circ 0=\{1\},\quad 2\circ 0=\{1,2\},\quad 0\in 2\circ 1,\quad 0\not\in 1\circ 2.$$
(1)

Now we consider two cases: (a) $0\circ 0\subseteq\{0,1\}$ (b) $2\in 0\circ 0$

(a) Let $0\circ 0\subseteq\{0,1\}$. Then $(2\circ 1)\circ 0\subseteq I_1$. Since I_1 is a commutative hyper K-ideal of type 1, so $2\circ(1\circ(1\circ 2))\subseteq I_1$, hence $1\not\in 1\circ 2$, therefore $1\circ 2=\{2\}$. Now by (1) we have $(1\circ 0)\circ 2=1\circ 2=\{2\}$ and $(1\circ 2)\circ 0=2\circ 0=\{1,2\}$, which contradicts (HK2).

(b) Let $2\in 0\circ 0$. Since $(1\circ 0)\circ 0\subseteq I_1$ and I_1 is a commutative hyper K-ideal of type 1, then $1\circ 2\subseteq 1\circ(0\circ(0\circ 1))\subseteq I_1$. Hence $2\not\in 1\circ 2$, thus $1\circ 2=\{1\}$. Also we have

$$(1\circ 2)\circ 0=1\circ 0=\{1\}\subseteq I_1$$
(2)

By (HK2) we get that

$$2\in 0\circ 0\subseteq(1\circ 1)\circ 0=(1\circ 0)\circ 1=1\circ 1$$
(3)

Since I_1 is a commutative hyper K-ideal of type 1, then by (2) and (3) we conclude that $2\in 1\circ 1\subseteq 1\circ(2\circ 0)\subseteq 1\circ(2\circ(2\circ 1))\subseteq I_1$, which is a contradiction.

Now let I_2 be a commutative hyper K-ideal of type 1, we prove that I_2 is a weak hyper K-ideal. On the contrary, let I_2 do not be a weak hyper K-ideal. Then we must have $1\circ 2\subseteq\{0,2\}$, so by (1) we have $1\circ 2=\{2\}$. Thus similar to the proof of case (i-a) we obtain a contradiction.

(ii) The proof is the same as the proof of (i).

(iii) Let $2\circ 0=\{2\}$ and $1\circ 0=\{1\}$. Then we prove that if $I_1=\{0,1\}(I_2=\{0,2\})$ is a commutative hyper K-ideal of type 1, then $I_1(I_2)$ is a weak hyper K-ideal. On the contrary, let I_1 do not be a weak hyper K-ideal. Then we must have $2\circ 1\subseteq I_1$. Since $2\in 2\circ(0\circ(0\circ 2))$, then $2\circ(0\circ(0\circ 2))\not\subseteq I_1$. On the other hand $(2\circ 0)\circ 1=2\circ 1\subseteq I_1$. Now since I_1 is a commutative hyper K-ideal of type 1, we must have $2\circ(0\circ(0\circ 2))\subseteq I_1$, which is a contradiction. Similarly, $1\circ 0=\{1\}$ implies that if $I_2=\{0,2\}$ is a commutative hyper K-ideal of type 1, then I_2 is a weak hyper K-ideal.

6 Quotient Hyper K-algebras

Definition 6. 1. Let $\sim$ be an equivalence relation on H and $A,B\subseteq H$. Then

(i) $A\sim B$ if and only if there exist $a\in A$ and $b\in B$ such that $a\sim b$,

(ii) $A\approx B$ if and only if for all $a\in A$ there exists $b\in B$ such that $a\sim b$, and for all $b\in B$ there exists $a\in A$ such that $a\sim b$,

(iii) $A\simeq B$ if and only if for all $a\in A$ and for all $b\in B$ we have $a\sim b$,

(iv) $\sim$ is called regular to the right if $a\sim b$ implies that $a\circ c\approx b\circ c$, for any $a,b,c\in H$,

(v) $\sim$ is called strongly regular to the right if $a \sim b$ implies that $a \circ c \simeq b \circ c$, for any $a, b, c \in H$,

(vi) $\sim$ is called good, if $a \circ b \sim \{0\}$ and $b \circ a \sim \{0\}$ implies that $a \sim b$, for all $a, b \in H$.

Analogously we define the regularity (strong regularity) of an equivalence to the left. A regular equivalence (strongly regular) to the right and to the left is called regular (strongly regular).

Lemma 6. 2. *Let $\sim$ be an equivalence relation on H and $A, B \subseteq H$. If $A \approx B$ and $B \approx C$, then $A \approx C$.*

From now on $\sim$ is a good regular relation. For any x in H by C_x we mean that equivalence class of x under $\sim$, and $I = C_0$.

Proposition 6. 3. *If $\sim$ be a good regular relation on H, then $I = C_0$ is a hyper K-ideal of H.*

Proof. Let $x \circ y < I$ and $y \in I$, we must show that $x \in I$. Since $x \circ y < I$ then There exists a $a \in x \circ y$ and $b \in I$ such that $a < b$, then $0 \in a \circ b$, thus $a \circ b \sim \{0\}$. Since $b \in I$ then $b \sim 0$ and $\sim$ is a regular relation then we get that $b \circ a \approx 0 \circ a$, then we have $b \circ a = \{0\}$. Since $\sim$ is a good relation we get that $a \sim b$, which means that $x \circ y \sim \{0\}$. Since $y \in I$ implies that $y \sim 0$ and $\sim$ is a regular relation then we get that $y \circ x \approx 0 \circ x$. Thus we get that $x \sim y$, and $y \sim \{0\}$ implies that $x \sim \{0\}$ therefore $x \in I$.

Denote $H/I = \{C_x : x \in H\}$ where $I = C_0$ and define

$$* : H/I \times H/I \to H/I$$

$$(C_x, C_y) \mapsto \{C_t \mid t \in x \circ y\}$$

Now we show that $*$ is well-defined. Let $C_x = C_{x'}$ and $C_y = C_{y'}$ we must show that $C_x * C_y = C_{x'} * C_{y'}$. Since $C_x = C_{x'}$ and $C_y = C_{y'}$ then $x \sim x'$ and $y \sim y'$. So $x \circ y \approx x' \circ y$ and $x' \circ y \approx x' \circ y'$, since $\sim$ is a regular relation, by Lemma 3.2.2 we have $x \circ y \approx x' \circ y'$. Now let $C_t \in C_x * C_y$. Then $C_t = C_s$ where $s \in x \circ y$, from $x \circ y \approx x' \circ y'$ we get that $s \sim u$ for some $u \in x' \circ y'$, hence $C_t = C_u$. Therefore $C_x * C_y \subseteq C_{x'} * C_{y'}$, and similarly $C_{x'} * C_{y'} \subseteq C_x * C_y$. Hence $*$ is well-defined.

Now we define the relation $<$ on H/I by $C_x < C_y$ if and only if $C_0 \in C_x * C_y$. Hence we have

$$x < y \Leftrightarrow 0 \in x \circ y \Longrightarrow C_0 \in C_x * C_y \Leftrightarrow C_x < C_y$$

Theorem 6.4. *Let $I = C_0$. Then $(H/I, *, C_0)$ is a hyper K-algebra.*

Theorem 6.5. *Suppose H is a bounded hyper K-algebra with the greatest element 1 and $I = C_0$. Then $(H/I, *, C_0)$ is also a bounded hyper K-algebra with the greatest element C_1.*

The converse of the above theorem does not hold.

Example 6.6. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

we can easily check that $\sim= \{(0,0),(0,1),(1,0),(1,1),(2,2)\}$ is a good regular relation on H. Now consider the quotient hyper K-algebra H/I. We can see that

$C_0 = \{0, 1\} = C_1 = \{y \mid y \sim 1\}$, $C_2 = \{y \mid y \sim 2\} = \{2\}$. So $H/I = \{C_0, C_2 = \{2\}\}$ and the following table shows the hyper K-algebra structure on H/I

$*$	C_0	C_2
C_0	C_0	C_0
C_2	C_2	$\{C_0, C_2\}$

We can check that this hyper K-algebra is bounded with the greatest element C_2. But H is not bounded, because $2 \not< 1$ and $1 \not< 2$.
Furthermore $C_1 < C_2$ but $1 \not< 2$.

Theorem 6.7. *Let $\sim$ be an equivalence relation on H and $I = C_0$. Then:*

(i) If $\sim$ is a good and strongly regular relation on H, then H/I is a BCK-algebra.

(ii) If $\sim= \triangle H$, then $H/I \cong H$

(iii) If $\sim= H \times H$, then $H/I = C_0$.

Remark 6.8. *If $\sim= \triangle H$, then we have $C_x < C_y$ if and only if $x < y$.*

Theorem 6.9. *Let $\sim$ be a good regular relation on H. If $I = C_0$ and J be a hyper K-ideal of H and $I \subseteq J$, then quotient hyper K-algebra $J/I = \{C_t \mid t \in J\}$ is a hyper K-ideal of H/I.*

Theorem 6.10. *If L is a hyper K-ideal of H/I, then $J = \{x \mid C_x \in L\}$ is a hyper K-ideal of H and $I \subseteq J$. Moreover $L = J/I$.*

Proof. Since $I = C_0 \in L$, then $0 \in J$. Let $x \circ y < J$ and $y \in J$. Then there exist $t \in x \circ y$ and $s \in J$ such that $t < s$. Hence $C_t < C_s$, which implies that $C_x * C_y < L$. Since $y \in J$, we get that $C_y \in L$, thus $C_x \in L$. Therefore $x \in J$, hence J is a hyper K-ideal of H. Let $x \in I = C_0$. Then $x \sim 0$, thus $C_x = C_0$ and hence $C_x \in L$. Therefore $x \in J$, that is $I \subseteq J$.

Theorem 6.11. *If I is a hyper K-ideal of H, then there is a bijection from $\mathcal{I}(H, I)$, the set of all hyper K-ideals of H containing I, to $\mathcal{I}(H/I)$, the set of all hyper K-ideals of H/I.*

Theorem 6.12. *Let I be a hyper K-ideal of H. Then there exists a canonical surjective homomorphism $\varphi : H \longrightarrow H/I$ by $\varphi(x) = C_x$, and $\ker(\varphi) = I$.*

Proof. It is clear that φ is well-defined. Let $x, y \in H$. Then $\varphi(x \circ y) = \{\varphi(t) \mid t \in x \circ y\} = \{C_t \mid t \in x \circ y\} = C_x * C_y = \varphi(x) * \varphi(y)$. Hence φ is homomorphism. Clearly φ is onto, and we have $\ker(\varphi) = \{x \in H \mid \varphi(x) = C_0\} = \{x \in H \mid C_x = C_0 = I\} = \{x \in H \mid x \in I\} = I$.

Theorem 6.13. *Let $f : H_1 \longrightarrow H_2$ be a homomorphism of hyper K-algebras, and let I be a hyper K-ideal of H_1 such that $I \subseteq \ker(f)$. Then there exists a unique homomorphism $\bar{f} : H_1/I \longrightarrow H_2$ such that $\bar{f}(C_x) = f(x)$ for all $x \in H_1$, $Im(\bar{f}) = Im(f)$ and $\ker(\bar{f}) = \ker(f)/I$. Moreover $\bar{f}$ is an isomorphism if and only if f is surjective and $I = \ker(f)$.*

Proof. Let $C_x = C_{x'}$. Then $x \sim x'$, which implies that $x \circ x' \sim x' \circ x'$, since $\sim$ is a regular relation. Thus there exists a $t \in (x \circ x') \bigcap I$. Then $0 = f(t) \in f(x \circ x') = f(x) \circ f(x')$, hence $f(x) < f(x')$. Similarly $f(x') < f(x)$, therefore $\bar{f}$ is well-defined.

Consider $\bar{f}(C_x * C_y) = \bar{f}(\{C_t \mid t \in x \circ y\}) = \{\bar{f}(C_t) \mid t \in x \circ y\} = \{f(t) \mid t \in x \circ y\} = f(x \circ y) = f(x) \circ f(y) = \bar{f}(C_x) * \bar{f}(C_y)$.

$$C_x \in \ker(\bar{f}) \Longleftrightarrow \bar{f}(C_x) = 0 \Longleftrightarrow f(x) = 0 \Longleftrightarrow x \in \ker(f)$$

Note that $\bar{f}$ is unique, since it is completely determined by f. Finally it is clear that $\bar{f}$ is surjective if and only if f is surjective.

Theorem 6.14. **(First isomorphism theorem)** *Let $f : H_1 \longrightarrow H_2$ be a homomorphism of hyper K-algebras. Then $H_1/\ker f \cong Im(f)$.*

Theorem 6.15. *Let I, J be hyper K-ideals of H. Then there is a (natural) homomorphism of hyper K-algebras between $I/(I \bigcap J)$ and $< I \bigcup J > /J$, where by $< I \bigcup J >$ we mean that the hyper K-ideal generated by $I \bigcup J$.*

Proof. Define $\varphi : I \to (I \bigcup J)/J$ by $\varphi(x) = C_x^J$. If $x_1 = x_2$, then it is clear that $C_{x_1}^J = C_{x_2}^J$, which means that φ is well-defined. Also we have

$$\varphi(x \circ y) = \{\varphi(t) \mid t \in x \circ y\} = \{C_t^J \mid t \in x \circ y\} = C_x^J * C_y^J = \varphi(x) * \varphi(y).$$

So that φ is a homomorphism. Moreover

$$\ker \varphi = \{x \in I \mid \varphi(x) = C_0^J\} = \{x \in I \mid C_x^J = C_0^J\} = \{x \in I \mid x \in J\} = I \cap J.$$

Thus by Theorem 6.14 the proof is completed.

Open Problem Under what suitable conditions the defined homomorphism in Theorem 6.15, is an isomorphism? In other word does the second isomorphism theorem hold?.

Theorem 6. 16. (Third isomorphism theorem) *Let I, J be hyper K-ideals of H such that $I \subseteq J$. Then $(H/I)/(J/I) \cong H/J$.*

Proof. It is clear that $J/I \subseteq H/I$. Define $f : H/I \longrightarrow H/J$ by $C_x^I \mapsto C_x^J$, where $C_x^I \in H/I$ and $C_x^J \in H/J$.

If $C_x^I = C_y^I$, then $x \sim y$ which implies that $x \circ y < I$. Since $I \subseteq J$ hence $x \circ y < J$. Thus $x \sim_J y$ then $C_x^J = C_y^J$ which means that f is well-defined.

$$\begin{aligned} f(C_x^I * C_y^I) &= f(\{C_t^I \mid t \in x \circ y\}) \\ &= \{C_t^J \mid t \in x \circ y\} \\ &= C_x^J * C_y^J = f(C_x^I) * f(C_y^I). \end{aligned}$$

Clearly f is onto.

$$\begin{aligned} \ker f &= \{C_x^I \in H/I \mid f(C_x^I) = C_0^J\} \\ &= \{C_x^I \in H/I \mid C_x^J = C_0^J\} \\ &= \{C_x^I \in H/I \mid x \in J\} \\ &= J/I. \end{aligned}$$

Now by Theorem 6.14 the proof is completed.

7 Uniform Topology On Hyper K-algebras

In this section I is an s-reflexive hyper K-ideal of H.

Definition 7. 1. We define the *relation* $\sim_I$ on H as follows:

$$x \sim_I y \text{ if and only if } x \circ y < I \text{ and } y \circ x < I.$$

If A, B are subsets of H, then we define $A \sim_I B$ if and only if $\exists a \in A$, $\exists b \in B$ such that $a \sim_I b$.

Proposition 7. 2. *The relation $\sim_I$ is an equivalence relation on H.*

Proof. (i) Since $0 \in x \circ x$, then $x \circ x < I$. Hence $x \sim_I x$.

(ii) Clearly $\sim_I$ is symmetric.

(iii) Let $x \sim_I y$ and $y \sim_I z$. Then $x \circ y < I$, $y \circ x < I$, $y \circ z < I$ and $z \circ y < I$. Since I is an s-reflexive hyper K-ideal of H we get that $x \circ y \subseteq I$, $y \circ x \subseteq I$, $y \circ z \subseteq I$ and $z \circ y \subseteq I$. We have $(x \circ z) \circ (x \circ y) < y \circ z$ and $y \circ z \subseteq I$, $(x \circ z) \circ (x \circ y) < I$. Thus from $(x \circ z) \circ (x \circ y) < I$, $x \circ y \subseteq I$ and Proposition 2.1.1 we conclude that $x \circ z < I$. Similarly $z \circ x < I$, therefore $x \sim_I z$.

Let X be a non-empty set and U and V be subsets of $X \times X$. We let

$U \diamond V = \{(x, y) \in X \times X \mid \exists z \in X$, such that $(z, y) \in U$ and $(x, z) \in V\}$,

$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\}$,

$\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

Definition 7. 3. [25] By a *uniformity* on X we shall mean a non empty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

(U_1) $\Delta \subseteq U$ for any $U \in \mathcal{K}$,

(U_2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{S}$,

(U_3) if $U \in \mathcal{K}$, then there exist a $V \in \mathcal{K}$, such that $V \diamond V \subseteq U$,

(U_4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,

(U_5) if $U \in \mathcal{K}$, and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{K}$.

The pair (X,$\mathcal{K}$) is called a *uniform structure* (uniform space).

Theorem 7. 4. *Let I be an s-reflexive hyper K-ideal of H and $U_I = \{(x, y) \in X \times X \mid x \sim_I y\}$. If*

$$\mathcal{K}^* = \{U_I \mid I \text{ is an s-reflexive hyper } K\text{-ideal of } H\}$$

then $\mathcal{K}^*$ satisfies the conditions $(U_1) - (U_4)$.

Proof. (U_1): Since $0 \in x \circ x$, hence $x \circ x < I$ for any s-reflexive hyper K-ideal I of H. Thus $x \sim_I x$ for any x in H, hence $\Delta \subseteq U_I$ for all $U_I \in \mathcal{K}^*$.

(U_2): For any $U_I \in \mathcal{K}^*$, we have

$$(x, y) \in (U_I)^{-1} \Longleftrightarrow (y, x) \in U_I \Longleftrightarrow y \sim_I x \Longleftrightarrow x \sim_I y \Longleftrightarrow (x, y) \in U_I.$$

Hence $(U_I)^{-1} = U_I \in \mathcal{K}^*$.

(U_3): For any $U_I \in \mathcal{K}^*$, the transitivity of $\sim_I$ implies that $U_I \diamond U_I \subseteq U_I$.

(U_4): For any $U_I, U_J \in \mathcal{K}^*$, we claim that $U_I \cap U_J = U_{I \cap J}$. Let $(x, y) \in U_I \cap U_J$. Then $x \sim_I y$ and $x \sim_J y$. So we have $x \circ y < I$, $y \circ x < I$, $x \circ y < J$ and $y \circ x < J$. Since I is an s-reflexive hyper K-ideal of H then we get that $x \circ y \subseteq I$, $y \circ x \subseteq I$, $x \circ y \subseteq J$ and $y \circ x \subseteq J$. So $x \circ y \subseteq I \cap J$.

Similarly $y \circ x \subseteq I \cap J$, therefore $x \sim_{I\cap J} y$. Thus $(x, y) \in U_{I\cap J}$. Conversely, let $(x, y) \in U_{I\cap J}$. Then $x \sim_{I\cap J} y$, hence $x \circ y < I \cap J$ and $y \circ x < I \cap J$. Therefore $x \circ y \subseteq I \cap J$ and $y \circ x \subseteq I \cap J$. We have $I \cap J \subseteq I, J$. So $x \circ y < I$, $y \circ x < I$, $x \circ y < J$ and $y \circ x < J$. Thus $x \sim_I y$ and $x \sim_J y$, therefore $(x, y) \in (U_I \cap U_J)$. So $U_I \cap U_J = U_{I\cap J}$. Since I, J are s-reflexive hyper K-ideals of H then $I \cap J$ is an s-reflexive hyper K-ideal of H, thus $U_I \cap U_J \in \mathcal{K}^*$.

Theorem 7. 5. *Let* $\mathcal{K} = \{U \subseteq X \times X \mid U_I \subseteq U$ *for some* $U_I \in \mathcal{K}^*\}$. *Then* $\mathcal{K}$ *satisfies a uniformity on* H *and the pair* $(H,\mathcal{K})$ *is a uniform structure.*

Proof. By applying Theorem 7.4 we can show that $\mathcal{K}$ satisfies the conditions $(U_1)-(U_4)$. Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$. Then there exists a $U_I \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem.

Consider $x \in H$ and $U \in \mathcal{K}$, we define

$$U[x] := \{y \in H \mid (x, y) \in U\}.$$

Theorem 7. 6. *Let* H *be a hyperK algebra, and*

$$\mathcal{U} = \{G \subseteq H \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}.$$

Then $\mathcal{U}$ *is a topology on* H.

Proof. It is clear that $\emptyset$ and H are in $\mathcal{U}$, and $\mathcal{U}$ is closed under arbitrary union. To show that $\mathcal{U}$ is closed under finite intersection, let $G, J \in \mathcal{U}$ and suppose $x \in G \cap J$. Then there exist $U, V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq J$. Let $W = U \cap V$. Then $W \in \mathcal{K}$ by (U_4). Also it is easy to show that $W[x] \subseteq U[x] \cap V[x]$ and hence $W[x] \subseteq G \cap J$. So $G \cap J \in \mathcal{U}$. Thus $\mathcal{U}$ is a topology on H.

Note that for any x in H, $U[x]$ is an open neighborhood of x.

Definition 7. 7. Let $(X,\mathcal{K})$ be a uniform space. Then the topology $\mathcal{U}$ is called the uniform topology on X induced by $\mathcal{K}$.

Theorem 7. 8. [25] *If* (X, τ) *is a uniform space then the corresponding topological space is completely regular.*

Proposition 7. 9. *The topological space* $(H,\mathcal{U})$ *is completely regular.*

Proposition 7. 10. *Every hyper* K*-ideal* I *of* H *is a clopen set in* $(H,\mathcal{U})$.

Proof. Let I be a hyper K-ideal of H. To prove that I is closed we shall show that $I^c = \bigcup_{x \notin I} U_I[x]$. Indeed, assume $y \in I^c$, then from $y \in U_I[y]$ it follows that $y \in \bigcup_{x \notin I} U_I[x]$. Hence $I^c \subseteq \bigcup_{x \notin I} U_I[x]$. Conversely, let $y \in \bigcup_{x \notin I} U_I[x]$. Then there is $z \in I^c$ such that $y \in U_I[z]$. Hence we get $y \circ z < I$ and $z \circ y < I$. In this case, assume that $y \in I$ we easily get $z \in I$, which is a contradiction. Hence $y \in I^c$, and then $\bigcup_{x \notin I} U_I[x] \subseteq I^c$. Thus $I^c = \bigcup_{x \notin I} U_I[x]$, that is I^c is an open set and I a closed set. To prove that I is open we shall show that $I = \bigcup_{x \in I} U_I[x]$. If $y \in I$ then we get $y \in U_I[y]$, so $y \in \bigcup_{x \in I} U_I[x]$. Hence $I \subseteq \bigcup_{x \in I} U_I[x]$. On the other hand, let $y \in \bigcup_{x \in I} U_I[x]$, then there is $z \in I$ such that $y \in U_I[z]$. Thus $z \circ y < I$ and $y \circ z < I$. From these facts we easily obtain $y \in I$. Hence $\bigcup_{x \in I} U_I[x] \subseteq I$, and the $I = \bigcup_{x \in I} U_I[x]$. Then I is open.

Proposition 7.11. *Let I be a hyper K-ideal of H, and $A, B \subseteq H$. If $A \circ B < I$ and $B \subseteq I$, then $A < I$.*

Theorem 7.12. *Each $U_I[x]$ is a clopen set for any s-reflexive hyper K-ideal I of H.*

Proof. Let I be an arbitrary hyper K-ideal of H and x an element of H. We want to show that $U_I[x]$ is a closed subset of H. Let $y \in (U_I[x])^c$. We claim that for the given element y we have that $U_I[y] \subseteq (U_I[x])^c$. Let $z \in U_I[y]$, then $z \sim_I y$, so $z \circ y \subseteq I$ and $y \circ z \subseteq I$. If we have $x \sim_I z$, then $x \circ z \subseteq I$ and $z \circ x \subseteq I$. Now consider $(x \circ y) \circ (x \circ z) < z \circ y$ since $z \circ y \subseteq I$ then we get that $(x \circ y) \circ (x \circ z) < I$. From $x \circ z \subseteq I$ and Proposition 7.11 we have $x \circ y < I$. Similarly from $(y \circ x) \circ (z \circ x) < (y \circ z)$ and $y \circ z \subseteq I$ we get that $y \circ x < I$. Therefore $x \sim_I y$, a contradiction. Hence it must be $z \in (U_I[x])^c$, so $U_I[y] \subseteq (U_I[x])^c$. Hence $(U_I[x])^c$ is open , that is $U_I[x]$ is closed.

Proposition 7.13. *If the hyper K-ideal $< 0 >$ is an s-reflexive hyper K-ideal, then $\mathcal{K}$ is a discrete topology.*

Proof. It is clear that

$$\begin{aligned} U_{<0>}[x] &= \{y \in H \mid x \sim_{<0>} y\} \\ &= \{y \in H \mid x \circ y << 0 >, y \circ x << 0 >\} \\ &= \{y \in H \mid x = y\} = \{x\} \end{aligned}$$

therefore each $\{x\}$ is clopen set. Thus the topology is discrete.

In the forthcoming sections we let H to be a bounded hyper K-algebra with unit 1, so Nx means that $1 \circ x$.

8 Dual Positive Implicative Hyper K-ideals of type 3

Definition 8. 1. A non-empty subset D of H is called *a dual positive implicative hyper K-ideal of type 3 (DPIHKI-T3)* if it satisfies:
(i) $1 \in D$
(ii) $N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) < D$ imply $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Example 8. 2. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

Then $I = \{0, 1\}$ is a $DPIHKI - T3$.

Theorem 8. 3. *Let $1 \in D \subseteq H$. Then D is a $DPIHKI - T3$ if and only if $N(Nx \circ Nz) \subseteq D$, for all $x, z \in H$.*

Proof. Let D be a $DPIHKI - T3$. Then by Definition 8.1 and Theorem 2.3(xii) we conclude that $N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) < D$, for all $x, y, z \in H$. So by hypothesis we get that $N(Nx \circ Nz) \subseteq D$, for all $x, z \in H$.
The proof of the converse is trivial. □

In the sequel of this chapter we let $H = \{0, 1, 2\}$ to be a bounded hyper K-algebra of order 3 with unit 1.

Theorem 8. 4. *In H we have $1 \circ 0 = \{1\}$.*

Proof. On the contrary let $1 \circ 0 \neq \{1\}$. Then we must have $1 \circ 0 = \{1, 2\}$. By $(HK2)$ we have $(1 \circ 0) \circ 2 = (1 \circ 2) \circ 0$, so $0 \in 2 \circ 2 \subseteq (1 \circ 0) \circ 2 = (1 \circ 2) \circ 0$. Thus there exists $x \in 1 \circ 2$ such that $0 \in x \circ 0$, which implies that $x < 0$, thus from (HK4) and $(HK5)$ we get that $x = 0$. Hence $0 \in 1 \circ 2$, that is $1 < 2$. Since $2 < 1$, thus $2 = 1$, which is a contradiction. □

Theorem 8.5. *For all x in H we have $NNx = x$ if and only if $1 \circ 1 = \{0\}$ and $1 \circ 2 = \{2\}$.*
Proof. Let $NNx = x$, i.e. $1 \circ (1 \circ x) = x$, for all x in H. Since $1 \circ (1 \circ 2) = 2$, we get that $0 \notin 1 \circ 2$ and $1 \notin 1 \circ 2$. So $1 \circ 2 = \{2\}$. Now since $1 \circ (1 \circ 1) = 1$, we conclude that $1 \notin 1 \circ 1$ and $2 \notin 1 \circ 1$. Thus $1 \circ 1 = \{0\}$.
Conversely, the proof follows from Theorem 8.4 and hypothesis. □

Theorem 8.6. *Let $D_1 = \{1\}$ and $D_2 = \{1, 2\}$ in H. Then D_1 and D_2 are not $DPIHKI - T3$.*

Proof. Since $0 \in 1 \circ ((1 \circ 0) \circ (1 \circ 1)) = N(N0 \circ N1)$, $0 \notin D_1$ and $0 \notin D_2$, then D_1 and D_2 are not $DPIHKI - T3$, by Theorem 8.3.

Theorem 8.7. *Let $D = \{0, 1\}$ in H.Then the following statements hold:*
(i) If $2 \in 1 \circ 2$, then D is not a $DPIHKI - T3$,
(ii) If $2 \in 1 \circ 1$, then D is not a $DPIHKI - T3$.

Proof. (i) Since $2 \in 1 \circ 2 \subseteq 1 \circ ((1 \circ 2) \circ (1 \circ 1)) = N(N2 \circ N1)$ and $2 \notin D$, then D is not $DPIHKI - T3$, by Theorem 8.3.
(ii) The proof is similar to (i). □

Theorem 8.8. *Let $D = \{0, 1\}$ in H. Then D is a $DPIHKI - T3$ if and only if $2 \notin 1 \circ 2$ and $2 \notin 1 \circ 1$.*

Proof. Let $2 \notin 1 \circ 2$ and $2 \notin 1 \circ 1$. Thus $1 \circ 2 = \{1\}$ and $1 \circ 1 = \{0, 1\}$ or $1 \circ 1 = \{0\}$. Now by some calculations we can get that $N(Nx \circ Nz) \subseteq D$, for all $x, z \in H$.
Conversely, on the contrary let $2 \in 1 \circ 2$ or $2 \in 1 \circ 1$. Then Theorem 8.7(i,ii) gives a contradiction. Thus $2 \notin 1 \circ 2$ and $2 \notin 1 \circ 1$. □

Remark 8.9. *In this chapter let $D = \{0, 1\}$ be a $DPIHKI - T3$. Thus:*
(i) From Theorem 8.8 we conclude that $1 \circ 2 = \{1\}$ and $1 \circ 1 = \{0, 1\}$ or $1 \circ 1 = \{0\}$.
(ii) By (HK2) we have $(1 \circ 1) \circ 0 = (1 \circ 0) \circ 1$ and $(1 \circ 1) \circ 2 = (1 \circ 2) \circ 1$. Thus by (i) and Theorem 8.4 we conclude that $0 \circ 0 \subseteq \{0, 1\}$ and $0 \circ 2 \subseteq \{0, 1\}$.

Theorem 8.10. *If $2 \circ 2 = \{0\}$ and $0 \circ 0 = \{0\}$ in H, then $2 \circ 0 = \{2\}$.*

Theorem 8.11. *If $2 \circ 2 = \{0\}$ and $0 \circ 1 = \{0\}$ in H, then $1 \notin 2 \circ 1$.*

Theorem 8.12. *If $2 \circ 1 = \{0, 2\}$ and $1 \circ 1 = \{0\}$ in H, then $2 \circ 0 = \{2\}$.*

Proof. On the contrary let $2 \circ 0 \neq \{2\}$. Then we must have $2 \circ 0 = \{1, 2\}$. By (HK2) we have $(2 \circ 1) \circ 0 = (2 \circ 0) \circ 1$ and also by hypothesis we have $(2 \circ 1) \circ 0 = \{0, 1, 2\}$ and $(2 \circ 0) \circ 1 = \{0, 2\}$, which is a contradiction.

Theorem 8. 13. *Let* $0 \circ 1 = \{0, 1\}$ *and* $0 \circ 2 = \{0\}$ *in* H*. Then the following statements hold:*
(i) If $2 \circ 2 \subseteq \{0, 2\}$*, then* $2 \circ 1 \nsubseteq \{0, 2\}$*,*
(ii) If $2 \circ 2 = \{0, 1\}$ *or* $2 \circ 2 = \{0, 1, 2\}$*, then* $2 \circ 1 \neq \{0\}$*.*

Proof. (i) On the contrary let $2 \circ 1 \subseteq \{0, 2\}$. If $2 \circ 1 = \{0, 2\}$, then by (HK2) we have $(2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. If $2 \circ 2 = \{0\}$, then by hypothesis we have $(2 \circ 2) \circ 1 = \{0, 1\}$ and if $2 \circ 2 = \{0, 2\}$, then $(2 \circ 2) \circ 1 = \{0, 1, 2\}$. On the other hand if $2 \circ 2 = \{0\}$, then $(2 \circ 1) \circ 2 = \{0\}$ and if $2 \circ 2 = \{0, 2\}$, then $(2 \circ 1) \circ 2 = \{0, 2\}$, which is a contradiction. If $2 \circ 1 = \{0\}$, then the proof is similar to the case of $2 \circ 1 = \{0, 2\}$.
The proof of (ii) is the same as (i).

Theorem 8. 14. *Let* $0 \circ 1 = \{0, 2\}$ *in* H*. Then the following statements hold:*
(i) $2 \circ 2 \nsubseteq \{0, 1\}$*,*
(ii) $2 \circ 1 \nsubseteq \{0, 1\}$*,*
(iii) If $1 \circ 1 = \{0\}$*, then* $2 \circ 2 \neq \{0, 1, 2\}$*,*
(iv) If $0 \circ 0 = \{0\}$*, then* $2 \circ 0 = \{2\}$*,*
(v) If $2 \circ 2 = \{0, 1, 2\}$*, then* $0 \circ 2 = \{0, 1\}$*.*

Proof. (i) On the contrary let $2 \circ 2 \subseteq \{0, 1\}$. By (HK2) we have $(0 \circ 2) \circ 1 = (0 \circ 1) \circ 2$. If $2 \circ 2 = \{0\}$, then by hypothesis and Remark 8.9 we get that $(0 \circ 1) \circ 2 \subseteq \{0, 1\}$ and $(0 \circ 2) \circ 1 = \{0, 2\}$ or $\{0, 1, 2\}$, which is a contradiction. If $2 \circ 2 = \{0, 1\}$, then the proof is similar to the case of $2 \circ 2 = \{0\}$.
The proof of the other cases are similar to above by considering the suitable modifications.

Theorem 8. 15. *Let* $0 \circ 1 = \{0, 1, 2\}$ *in* H*. Then the following statements hold:*
(i) $2 \circ 2 \nsubseteq \{0, 1\}$*,*
(ii) $2 \circ 1 \nsubseteq \{0, 1\}$*,*
(iii) If $2 \circ 2 = \{0, 2\}$ *and* $0 \circ 2 = \{0\}$*, then* $2 \circ 1 \neq \{0, 2\}$*,*
(iv) If $2 \circ 1 = \{0, 2\}$ *and* $1 \notin 2 \circ 2$*, then* $0 \circ 2 = \{0, 1\}$*.*

Proof. (i) On the contrary let $2 \circ 2 \subseteq \{0, 1\}$. By (HK2) we have $(0 \circ 2) \circ 1 = (0 \circ 1) \circ 2$. If $2 \circ 2 = \{0\}$, then by hypothesis and Remark 8.9 we get that $(0 \circ 1) \circ 2 = \{0, 1\}$ and $(0 \circ 2) \circ 1 = \{0, 1, 2\}$, which is a contradiction. If $2 \circ 2 = \{0, 1\}$, then the proof is similar to the case of $2 \circ 2 = \{0\}$.
The proof of the other cases are the same as above by considering the suitable modifications. □

Theorem 8. 16. *If* $2 \circ 1$*,* $2 \circ 2$ *and* $0 \circ 1 \subseteq \{0, 2\}$*, then* $0 \circ 2 = \{0\}$*.*

Proof. By (HK2) we have $(2 \circ 1) \circ 2 = (2 \circ 2) \circ 1 \subseteq \{0, 2\}$. Since $0 \circ 2 \subseteq (2 \circ 1) \circ 2 \subseteq \{0, 2\}$ and also $2 \notin 0 \circ 2$, by Remark 8.9(ii), then we get that $0 \circ 2 = \{0\}$. □

Theorem 8.17. *Let $2 \circ 1 = \{0\}$ in H. Then the following statements hold:*
(i) If $1 \circ 1 = \{0, 1\}$ and $2 \circ 2 = \{0, 1\}$ or $\{0, 1, 2\}$, then $0 \circ 2 = \{0, 1\}$,
(ii) If $0 \circ 0 = \{0\}$ and $1 \circ 1 = \{0, 1\}$, then $2 \circ 0 = \{2\}$,
(iii) If $0 \circ 1 = \{0, 1\}$, then $0 \circ 2 = \{0, 1\}$,
(iv) If $0 \circ 0 = \{0, 1\}$, then $2 \circ 0 = \{1, 2\}$.

Theorem 8.18. *Let $2 \circ 1 = \{0, 2\}$ in H. Then*
(i) If $1 \in 0 \circ 1$ and $1 \notin 2 \circ 2$, then $0 \circ 2 = \{0, 1\}$,
(ii) If $0 \circ 0 = \{0, 1\}$, then $2 \circ 0 = \{1, 2\}$.

Proof. (i) On the contrary let $0 \circ 2 \neq \{0, 1\}$. Then we must have $0 \circ 2 = \{0\}$, by Remark 8.9(ii). By (HK2) we have $(2 \circ 1) \circ 2 = (2 \circ 2) \circ 1$. Now by hypothesis we get that $(2 \circ 1) \circ 2 \subseteq \{0, 2\}$ and $1 \in (2 \circ 2) \circ 1$, which is a contradiction. The proof of (ii) is similar to (i).

Theorem 8.19. *If $2 \circ 2 \subseteq \{0, 2\}$ and $0 \circ 0 = \{0, 1\}$, then $2 \circ 0 = \{1, 2\}$.*

Proof. On the contrary let $2 \circ 0 \neq \{1, 2\}$. Then we must have $2 \circ 0 = \{2\}$. By (HK2) we have $(2 \circ 2) \circ 0 = (2 \circ 0) \circ 2$. Now by hypothesis we get that $1 \in (2 \circ 2) \circ 0$ and $1 \notin (2 \circ 0) \circ 2$, which is a contradiction.

Now we are ready to determine all of hyper K-algebras of order 3, in which $D = \{0, 1\}$ is a $DPIHKI - T3$.

Theorem 8.20. (Main theorem) *There are 220 non-isomorphic bounded hyper K-algebras of order 3, to have $D = \{0, 1\}$ as a $DPIHKI - T3$.*

Proof. Let $H = \{0, 1, 2\}$ and 1 be its unit. The following table shows a probable hyper K-algebra structure on H, in which $D = \{0, 1\}$ is a $DPIHKI - T3$:

$\circ$	0	1	2
0	a_{11}	a_{12}	a_{13}
1	a_{21}	a_{22}	a_{23}
2	a_{31}	a_{32}	a_{33}

By Remark 2.1.9 we have $a_{21} = 1 \circ 0 = \{1\}$, $a_{22} = 1 \circ 1 = \{0\}$ or $\{0, 1\}$, $a_{23} = 1 \circ 2 = \{1\}$, $a_{11} = 0 \circ 0 \subseteq \{0, 1\}$ and $a_{13} = 0 \circ 2 \subseteq \{0, 1\}$.
Also since H is bounded, then by $(HK3)$ and $(HK5)$ we have $0 \in a_{12} \bigcap a_{22} \bigcap a_{32}$.
There are two cases for $a_{22} = 1 \circ 1$. Let $1 \circ 1 = \{0\}$. Then by (HK2) we have $(1 \circ 1) \circ 0 = (1 \circ 0) \circ 1$, so $0 \circ 0 = \{0\}$. Similarly $(1 \circ 1) \circ 2 = (1 \circ 2) \circ 1$

implies that $0 \circ 2 = \{0\}$. We will show that in this case there exist exactly 40 non-isomorphic hyper K-algebras. On the other hand if $1 \circ 1 = \{0, 1\}$, then by Remark 8.9(ii) we get that $0 \circ 0 \subseteq \{0, 1\}$ and $0 \circ 2 \subseteq \{0, 1\}$ and in this situation we will obtain exactly 180 non-isomorphic hyper K-algebras other than the previous 40 ones. So totally we have 220 different non-isomorphic bounded hyper K-algebras of order 3, to have $D = \{0, 1\}$ as a $DPIHKI{-}T3$. Now we give the details. To do this we consider two main cases $1 \circ 1 = \{0\}$ and $1 \circ 1 = \{0, 1\}$, and many subcases of them.

1: $1 \circ 1 = \{0\}$
We consider some subcases as follows:
1.1: $0 \circ 1 = \{0\}$
In this case also we consider 4 states as follows:
1.1.1: $2 \circ 2 = \{0\}$
By Theorems 8.10 and 8.11 we must have $2 \circ 0 = \{2\}$ and $2 \circ 1 \subseteq \{0, 2\}$. So there exist 2 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

1.1.2: $2 \circ 2 = \{0, 1\}$
By $(HK2)$ We have $(2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. So by hypothesis we get that $(2 \circ 1) \circ 2 = \{0\}$. If $1 \in 2 \circ 1$ or $2 \in 2 \circ 1$, then $1 \in (2 \circ 1) \circ 2 = \{0\}$, which is a contradiction. Thus $1 \notin 2 \circ 1$ and $2 \notin 2 \circ 1$, hence $2 \circ 1 = \{0\}$. So there exists two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0, 1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1, 2\}$	$\{0\}$	$\{0, 1\}$

1.1.3: $2 \circ 2 = \{0, 2\}$
If $2 \circ 1 = \{0, 2\}$, then by Theorem 8.12 we have $2 \circ 0 = \{2\}$. So there exists

seven hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

1.1.4: $2\circ 2=\{0,1,2\}$

We prove that $2\circ 1\neq\{0,2\}$. On the contrary, let $2\circ 1=\{0,2\}$. By $(HK2)$ we have $(2\circ 2)\circ 1=(2\circ 1)\circ 2$, while $(2\circ 2)\circ 1=\{0,2\}$ and $(2\circ 1)\circ 2=\{0,1,2\}$, which is a contradiction . So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

1.2: $0\circ 1=\{0,1\}$

In this case also we consider four states as follows:

1.2.1: $2\circ 2=\{0\}$

By Theorems 8.10 and 8.13(i) we must have $2\circ 0=\{2\}$ and $2\circ 1\not\subseteq\{0,2\}$.

So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

1.2.2: $2 \circ 2 = \{0,1\}$
By Theorem 8.13(ii) we have $2 \circ 1 \neq \{0\}$. If $2 \circ 1 = \{0,2\}$, then by Theorem 8.12 we have $2 \circ 0 = \{2\}$. So there exist five hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

1.2.3: $2 \circ 2 = \{0,2\}$
By Theorem 8.13(i) we have $2 \circ 1 \not\subseteq \{0,2\}$. So there exist four hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

1.2.4: $2 \circ 2 = \{0,1,2\}$
This case is similar to the case 1.2.2. So there exist five hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

1.3: $0 \circ 1 = \{0,2\}$

In this case we have only one state, since by Theorem 8.14(i,v) we have $2 \circ 2 \not\subseteq \{0,1\}$ and $2 \circ 2 \neq \{0,1,2\}$.

1.3.1: $2 \circ 2 = \{0,2\}$

By Theorem 2.1.14(i,iv) we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

1.4: $0 \circ 1 = \{0,1,2\}$

In this case we have two states, since by Theorem 8.15(i) we have $2 \circ 2 \not\subseteq \{0,1\}$.

1.4.1: $2 \circ 2 = \{0,2\}$

By Theorem 8.15(ii,iii) we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 1 \neq \{0,2\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

1.4.2: $2 \circ 2 = \{0,1,2\}$

By Theorem 2.1.15(ii) we have $2 \circ 1 \not\subseteq \{0,1\}$. If $2 \circ 1 = \{0,2\}$, then by Theorem 8.12 we have $2 \circ 0 = \{2\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we consider the following case:

2: $1 \circ 1 = \{0,1\}$
This case has two subcases $0 \circ 0 = \{0\}$ or $\{0,1\}$.
2.1: $0 \circ 0 = \{0\}$
We consider the following subcases as :
2.1.1: $0 \circ 1 = \{0\}$
In this case also we consider four states as follows:
2.1.1.1: $2 \circ 2 = \{0\}$
By Theorems 8.10 and 8.11 we have $2 \circ 0 = \{2\}$ and $1 \notin 2 \circ 1$. If $2 \circ 1 \subseteq \{0,2\}$, then by Theorem 8.16 we have $0 \circ 2 = \{0\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

2.1.1.2: $2 \circ 2 = \{0,1\}$
If $2 \circ 1 = \{0\}$, then by Theorem 8.17(i,ii) we have $0 \circ 2 = \{0,1\}$ and $2 \circ 0 = \{2\}$.

So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

2.1.1.3: $2\circ 2=\{0,2\}$
If $2\circ 1\subseteq\{0,2\}$, then by Theorem 8.16 we have $0\circ 2=\{0\}$. If $2\circ 1=\{0\}$, then by Theorem 8.17(ii) we have $2\circ 0=\{2\}$. So there exist 11 hyper K-algebras

as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

2.1.1.4: $2 \circ 2 = \{0,1,2\}$

If $2 \circ 1 = \{0\}$, then by Theorem 8.17(i,ii) we have $0 \circ 2 = \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

2.1.2: $0 \circ 1 = \{0,1\}$
In this case also we consider four states as follows:
2.1.2.1: $2 \circ 2 = \{0\}$
By Theorem 2.1.10 we have $2 \circ 0 = \{2\}$. If $0 \circ 2 = \{0\}$, then by Theorem 8.13(i) we have $2 \circ 1 \not\subseteq \{0,2\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

2.1.2.2: $2 \circ 2 = \{0, 1\}$
If $2 \circ 1 = \{0\}$, then by Theorem 8.17(i,ii) we have $0 \circ 2 = \{0, 1\}$ and $2 \circ 0 = \{2\}$. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

2.1.2.3: $2 \circ 2 = \{0, 2\}$
If $2 \circ 1 = \{0\}$, then by Theorem 8.17(ii,iii) we have $0 \circ 2 = \{0, 1\}$ and $2 \circ 0 = \{2\}$ If $2 \circ 1 = \{0, 2\}$, then by Theorem 8.18(i) we have $0 \circ 2 = \{0, 1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.1.2.4: $2\circ 2=\{0,1,2\}$
This case is similar to the case of 2.1.2.2. So there exist 13 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.1.3: $0 \circ 1 = \{0,2\}$
In this case we have two states since by Theorem 8.14(i) we obtain $2 \circ 2 \not\subseteq \{0,1\}$.
2.1.3.1: $2 \circ 2 = \{0,2\}$
If $2 \circ 1 = \{0,2\}$, then by Theorem 8.16 we have $0 \circ 2 = \{0\}$. By Theorem 2.1.14(ii,iv) we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 0 = \{2\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

2.1.3.2: $2 \circ 2 = \{0,1,2\}$
By Theorem 8.14(ii,iv,v) we must have $2 \circ 1 \not\subseteq \{0,1\}$, $2 \circ 0 = \{2\}$ and

$0 \circ 2 = \{0, 1\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.1.4: $0 \circ 1 = \{0, 1, 2\}$
By Theorem 8.15(i) we have $2 \circ 2 \not\subseteq \{0, 1\}$, thus in this case we have only two states as follows:
2.1.4.1: $2 \circ 2 = \{0, 2\}$
By Theorem 8.15(ii) we have $2 \circ 1 \not\subseteq \{0, 1\}$. If $2 \circ 1 = \{0, 2\}$, then by Theorem 8.18(i) we have $0 \circ 2 = \{0, 1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

2.1.4.2: $2 \circ 2 = \{0, 1, 2\}$
By Theorem 8.15(ii) we have $2 \circ 1 \not\subseteq \{0, 1\}$. So there exist eight hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we consider the following case:

2.2: $0 \circ 0 = \{0,1\}$

We consider some subcases as follows:

2.2.1: $0 \circ 1 = \{0\}$

In this case also we consider four states as follows:

2.2.1.1: $2 \circ 2 = \{0\}$

By Theorems 8.19 and 2.1.11 we have $2 \circ 0 = \{1,2\}$ and $1 \notin 2 \circ 1$. Since $1 \notin 2 \circ 1$, hence $2 \circ 1 \subseteq \{0,2\}$ and by Theorem 8.16 we have $0 \circ 2 = \{0\}$. So there exist two hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0\}$

2.2.1.2: $2 \circ 2 = \{0,1\}$

If $2 \circ 1 \subseteq \{0,2\}$, then by Theorems 8.17(iv) and 8.18(ii) we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 = \{0\}$, then by Theorem 8.17(i) we have $0 \circ 2 = \{0,1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

2.2.1.3: $2 \circ 2 = \{0,2\}$

By Theorem 8.19 we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 \subseteq \{0,2\}$, then by Theorem 8.16 we have $0 \circ 2 = \{0\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.2.1.4: $2 \circ 2 = \{0, 1, 2\}$
This case is similar to the case of 2.2.1.2. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

2.2.2: $0 \circ 1 = \{0, 1\}$
Consider the following four states:
2.2.2.1: $2 \circ 2 = \{0\}$
By Theorem 8.19 we have $2 \circ 0 = \{1, 2\}$. If $2 \circ 1 \subseteq \{0, 2\}$, then by Theorems 8.17(iii) and 8.18(i) we have $0 \circ 2 = \{0, 1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0\}$

2.2.2.2: $2\circ 2=\{0,1\}$
If $2\circ 1\subseteq\{0,2\}$, then by Theorems 8.17(iv) and 8.18(ii) we have $2\circ 0=\{1,2\}$. If $2\circ 1=\{0\}$, then by Theorem 8.17(iii) we have $0\circ 2=\{0,1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1\}$

2.2.2.3: $2 \circ 2 = \{0,2\}$
By Theorem 8.19 we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 \subseteq \{0,2\}$, then by Theorems 8.18(i) and 8.17(iii) we have $0 \circ 2 = \{0,1\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,2\}$

2.2.2.4: $2 \circ 2 = \{0,1,2\}$
If $2 \circ 1 \subseteq \{0,2\}$, then by Theorems 8.17(iv) and 8.18(ii) we have $2 \circ 0 = \{1,2\}$. If $2 \circ 1 = \{0\}$, then by Theorem 8.17(i) we have $0 \circ 2 = \{0,1\}$. So there exist 11 hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

2.2.3: $0 \circ 1 = \{0,2\}$
By Theorem 8.14(i) we have $2 \circ 2 \not\subseteq \{0,1\}$. Thus this case has only two states as follows:
2.2.3.1: $2 \circ 2 = \{0,2\}$
By Theorems 8.14(ii) and 8.19 we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 0 = \{1,2\}$. If $2 \circ 1 = \{0,2\}$, then by Theorem 8.16 we have $0 \circ 2 = \{0\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.2.3.2 $2 \circ 2 = \{0,1,2\}$.
By Theorem 8.14(ii,v) we have $2 \circ 1 \not\subseteq \{0,1\}$ and $0 \circ 2 = \{0,1\}$. If $2 \circ 1 = \{0,2\}$, then by Theorem 8.18(ii) we have $2 \circ 0 = \{1,2\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

2.2.4: $0 \circ 1 = \{0,1,2\}$
By Theorem 8.15(ii) we have $2 \circ 2 \not\subseteq \{0,1\}$. Thus this case has only two states

as follows:
2.2.4.1: $2 \circ 2 = \{0,2\}$
By Theorems 8.15(ii) and 2.1.19 we have $2 \circ 1 \not\subseteq \{0,1\}$ and $2 \circ 0 = \{1,2\}$. If $2 \circ 1 = \{0,2\}$, then by Theorem 8.18(i) we have $0 \circ 2 = \{0,1\}$. So there exist three hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

2.2.4.2: $2 \circ 2 = \{0,1,2\}$
By Theorem 8.15(ii) we have $2 \circ 1 \not\subseteq \{0,1\}$. If $2 \circ 1 = \{0,2\}$, then by Theorem 8.18(ii) we have $2 \circ 0 = \{1,2\}$. So there exist six hyper K-algebras as follows:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Now we show that each pair of the above 220 hyper K-algebras are not isomorphic together. On the contrary let $(H_1, \circ_1, 0)$ and $(H_2, \circ_2, 0)$ be isomorphic. then there exists an isomorhpism $f : H_1 \to H_2$. So $f(x \circ_1 y) = f(x) \circ_2 f(y)$, for all $x, y \in H$, thus we have $f(0_1) = 0_2$, $f(1) = 2$, $f(2) = 1$. But $f(1 \circ_1 2) = f(\{1\}) = \{2\}$ and $f(1) \circ_2 f(2) = 2 \circ 1 \supseteq \{0\}$, which is a contradiction, since $0 \notin f(1 \circ_1 2) = \{2\}$. □

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