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SPECTRUM OF THE FOURIER-STIELTJES ALGEBRA OF A SEMIGROUP

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Abstract. For a unital foundation topological $*$-semigroup S whose representations separate points of S, we show that the spectrum of the Fourier-Stieltjes algebra $B(S)$ is a compact semitopological semigroup. We also calculate $B(S)$ for several examples of S.

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1. Introduction

In [3] Lau studied the subalgebra $F(S)$ of $\mathrm{WAP}(S)$ of a topological semigroup S with involution. If G is an abelian topological group, then $F(G) \simeq M(\hat{G})$ where $\hat{G}$ is the dual group of G. If S is a topological $*$- semigroup with an identity, then $F(S)$ is the linear span of positive definite functions on S. The authors introduced and studied Fourier and Fourier-Stieltjes algebras $A(S)$ and $B(S)$ of a foundation topological $*$-semigroup S in [1]. When S is unital, $B(S) = F(S)$.

Let S be a locally compact topological semigroup and $M(S)$ be the Banach algebra of all bounded regular Borel measures on S. We consider the mappings

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L_μ and R_μ of S into $M(S)$ defined by

$$L_\mu(x) = \mu * \delta_x, \quad R_\mu(x) = \delta_x * \mu \quad (x \in S, \mu \in M(S)),$$

where δ_x is the point mass at x. Then the semigroup algebra $L(S)$ consists of those $\mu \in M(S)$ for which $L_{|\mu|}$ and $R_{|\mu|}$ are continuous with respect to the weak topology of $M(S)$, and $L(S)$ is a Banach subalgebra of $M(S)$. The semigroup S is called foundation if $\cup\{\text{supp}(\mu) : \mu \in L(S)\}$ is dense in S [6].

A representation of S is a pair $\{\pi, H_\pi\}$ of a Hilbert space H_π and a semigroup homomorphism $\pi : S \to B(H_\pi)$ such that π is (weakly) continuous, i.e. the mappings $x \mapsto \langle \pi(x)\xi, \eta\rangle$ are continuous on S, for all $\xi, \eta \in H_\pi$, and that π is bounded if $\|\pi\| = \sup_{x\in S}\|\pi(x)\| < \infty$. Also π is called a $*$-representation if moreover $\pi(x^*) = \pi(x)^*(x \in S)$, where the right hand side is the adjoint operator. A $*$-representation $\{\sigma, H\}$ of $L(S)$ is called non-vanishing if for every $0 \neq \xi \in H$, there exists $\mu \in L(S)$ with $\sigma(\mu)\xi \neq 0$. Let $\Sigma(L(S))$ be the family of all $*$-representations of $L(S)$ on a Hilbert space which are non-vanishing, and $\Sigma(S)$ be the family of all continuous $*$-representations π of S with $\|\pi\| \leq 1$, then one has a bijective correspondence between $\Sigma(S)$ and $\Sigma(L(S))$ via

$$\langle \tilde{\pi}(\mu)\xi, \eta\rangle = \int_S \langle \pi(x)\xi, \eta\rangle d\mu(x) \quad (\mu \in L(S),\ \xi, \eta \in H_\pi = H_{\tilde{\pi}}).$$

Given $\rho \subseteq \Sigma = \Sigma(S)$ and $\mu \in L(S)$, define $\|\mu\|_\rho = \sup\{\|\tilde{\pi}(\mu)\| : \pi \in \rho\}$ and $I_\rho = \{\mu \in L(S) : \|\mu\|_\rho = 0\}$. Then I_ρ is clearly a closed two-sided ideal of $L(S)$ and $\|\mu + I_\rho\| = \|\mu\|_\rho$ defines a C^*-norm on $L(S)/I_\rho$. The completion of this quotient space in this norm is a C^*-algebra which is denoted by $C^*_\rho(S)$. When $\rho = \Sigma$, then the C^*-algebra $C^*(S) = C^*_\Sigma(S)$ is called the (full) semigroup C^*-algebra of S. If S is foundation and Σ separates the points of S, then $L(S)$ is $*$-semisimple and so $I_\Sigma = \{0\}$. In this case $L(S)$ is a norm dense subalgebra of $C^*(S)$ (see [1] for more details).

A complex valued function $u : S \longrightarrow \mathbb{C}$ is called positive definite if for all positive integers n and all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, and $x_1, x_2, \ldots, x_n \in S$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \bar{\lambda}_j u(x_i x_j^*) \geq 0.$$

Let $P(S)$ denotes the set of all continuous positive definite functions on S. We denote the linear span of $P(S)$ by $B(S)$ and call it the Fourier-Stieltjes algebra of S. Let S be a topological *-semigroup and $C_c(S)$ be the algebra of all continuous functions on S with compact support. Then the closed subalgebra $\overline{(B(S) \cap C_c(S))} \subseteq B(S)$ is denoted by $A(S)$ and is called the Fourier algebra of S.

2. Fourier-Stieljes algebra

It is well known that for an abelian topological group G, the Fourier and Fourier-Stieltjes algebras $A(G)$ and $B(G)$ are isometrically isomorphic to the group and measure algebras $L^1(\hat{G})$ and $M(\hat{G})$ of the dual group $\hat{G}$. For a class of commutative foundation topological $*$-semigroup with identity we show that

$B(S)$ is isometrically isomorphic to $M(\hat{S})$. Here $\hat{S}$ is the set of continuous semi-characters on S which is a locally compact topological semigroup [3].

Theorem 2.1. *Let S be a commutative foundation topological $*$-semigroup with identity. For $\lambda \in L(\hat{S})$, define $\hat{\lambda} : S \to \mathbb{C}$ by*

$$\hat{\lambda}(x) = \int_{\hat{S}} \chi(x) d\lambda(\chi) \quad (x \in S).$$

Then the map $\lambda \mapsto \hat{\lambda}$ is a continuous monomorphism from $L(\hat{S})$ into $B(S)$.

Proof. $\hat{S}$ is a locally compact topological semigroup [3]. Also for each $\lambda \in L(\hat{S})$ there is a probability measure γ on $\hat{S}$ and $\phi \in L^1(\hat{S}, \gamma)$ such that $d\lambda = \phi d\gamma$. We can decompose ϕ as

$$\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4),$$

where $\phi_i \geq 0$, for $i = 1, \ldots, 4$. Put $d\lambda_i = \phi_i d\gamma$. Then for each $n \geq 1$, $c_1, \ldots, c_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in S$,

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \hat{\lambda}_k(x_i x_j^*) = \int_{\hat{S}} \sum_{i,j=1}^{n} c_i \bar{c}_j \hat{\chi}(x_i x_j^*) d\lambda_k(\chi) \geq 0,$$

for $k = 1, \ldots, 4$. Next we show that $\hat{\lambda}_k$ is also continuous. Given $\varepsilon > 0$, there is a measurable subset $K \subseteq \hat{S}$ such that

$$\int_{\hat{S} \setminus K} \phi_k(\chi) d\gamma(\chi) < \varepsilon.$$

By Ascoli's Theorem, K is equicontinuous. Now given $x_0 \in S$, there is a neighborhood U of x_0 in S such that

$$|\chi(x) - \chi(x_0)| < \varepsilon \quad (\chi \in K, x \in U).$$

For each $x \in U$,

$$\begin{aligned}
|\hat{\lambda}_k(x) - \hat{\lambda}_k(x_0)| &\leq \int_{\hat{S}} |\chi(x) - \chi(x_0)| d\lambda_k(\chi) \\
&\leq \int_K |\chi(x) - \chi(x_0)| d\lambda_k(\chi) + \int_{\hat{S} \setminus K} |\chi(x) - \chi(x_0)| d\lambda_k(\chi) \\
&\leq \varepsilon \lambda_k(K) + 2\varepsilon \leq (2 + \lambda_k(\hat{S}))\varepsilon.
\end{aligned}$$

This shows that $\hat{\lambda}_k \in P(S)$, for $k = 1, \ldots, 4$, and so $\hat{\lambda} \in B(S)$. Next we have

$$\begin{aligned}
\|\hat{\lambda}\|_{B(S)} = sup\Big| \int_S \hat{\lambda}(x) d\mu(x) \Big| &= sup\Big| \int_S \int_{\hat{S}} \chi(x) d\lambda(\chi) d\mu(x) \Big| \\
&\leq \int_{\hat{S}} \Big| \int_S \chi(x) d\mu(x) \Big| d|\lambda|(\chi) \leq \|\lambda\|,
\end{aligned}$$

where the supremum is taken over all $\mu \in L(S)$ with $\|\mu\|_\Sigma \leq 1$ (see [1]). Also the last inequality follows from the fact that each semi-character $\chi \in \hat{S}$ could be regarded as a representation of S.

When λ is positive, we also have

$$\|\lambda\| = \int_{\hat{S}} \chi(e) d\lambda(\chi) = \hat{\lambda}(e) \leq \|\hat{\lambda}\|_{B(S)},$$

since $\chi(e) = 1$, for each $\chi \in \hat{S}$, where e is the identity of S. In general, $\lambda = (\lambda_1 - \lambda_2) + i(\lambda_3 - (\lambda_4)$, with λ_k's positive, and we have

$$\|\lambda\| \leq \sum_{i=1}^{4} \|\lambda_i\| \leq \sum_{i=1}^{4} \|\hat{\lambda}_i\|.$$

In particular the map $\lambda \mapsto \hat{\lambda}$ is injective.

Finally, for $\lambda, \mu \in L(S)$ and $x \in S$ we have

$$(\lambda * \mu)\hat{}(x) = \int_S \chi(x) d(\lambda * \mu)(\chi) = \int_S \int_S \chi(x)\zeta(x) d\lambda(\chi) d\mu(\zeta) = \hat{\lambda}(x)\hat{\mu}(x),$$

and we are done. □

Remark 2.2. In the group case, the range of the above map is $A(S)$. We don't know if this is the case for foundation semigroups.

Following [4] we say that S is of type $\mathcal{U}$ if it has a dense subsemigroup U which is a union of groups. Then to each $x \in U$ there corresponds an element $x^{'} \in U$ (the inverse of x in the group to which x belongs) such that $xx^{'}$ and $x^{'}x$ are idempotents and

$$xx^{'}x = x, \quad x^{'}xx^{'} = x^{'}.$$

In [5] a concept of positive definite functions is defined for semigroups of type $\mathcal{U}$. We denote the set of positive definite functions on U by $P(U)$. When U is an increasing union or a disjoint union of groups, this element $x^{'}$ is unique for each $x \in U$. When the latter holds and the map $x \mapsto x^{'}$ is continuous we say that S is of type $\bar{\mathcal{U}}$. In this case the map $x \mapsto x^{'}$ on U extends to a continuous map $x \to x^*$ on S and S becomes a topological $*$-semigroup. In this case we can talk about positive definite functions on S in the sense of section 1. If U is an increasing union or a disjoint union of groups, each open in S, then S is of type $\bar{\mathcal{U}}$. If S is of type $\bar{\mathcal{U}}$, then it is easy to see that for each $f \in C_b(S)$, $f \in P(S)$ if and only if $f_{|U} \in P(U)$. In particular for a unital commutative semigroup S of type $\bar{\mathcal{U}}$ we have $B(S) = R(S)$ [5, 7.2.5]. Now the following result follows from [5] immediately.

Proposition 2.3. *If S is a commutative foundation $*$-semigroup of type $\bar{\mathcal{U}}$ with identity, then the map $\lambda \mapsto \hat{\lambda}$ is a linear isometry of $M(\hat{S})$ onto $B(S)$.*

Note that if we consider the semigroup of integers $\mathbb{Z}$ with trivial involution $n^* = n$, then we have $B(\mathbb{Z}) \neq R(\mathbb{Z})$ [7].

3. Spectrum of the Fourier algebra

In this section we show that for a unital foundation topological $*$-semigroup S, the spectrum of $B(S)$ is a compact unital semitopological semigroup. Let S be a unital foundation topological $*$-semigroup with identity e and $\Omega = \Omega(S)$ be the family of all continuous $*$-representations ω of S in a W*-algebra M_ω with $\|\omega\| \leq 1$. Let ω_Ω be the universal representation of S in the ℓ^∞ direct sum $M_\Omega = \sum_{\omega\in\Omega} \oplus M_\omega$. Then the predual $(M_\Omega)_*$ is the ℓ^1 direct sum $\sum_{\omega\in\Omega} \oplus (M_\omega)_*$ and for each $\psi \in (M_\Omega)_*$ we have $u = \psi \circ \omega_\Omega \in B(S)$ and $\|u\| \leq \|\psi\|$ [1, 3.1, 3.4], [7].

For $u \in B(S)$ and $x, y \in S$ let $u_x(y) = u(yx)$ then $u_x \in B(S)$ with $\|u_x\| \leq \|u\|$ [1, 3.4]. This means that the right translation operators $\tau_x : B(S) \to B(S)$ defined by

$$\tau_x(u) = u_x \quad (x \in S, u \in B(S)),$$

are bounded with $\|\tau_x\| \leq 1$.

Definition 3.1. For $u \in B(S)$ and $f \in B(S)^* = W^*_\Omega(S)$ define $E_f(u) : S \to \mathbb{C}$ by

$$E_f(u)(x) = \langle f, u_x\rangle \quad (x \in S).$$

Lemma 3.2. *For $f \in W^*_\Omega(S)$, $E_f : B(S) \to B(S)$ is a bounded linear operator which commutes with right translation operators and $\|E_f\| = \|f\|$.*

Proof. Let $u \in B(S)$ and choose $\psi \in (M_\Omega)_*$ with $u = \psi \circ \omega_\Omega$ and $\|u\| = \|\psi\|$, then $u(x) = \langle \omega_\Omega(x), \psi\rangle$, for $x \in S$. Given $\zeta \in (M_\Omega)_*$ and $m \in M_\Omega$ define $\zeta.m \in (M_\Omega)_*$ by $\langle n, \zeta.m\rangle = \langle mn, \zeta\rangle$ for $n \in M_\Omega$. Also $m.\zeta$ is defined similarly. For each $x, y \in S$,

$$u_x(y) = u(yx) = \langle \omega_\Omega(yx), \psi\rangle = \langle \omega_\Omega(y), \psi.\omega_\Omega(x)\rangle,$$

hence $u_x = (\psi.\omega_\Omega(x)) \circ \omega_\Omega$. To each $f \in W^*_\Omega(S)$ there corresponds $f^\circ \in M_\Omega$ defined by $\langle f^\circ, \zeta\rangle = \langle f, \zeta \circ \omega_\Omega\rangle$, for $\zeta \in (M_\Omega)_*$. Then

$$E_f(u)(x) = \langle f, u_x\rangle = \langle f, (\psi.\omega_\Omega(x)) \circ \omega_\Omega\rangle = \langle f^\circ, \psi.\omega_\Omega(x)\rangle = \langle \omega_\Omega(x), f^\circ.\psi\rangle,$$

so $E_f(u) = (f^\circ.\psi) \circ \omega_\Omega \in B(S)$ with $\|E_f(u)\| \leq \|f^\circ.\psi\| \leq \|u\|\,\|f\|$, that is $\|E_f\| \leq \|f\|$. On the other hand $|\langle f, u\rangle| = |E_f(u)(e)| \leq \|E_f(u)\| \leq \|E_f\|\,\|u\|$, hence $\|E_f\| = \|f\|$. Finally, for $x, y \in S$,

$$(E_f(u))_x(y) = E_f(u)(yx) = \langle f, u_{yx}\rangle = \langle f, (u_x)_y\rangle = E_f(u_x)(y),$$

and so E_f commutes with right translation operators. □

Let $L(B(S))$ be the space of all bounded linear operators on $B(S)$ and $L_0(B(S))$ be the closed subspace of $L(B(S))$ consisting of those operators which commute with all right translation operators τ_x on $B(S)$.

Theorem 3.3. *Let S be a unital foundation topological $*$-semigroup with identity e, then $B(S)^*$ is isometrically isomorphic to $L_0(B(S))$ and $B(S)\hat{}$ is homeomorphic to the space $End(L_0(B(S)))$ consisting of non-zero endomorphisms of $L_0(B(S))$. In particular $B(S)\hat{}$ is a compact unital semitopological semigroup.*

Proof. By above lemma, the map $f \mapsto E_f$ is an isometric isomorphism from $B(S)^*$ into $L_0(B(S))$. Given $E \in L_0(B(S))$ define $f \in B(S)^*$ by $\langle f, u\rangle = E(u)(e)$, for $u \in B(S)$. Then

$$E_f(u)(x) = \langle f, u_x\rangle = E(u_x)(e) = E(u)_x(e) = E(u)(x),$$

for $x \in S$ and $u \in B(S)$. Therefore $E_f = E$. Now it is easy to check that f is multiplicative if and only if E_f is an endomorphism. Next $B(S)^*$ is isomorphic with the w^*-closed linear span of $\{\omega_\Omega(x) : x \in S\}$ in M_Ω [1, 2.1]. Now for each net $\{f_\alpha\} \subseteq B(S)^*$, $E_{f_\alpha} \to E_f$ in WOT if and only if $E_{f_\alpha}(u) \to E_f(u)$ weakly, for each $u \in B(S)$, that is $\langle m, E_{f_\alpha}(u)\rangle \to \langle m, E_f(u)\rangle$, for $m \in B(S)^*$, which in turn is equivalent to $\langle f_\alpha^\circ, \psi.m\rangle = \langle m, f_\alpha^\circ.\psi\rangle \to \langle f^\circ, \psi.m\rangle = \langle m, f^\circ.\psi\rangle$, for $m \in B(S)^*$ and $\psi \in (M_\Omega)_*$. But $B(S)$ is unital and so $(M_\Omega)_*.B(S) = (M_\Omega)_*$, hence the latter is equivalent to $\langle f_\alpha^\circ, \psi\rangle \to \langle f, \psi\rangle$, for $\psi \in (M_\Omega)_*$, that is $f_\alpha \to f$ in w^*-topology. □

4. Examples

In this section we calculate the algebras $A(S)$ and $B(S)$ in various examples. One class of examples are semigroups of type $\mathcal{U}$ [4].

The following example shows that the existence of an identity is needed in Proposition 2.3.

Example 4.1. Let $S = \mathbb{N} \cup \{0\}$ with discrete topology and multiplication $n.m = \delta_{nm} n$, for $n, m \in S$. Then each singleton $\{n\}$ is the trivial group and S is of type $\bar{\mathcal{U}}$. In this case $R(S) = \ell^1(\mathbb{N}) \cup \mathbb{C}$ [5, 3.1.6], whereas $B(S) = span\{f \in c_b(S) : f(n) \geq f(0) \geq 0\}$.

Example 4.2. Let S be the unit ball of $L^\infty(\Omega, \mu)$ with pointwise multiplication and w^*-topology. We assume that μ is a finite measure on Ω. Put

$$U = \{f \in S : |f| = 1\, or\, 0\}.$$

In this case $f' = \bar{f}$ if $f \neq 0$ and $0' = 0$. We claim that the map $f \mapsto f' = \bar{f}$ is continuous on U. Let $f_\alpha \to f$ in w^*-topology, i.e.

$$\int_\Omega g f_\alpha d\mu \to \int_\Omega g f d\mu \quad (g \in L^1(\Omega, \mu)).$$

Then we have

$$\int_\Omega g(\bar{f}_\alpha - \bar{f}) d\mu = \Big(\int_\Omega \bar{g}(\bar{f}_\alpha - \bar{f}) d\mu\Big)^{-} \to 0,$$

for each $g \in L^1(\Omega, \mu)$. This shows that S is of type $\bar{\mathcal{U}}$. In particular $B(S) = R(S)$.

Example 4.3. Let $S = G \cup \{\infty\}$ be a one-point compactification of a locally compact group G. If $\{g_\alpha\}$ is a net in G and $g_\alpha \to \infty$ in S, then $g_\alpha^{-1} \to \infty$ in S. If $g_\alpha \to g$ in G then $g_\alpha^{-1} \to g^{-1}$ in G. Hence S is of type $\bar{\mathcal{U}}$. Also S is unital with identity ∞. If G is abelian, then $B(S) = R(S) = M_0(\hat{G})\hat{} \oplus \mathbb{C}$, where $M_0(\hat{G}) = \{\mu \in M(\hat{G}) : \hat{\mu} \in C_0(G)\}$ [5, 5.1.3].

Example 4.4. Let $S = ([0,1], max)$ with involution $x^* = x$. Then S is a compact abelian unital semigroup and $\hat{S}$ is an idempotent semigroup. Indeed

$$\hat{S} = \{\chi_{[0,x]} : x \in S\}.$$

In particular $\hat{S}$ separates the points of S (and so does $\Sigma(S)$.) Also

$$L^1(S) = \{f : S \to \mathbb{C} : f \text{ measurable and } \int_0^1 |f(x)|dx < \infty\}$$

is a Banach algebra with convolution

$$f * g(x) = f(x) \int_0^x g(t)dt + g(x) \int_0^x f(t)dt.$$

$L^1(S)$ has a bounded approximate identity. Let $f : S \to \mathbb{C}$ be positive definite, then

$$\sum_{i,j=1}^{n} c_i \bar{c}_j f(x_i x_j^*) \geq 0,$$

for each $n \geq 1, c_1, \ldots, c_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in S$. Once put $n = 1, c_1 = 1$, and $x_1 = x$, and then put $n = 2, c_1 = c_2 = \sqrt{-1}$, and $x_1 = x, x_2 = y$ to get

$$f(x) \geq 0, \ f(x) - 2f(xy) + f(y) \geq 0,$$

for each $x, y \in S$. This shows that f is non-negative and non-increasing. Conversely all such functions are positive definite, and so $A(S) = B(S) = BV[0,1]$. In particular $A(S)$ is regular and natural [4, 4.4.35]. Also $B(S)$ is not a dual space [5]. Note that in this case S is not foundation [7] (compare with [1].) The convolution product of two elements in $L^2(S)$ is defined as above. In particular for $g(x) = 1$ and

$$f(x) = \begin{cases} xsin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0, \end{cases},$$

we have $f, g \in L^2(S)$, but

$$f * g(x) = x^2 sin(\frac{1}{x}) + \int_0^x t sin(\frac{1}{t})dt,$$

for $x \neq 0$ and $f * g(0) = 0$. It is easy to see that $f * g \notin BV[0,1]$. In particular $A(S) \neq L^2(S) * L^2(S)$.

Example 4.5. Let $S = (\mathbb{R}^+, +)$ with involution $x^* = x$. Then S is a locally compact commutative unital $*$-semigroup. If $f : S \to \mathbb{C}$ is continuous and positive definite, then in the corresponding inequality, once put $n = 1, c_1 = 1$, and $x_1 = \frac{x}{2}$, and then put $n = 2, c_1 = 1, c_2 = -1$, and $x_1 = \frac{x}{2}, x_2 = \frac{y}{2}$ to get

$$f(x) \geq 0, \ f(x) - 2f(\frac{x}{2} + \frac{y}{2}) + f(y) \geq 0,$$

for each $x, y \in S$. This shows that f is non negative and convex. Conversely we know that $\hat{\mathbb{R}}^+ \simeq \mathbb{R}^+$ [4] and we have the Laplace transform

$$\hat{\mu}(x) = \int_0^\infty e^{-xt} d\mu(t),$$

for $\mu \in M(\mathbb{R}^+)$, and these are exactly the elements of $B(\mathbb{R}^+)$ [2].

Example 4.6. Let $S = (\mathbb{N} \cup \{0\}, +)$ with involution $x^* = x$. Then S is a discrete abelian unital semigroup. If $f : S \to \mathbb{C}$ is positive definite, then in the corresponding inequality, once put $n = 1, c_1 = 1$, and $x_1 = n$, and then put $n = 2, c_1 = c_2 = 1$ and $x_1 = 0, x_2 = n$, or $c_1 = 1, c_2 = -1$ and $x_1 = m, x_2 = n$ to get

$$f(2n) \geq 0,\ f(0) - 2f(n) + f(2n) \geq 0,\ f(2m) - 2f(m+n) + f(2n) \geq 0,$$

for each $m, n \in S$. It follows from the first and second inequality that f is real valued. In this case $\hat{S} \simeq [-1, 1]$ with multiplication. Hence $B(S) \simeq M[-1, 1]$.

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