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BLOW-UP AND NONGLOBAL SOLUTION FOR A FAMILY OF NONLINEAR HIGHER-ORDER EVOLUTION PROBLEM

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ABSTRACT. In this paper we consider a kind of higher-order evolution equation as $\frac{\partial^k u}{\partial t^k}+\frac{\partial^{k-1}u}{\partial t^{k-1}}+\cdots+u_t-\triangle u=f(u,\bigtriangledown u,x)$. For this equation, we investigate nonglobal solution, blow-up in finite time and instantaneous blow-up under some assumption on k, f and initial data. In this paper we employ the Test function method, the Generalized convexity method and Galerkin's method for some of our proofs. Moreover, occasionally by changing P.D.E problems to some ordinary differential inequalities, we investigate this kind of higher-order evolution equations.

Keywords: Higher-order evolution equations, blow-up, nonglobal solution, instantaneous blow-up.

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1. INTRODUCTION

In the present paper, we investigate, blow-up in finite time, instantaneous blow-up and nonglobal solution for some nonlinear higher-order evolution equations, as follows

$$\frac{\partial^k u}{\partial t^k}+\frac{\partial^{k-1} u}{\partial t^{k-1}}+\cdots+u_t-\triangle u=f(u,\bigtriangledown u,x), \tag{1.1}$$

and some other similar equations.

In the case $k=1$, there are many results for blow-up and nonglobal solutions with Dirichlet boundary condition whenever $f(u)=u|u|^{q-1}$ or $f(u)=au|u|^{q-1}+b|\bigtriangledown u|^p$with $q,p>1$ ([5], [6], [7], [9], [10], [15]). In the case $k=2$, Souplet in [16] obtained some results for blow-up, when $f(u)=|u|^q+\lambda u,(q>1)$. On the other hand, Souplet has proved the nonglobal character of the solution for this problem, whenever the initial data have positive projection on the first eigenvector of $-\triangle$ operator. We are going to extend and complete these results. In the case $k\geq 1$, Laptev in [12] Considered a similar inequality, as follows

$$\frac{\partial^k u}{\partial t^k}-\triangle u\geq |x|^{\sigma}|u|^q,$$

and he proved that in the complement of a ball $\Omega=\mathbb{R}^N\backslash B_R$ for $\sigma>-2$, there exists a critical exponent q^* such that for $1<q\leq q^*$ the last inequality has no global solution under some assumption on the initial data. Moreover in the case $\Omega=B_R$ for $\sigma\leq -2$ he showed that for $1<q<\infty$ this inequality has no global solution. Laptev in [13] for the following inequalities

$$\frac{\partial^k u}{\partial t^k}-\triangle u^m\geq |u|^q \quad \frac{\partial^k u}{\partial t^k}-\triangle u\geq |x|^{\sigma}|u|^q \quad \frac{\partial^k u}{\partial t^k}-div(|x|^{\alpha}Du)\geq |u|^q$$

with $k=1,2,\ldots$ in a cone-like domains obtained a critical exponent q^* such that for $1<q\leq q^*$ the last inequalities have no global solution. In the present paper some of our result about these inequalities are independent on the geometry of the domain.

On the other hand, in this paper, we have investigated the equation (1.1) and similar equations and inequalities comprehensively for $k\geq 1$ and we obtain many results for blow-up in finite time, instantaneous blow-up and nonglobal solution under some suitable data.

In Section 2, we consider the equation (1.1) whenever $f(u)=|u|^q$ and Ω is a smooth domain in $\mathbb{R}^N$ (possibly unbounded), then we show that if the initial data are nonnegative and one of them is very large, then the solution cannot be global. Moreover for this problem with Neuman boundary condition on the bounded domain, the global solution does not exist if $\sum_{i=1}^{k}\int_\Omega \frac{\partial^{i-1}}{\partial t^{i-1}}u(0,x)>0$. In our proofs in this section, we will take advantage from a type of the Test function method.

In Section 3, we consider some ordinary differential inequalities of the following forms for $t>0$.

$$z^{(k)}+z^{(k-1)}+\cdots+z'\geq cz^q, \tag{P_1}$$

$$z^{(k)}+\lambda z\geq cz^q, \tag{P_2}$$

$$z^{(k)}+z^{(k-1)}+\lambda z\geq cz^q, \tag{P_3}$$

where c and λ are positive constants, and $z\geq 0$ is a real valued function in $C^k(R)$. For these inequalities, we investigate the nonexistence of global solution and blow-up in finite time. On the other hand, we show that for (P_1), if $\sum_{i=1}^{k} z(0)^{(i-1)} > 0$ then the solution blows-up in finite time. For (P_2) and (P_3), we show that if the initial data are nonnegative and $z(0)$ is large enough then the solution blows-up in finite time.

By using the results of Section 3, in Section 4, we prove the nonglobal character of the classical solutions for some nonlinear higher-order evolution equations of type (1.1). In the following there are some of them, moreover one can see the rest of these equations in Section 4.

$$\frac{\partial^k u}{\partial t^k} + \cdots + u_t - \triangle u = \lambda u + u^q, \tag{1.2}$$

$$\frac{\partial^{k-1} u}{\partial t^{k-1}} + \cdots + u_t - \triangle u + \lambda u = \int_0^t u^q, \tag{1.3}$$

$$\frac{\partial^k u}{\partial t^k} - \triangle u = u^q, \tag{1.4}$$

$$\frac{\partial^k u}{\partial t^k} + \frac{\partial^{k-1} u}{\partial t^{k-1}} - \triangle u = a|\bigtriangledown u|^p + bu^q \tag{1.5}$$

The basic idea for our proofs in Section 4, relies to use from the first eigenfunction to change P.D.E problems to O.D.E. problems. Indeed, by multiplying the equations by the first eigenfunction of $-\triangle$ operator and integrating over Ω, one can change the above equations to some ordinary differential inequalities. Then we can use the result of the Section 3 to investigate these equations.

In Section 5, we consider the equation (1.1) for $k = 2$ and $f(u) = u^p$. By using the Generalized Convexity method we show that, if

$$\int |\bigtriangledown u(0)|^2 dx + \int |u_t(0)|^2 dx - \frac{2}{p+1} \int u^{p+1}(0) dx < 0,$$

then the nonnegative solution blows-up in finite time. Then By using the Galerkin's method, we show that, if

$$\int |\bigtriangledown u(0)|^2 dx - \frac{2}{p+1} \int u^{p+1}(0) dx \geq 0,$$

and

$$\int |\bigtriangledown u(0)|^2 dx + \int |u_t(0)|^2 dx - \frac{2}{p+1} \int u^{p+1}(0) dx,$$

is positive and small enough, then this problem has a global weak solution.

2. Nonglobal Solution for a Higher-Order Evolution Problem

In this section we consider a higher-order evolution problem as follows,

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} + \cdots + u_t - \triangle u = |u|^q \\[2ex] \dfrac{\partial^i}{\partial t^i} u(0,x) = u_i(x), \quad 0 \leq i \leq k-1, x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$. For this problem, at first we assume that $\frac{\partial^{(i)}}{\partial t^i}u(0,x) = \lambda_i\psi_i$ such that $\psi_i \geq 0$ and $\lambda_i \geq 0$. In this situation we show that if there exists a j such that $\psi_j \not\equiv 0$ in a subdomain Ω' of Ω and λ_j is large, then global solution does not exist. Moreover, when Ω is bounded, we shall see that the above problem with the Neuman boundary condition does not admit a global solution, if $\sum_{i=0}^{k-1}\int_\Omega u_i dx > 0$.

Definition 2.1. By a solution of the problem (2.1) in $Q := \Omega\times(0,T]$ we mean a function $u \in C([0,T];H^1_{loc}(\Omega))$ such that for every test function $\zeta \in C^{\infty,k}_{x,t}(Q)$, $\zeta(.,t) \in C^\infty(\Omega)$, $\zeta(.,T) = \zeta_t(.,T) = \cdots = \frac{\partial^k}{\partial t^k}\zeta(.,T) = 0$ the following equality holds
(2.2)

$$\sum_{j=1}^{k}(-1)^j\int_Q u\frac{\partial^j}{\partial t^j}\zeta - \int_Q(\triangle\zeta)u = \int_Q |u|^p\zeta - \sum_{i=1}^{k}\sum_{j=1}^{i}(-1)^j\int_\Omega \frac{\partial^{(i-j)}}{\partial t^{i-j}}u(0,x)\frac{\partial^{j-1}}{\partial t^{j-1}}\zeta$$
$$+\int_0^T\int_{\partial\Omega}(\zeta\frac{\partial u}{\partial n} - u\frac{\partial\zeta}{\partial n})$$

In our proofs in this section, we will take advantage from the test function method. Test function method is used in different ways. In the first of these, Guedda and Kirane reconfigured the test function method of Pohozaev et. [8]. Their method enable them to find the critical exponent for equations of the form (1.1) for $k = 1$ and $f(u) = u|u|^{p-1}$ as well as the others. The basic idea of the Test function method can be found as far back as in the articles of Baras and Pierre [2] and Baras and Kersner [1].

In the first step,without considering boundary condition, we show that the problem (2.1) may have no global solution when one of the initial data is very large in a ball $B \subset\subset \Omega$.

Theorem 2.2. *Consider Problem (2.1). Let $\frac{\partial^{(i)}}{\partial t^i}u(0,x) = \lambda_i\psi_i$. If there exists a ball $B \subset\subset \Omega$ such that $\lambda_i\psi_i \geq 0$ for $0 \leq i \leq k-1$ and there exists a j with $\psi_j \not\equiv 0$ then there exists a $\Lambda = \Lambda(\Omega, q, \lambda_0\psi_0, \ldots, \lambda_{k-1}\psi_{k-1})$ such that for $\lambda \geq \Lambda$ the solution of the problem (2.1) is not global.*

Proof. Let $B = \{x : |x| < 1\} \subset\subset \Omega$. Let $\zeta = \xi(\tau - t)^\beta$ where $\beta \in \mathbb{N}$ is very large and $0 < \tau < 1$ and $0 \leq \xi \in C^\infty(B)$ is introduced as follows.
Consider $\phi : [0,1] \to [0,1]$ is a decreasing smooth function such that

$$\phi(r) = \begin{cases} 1 & 0 \leq r \leq \frac{2}{3} \\ (1-r)^\sigma & \frac{5}{6} \leq r \leq 1 \end{cases}$$

where σ is a large number. Let $\xi(x) = \phi(|x|)$. Notice that, since σ is large, one can easily deduce that $\xi_{|_{\partial D}} = 0$ and $\frac{\partial\xi}{\partial n} = 0$ on $\partial\Omega$.

Let $v(t) = \int_B \xi u dx$. By the definition of the solution which is introduced in (2.1) and applying integration by parts, we obtain easily

$$\begin{aligned}
& -\sum_{i=1}^{k}\sum_{p=1}^{i}\tau^{\beta-p+1}c_{\beta,p-1}\int_B \frac{\partial^{(i-p)}}{\partial t^{i-p}}u(0,x)\xi(x)\\
& +\sum_{i=1}^{k}c_{\beta,i}\int_0^\tau(\tau-t)^{\beta-i}\int_B \xi u\\
& -\int_0^\tau(\tau-t)^\beta\int_B(\triangle\xi)u = \int_0^\tau(\tau-t)^\beta\int_B|u|^q\xi,
\end{aligned} \tag{2.3}$$

or

$$\begin{aligned}
& -\sum_{i=1}^{k}\sum_{p=1}^{i}\tau^{\beta-p+1}c_{\beta,p-1}\int_B \frac{\partial^{(i-p)}}{\partial t^{i-p}}u(0,x)\xi(x)\\
& +\sum_{i=1}^{k}c_{\beta,i}\int_0^\tau(\tau-t)^{\beta-i}v(t)\\
& -\int_0^\tau(\tau-t)^\beta\int_B(\triangle\xi)u = \int_0^\tau(\tau-t)^\beta\int_B|u|^q\xi,
\end{aligned} \tag{2.4}$$

where $c_{\beta,i} = \beta(\beta-1)\ldots(\beta-i+1)$ and $c_{\beta,0} = 1$.

In this step, we obtain an estimate for the second term to the left-hand side of (2.4) as follows. Let $C(\xi) := (\int_B \xi)^{\frac{q}{q'}}$ where $\frac{1}{q}+\frac{1}{q'} = 1$. By using Hölder's inequality and Young's inequality, we get

$$\begin{aligned}
& C(\xi)c_{\beta,i}\int_0^\tau(\tau-t)^{\beta-i}v(t)\,dt = \int_0^\tau(\tau-t)^{\beta/q}v(t)C(\xi)c_{\beta,i}(\tau-t)^{\beta/q'-i}\\
& \le (\int_0^\tau(\tau-t)^\beta|v|^q)^{1/q}(\int_0^\tau(\tau-t)^{\beta-iq'}C(\xi)^{q'}c_{\beta,i}^{q'})^{1/q'}\\
& = (\int_0^\tau\frac{(\tau-t)^\beta|v|^q}{2k})^{1/q}(\int_0^\tau(2k)^{q'/q}C(\xi)^{q'}c_{\beta,i}^{q'}(\tau-t)^{\beta-iq'})^{1/q'}\\
& \le \frac{1}{2kq}\int_0^\tau(\tau-t)^\beta|v|^q+\frac{C(\xi)^{q'}}{q'}(2k)^{q'/q}c_{\beta,i}^{q'}\frac{\tau^{\beta-iq'+1}}{\beta-iq'+1}.
\end{aligned} \tag{2.5}$$

Here, we choose β so large that $\beta-iq'+1>1$, for $i=1,\ldots,k$.

By multiplying (2.4) with $C(\xi)$ and using (2.5), we obtain

$$\begin{aligned}
& C(\xi)(-\sum_{i=1}^{k}\sum_{p=1}^{i}\tau^{\beta-p+1}c_{\beta,p-1}\int_B \frac{\partial^{(i-p)}}{\partial t^{i-p}}u(0,x)\xi(x)\\
& +\sum_{i=1}^{k}\frac{C(\xi)^{q'}}{q'}(2k)^{q'/q}c_{\beta,i}^{q'}\frac{\tau^{\beta-iq'+1}}{\beta-iq'+1}\\
& -C(\xi)\int_0^\tau(\tau-t)^\beta\int_B(\triangle\xi)u \ge\\
& C(\xi)\int_0^\tau(\tau-t)^\beta\int_B|u|^q\xi-\frac{1}{2q}\int_0^\tau(\tau-t)^\beta|v|^q.
\end{aligned} \tag{2.6}$$

By using Young's inequality for the third term on the left-hand side of (2.6), we obtain

$$\begin{aligned}&-C(\xi)\int_0^{\tau}(\tau-t)^{\beta}\int_B(\triangle\xi)u\\&\leq C(\xi)(c(q)\int_0^{\tau}(\tau-t)^{\beta}\int_B\frac{|\triangle\xi|^{q'}}{\xi^{q'/q}}+\frac{1}{2}\int_0^{\tau}(\tau-t)^{\beta}\int_B|u|^q\xi).\end{aligned} \tag{2.7}$$

Thus

$$\begin{aligned}&-C(\xi)\int_0^{\tau}(\tau-t)^{\beta}\int_B(\triangle\xi)u\\&\leq\frac{\tau^{\beta+1}}{\beta+1}C_1(\xi)+\frac{C(\xi)}{2}\int_0^{\tau}(\tau-t)^{\beta}\int_B|u|^q\xi,\end{aligned} \tag{2.8}$$

where $C_1(\xi)=c(q)C(\xi)\int_B\frac{|\triangle\zeta|^{q'}}{\zeta^{\frac{q'}{q}}}dx$ for large values of σ. By considering (2.6) and (2.8), we get

$$\begin{aligned}&C(\xi)(-\sum_{i=1}^{k}\sum_{p=1}^{i}\tau^{\beta-p+1}c_{\beta,p-1}\int_B\frac{\partial^{(i-p)}}{\partial t^{i-p}}u(0,x)\xi(x))\\&+\sum_{i=1}^{k}\frac{C(\xi)^{q'}}{q'}(2k)^{q'/q}c_{\beta,i}^{q'}\frac{\tau^{\beta-iq'+1}}{\beta-iq'+1}\\&+\frac{\tau^{\beta+1}}{\beta+1}C_1(\xi)\geq\frac{C(\xi)}{2}\int_0^{\tau}(\tau-t)^{\beta}\int_D|u|^q\xi\\&\qquad-\frac{1}{2q}\int_0^{\tau}|v|^q(\tau-t)^{\beta}.\end{aligned} \tag{2.9}$$

On the other hand, by the definition of the constant $C(\xi)$ and Hölder's inequality, we obtain

$$C(\xi)\int_0^{\tau}(\tau-t)^{\beta}\int_D|u|^q\xi\geq\int_0^{\tau}(\tau-t)^{\beta}|v|^q.$$

Now, by considering the last inequality and $\frac{\partial^{(i)}}{\partial t^i}u(0,x)=\lambda_i\psi_i$ in (2.9), we have

$$\begin{aligned}&C(\xi)(-\sum_{i=1}^{k}\sum_{p=1}^{i}\tau^{\beta-p+1}c_{\beta,p-1}\lambda_{i-p}\int_B\psi_{i-p}\xi(x))\\&+\sum_{i=1}^{k}\frac{C(\xi)^{q'}}{q'}(2k)^{q'/q}c_{\beta,i}^{q'}\frac{\tau^{\beta-iq'+1}}{\beta-iq'+1}\\&+\frac{\tau^{\beta+1}}{\beta+1}C_1(\xi)\geq\frac{1}{2}(1-\frac{1}{q})\int_0^{\tau}|v|^q(\tau-t)^{\beta}.\end{aligned} \tag{2.10}$$

Hence, since $\psi_i\geq 0$ for $0\leq i\leq k-1$ and there exists a j such that $\psi_j\not\equiv 0$ then for large value of λ_j the left hand side of (2.10) will be negative, which is implies that T^*, the maximum existence time of solution, must be less than $\tau<1$.□

Remark 2.3. Notice that in the last Theorem, B is not necessary be the unit ball with center at the origin, because by linear transformation every ball in the space can be transfer to the mention ball.

Remark 2.4. In the argument about the proof of the last Theorem, one can see that it is not necessary all of the initial data are positive, but if one of them is positive and very large then the result of the Theorem remains valid.

Remark 2.5. In the proof of above Theorem, we assumed that for large value of σ, $\int_B \frac{|\triangle\xi|^{q'}}{\xi^{q'/q}}dx$ is finite. To see this ,notice that

$$\int_B \frac{|\triangle\xi|^{q'}}{\xi^{q'/q}}dx = \int_0^1 \frac{r^{n-1}|\frac{(n-1)}{r}\phi'(r)+\phi''(r)|^{q'}}{\phi(r)^{q'/q}}dr \le c\Big(\int_0^1 r^{n-1-q'}\frac{|\phi'(r)|^{q'}}{\phi(r)^{q'/q}}dr + \int_0^1 \frac{r^{n-1}|\phi''(r)|^{q'}}{\phi(r)^{q'/q}}dr\Big), \tag{2.11}$$

for some positive constant c. Hence, according to the choice of the function ϕ, it is sufficient that we consider the above integrals near the point 1.The first term on the right-hand side of (2.11) near the point 1,changes to

$$\int_{\frac{5}{6}}^1 \frac{r^{n-1-q'}[\sigma(1-r)^{\sigma-1}]^{q'}}{(1-r)^{\sigma q'/q}}dr.$$

Thus, by choosing $-(\sigma-1)q'+\sigma\frac{q'}{q}<1$ or $q'-\sigma<1$, the above integral becomes finite. Moreover, for the second term on the right-hand side of (2.11) near the point 1,we have

$$\int_{\frac{5}{6}}^1 r^{(n-1)}\frac{|\phi''(r)|^{q'}}{\phi(r)^{q'/q}}dr = \int_{\frac{5}{6}}^1 \frac{r^{n-1}(\sigma(\sigma-1)(1-r)^{(\sigma-2)q'})}{(1-r)^{\sigma q'/q}}dr.$$

Hence, by choosing $-(\sigma-2)q'+\sigma(q'-1)<1$ or $-\sigma+2q'<1$, the above integral should be finite too.

Remark 2.6. Notice that in the last theorem our result was not depend on the boundary condition and boundedness or unboundedness of domain Ω, but if we consider the Problem (2.1) in a bounded domain with Neuman boundary condition, then we have a better result which has appeared in the following Theorem.

Theorem 2.7. *Let Ω be a smooth bounded domain and consider the problem (2.1) with the Neuman boundary condition $\frac{\partial u}{\partial n}(t,x)\ge 0$ for $x\in\partial\Omega$. Then the problem (2.1) has no global solution if $\sum_{i=1}^{k}\int_\Omega \frac{\partial^{(i-1)}}{\partial t^{i-1}}u(0,x)dx>0$.*

Proof. Without lost of generality, we may assume that $0\in\Omega$. Let $\varphi_1>0$ be the first eigenfunction of $-\triangle$ in $H_0^1(\Omega)$, and $\xi(x)=\varphi_1(\varepsilon x)$ for small values of ε. By taking $\xi(t,x)=\varphi_1(x)(\tau-t)^\beta$ as a test function, similar to the proof of the last Theorem, we have

$$C(\xi)(-\sum_{i=1}^{k}\sum_{p=1}^{i}\tau^{\beta-p+1}c_{\beta,p-1}\int_{\Omega}\frac{\partial^{(i-p)}}{\partial t^{i-p}}u(0,x)\xi(x)$$
$$+\sum_{i=1}^{k}\frac{C(\xi)^{q'}}{q'}(2k)^{q'/q}c_{\beta,i}^{q'}\frac{\tau^{\beta-iq'+1}}{\beta-iq'+1}-C(\xi)\int_0^{\tau}\int_{\Omega}(\tau-t)^{B}(\triangle\xi)u \tag{2.12}$$
$$\geq C(\xi)\int_0^{\tau}(\tau-t)^{\beta}\int_{\Omega}|u|^q-\frac{1}{2q}\int_0^{\tau}(\tau-t)^{\beta}|v|^q$$
$$+C(\xi)\textstyle\int_0^{\tau}(\tau-t)^{B}\int_{\partial\Omega}(\xi\frac{\partial u}{\partial n}-u\frac{\partial\xi}{\partial n}),$$

where $v(t)=\int_{\Omega}u(x,t)\xi(x)dx$ and $C(\xi)=(\int_{\Omega}\xi)^{\frac{q}{q'}}$.

Notice that, if $\varepsilon\to 0$ then $\triangle\xi\to 0$, $\frac{\partial\xi}{\partial n}\to 0$ and $C(\xi)\to C(0):=(\varphi_1(0)mes(\Omega))^{\frac{q}{q'}}$. Here, $mes(\Omega)$ is the Lebesgue measure of the domain Ω. Now, if we let $\varepsilon\to 0$ in (2.12), we get
(2.13)

$$C(0)(-\sum_{i=1}^{k}\sum_{p=1}^{i}\tau^{\beta-p+1}c_{\beta,p-1}\int_{\Omega}\frac{\partial^{i-p}}{\partial t^{i-p}}u(0,x)\varphi_1(0))$$
$$+\sum_{i=1}^{k}\frac{C(0)^{q'}}{q'}(2k)^{q'/q}c_{\beta,i}^{q'}\frac{\tau^{\beta-iq'+1}}{\beta-iq'+1}\geq$$
$$C(0)\int_0^{\tau}(\tau-t)^{\beta}\int_{\Omega}|u|^q-\frac{1}{2q}\int_0^{\tau}(\tau-t)^{\beta}|v|^q+\varphi_1(0)C(0)\int_0^{\tau}(\tau-t)^{\beta}\int_{\partial\Omega}\frac{\partial u}{\partial n}d\sigma.$$

On the other hand,

$$C(0)\int_0^{\tau}(\tau-t)^{\beta}\int_B|u|^q\varphi_1(0)\geq\int_0^{\tau}(\tau-t)^{\beta}|\int_B u\varphi_1(0)|^q$$

Thus by the last inequality, the Neuman boundary condition and (2.13), we get

$$-\tau^{\beta}(C(0)\varphi_1(0)\sum_{i=1}^{k}\int_{\Omega}\frac{\partial^{i-1}}{\partial t^{i-1}}u(0,x)dx)+O(\tau^{\beta-q'+1})\geq 0 \tag{2.14}$$

Now, if $\int_{\Omega}\frac{\partial^{i-1}}{\partial t^{i-1}}u(0,x)dx>0$, for $1\leq i\leq k$, and $\tau\to+\infty$ then the left-hand side of (2.14) should be negative, which is a contradiction. □

Remark 2.8. Let we substitute the equation (2.1) with the following inequality

$$\sum_{i=1}^{k}a_i\frac{\partial^{(i)}u}{\partial t^i}-\triangle u\geq F(u,\bigtriangledown u,x),$$

where $a_i\geq 0$, $\sum_{i=1}^{k}a_i>0$ and $F(u,\bigtriangledown u)\geq C|u|^q$. For suitable choice of initial data, similar to the proof of the above theorems, we can show that this inequality has no global solution.

3. Some ordinary differential inequalities

In this section we consider the following ordinary differential inequalities (O.D.I.)

$$z^{(k)} + z^{(k-1)} + \cdots + z' \geq cz^q \tag{P_1}$$

$$z^{(k)} + \lambda z \geq cz^q \tag{P_2}$$

$$z^{(k)} + z^{(k-1)} + \lambda z \geq cz^q \tag{P_3}$$

where $\lambda, c \geq 0$ are constants and $z \in C^k(\mathbb{R})$ is a real valued function. For these inequalities, we shall prove that the function z cannot be global under certain assumptions on λ, c, k and initial data. These problems are very useful for investigation of nonglobal smooth solution for some higher-order evolution equations. In the next section, we shall see some application of them.

At first we consider (P_1). Multiplying (P_1) by $(\tau - t)^\beta$ and integrating with respect to t in the interval $[0, \tau]$, and using integration by parts, yields

$$\begin{aligned}
&-(\sum_{i=1}^{k} z^{(i-1)}(0))\tau^\beta \\
&-(\sum_{i=1}^{k}\sum_{p=2}^{i} \tau^{\beta-p+1} c_{\beta,p-1} z^{(i-p)}(0)) \\
&+\sum_{i=1}^{k} c_{\beta,i}\int_0^\tau (\tau-t)^{\beta-i} z(t)dt \\
&\geq \textstyle\int_0^\tau (\tau-t)^\beta z^q,
\end{aligned} \tag{3.1}$$

By using Hölder's inequality and Young's inequality, for the third term on the left-hand side of (3.1), we get

$$\begin{aligned}
&c_{\beta,i}\int_0^\tau (\tau-t)^{\beta-i} z(t) = \int_0^\tau (\tau-t)^{\beta/q} z(t) c_{\beta,i}(\tau-t)^{\beta/q'-i} \\
&\leq (\int_0^\tau (\tau-t)^\beta z^q(t)dt)^{1/q} (\int_0^\tau (\tau-t)^{\beta-iq'} c_{\beta,i}^{q'})^{1/q'} \\
&= (\int_0^\tau \frac{(\tau-t)^\beta z^q(t)}{k})^{1/q} (\int_0^\tau k^{q'/q} c_{\beta,i}^{q'} (\tau-t)^{\beta-iq'})^{1/q'} \\
&\leq \frac{1}{kq}\int_0^\tau (\tau-t)^\beta z^q(t) + c_{\beta,i}^{q'} \frac{k^{q'/q}}{q'} \frac{\tau^{\beta-iq'+1}}{\beta-iq'+1},
\end{aligned} \tag{3.2}$$

where $\beta \in \mathbb{N}$ is a large number such that $\beta - iq' + 1 > 1$ for $i = 1, \ldots, k$. Thus, from (3.1) and (3.2), we get

$$\begin{aligned}
&-(\sum_{i=1}^{k} z^{(i-1)}(0))\tau^\beta \\
&-(\sum_{i=1}^{k}\sum_{p=2}^{i} \tau^{\beta-p+1} c_{\beta,p-1} z^{(i-p)}(0)) \\
&+k\frac{1}{kq}\int_0^\tau (\tau-t)^\beta z^q(t) \\
&+\Lambda \sum_{i=1}^{k} \frac{\tau^{\beta-iq'+1}}{\beta-iq'+1} c_{\beta,i}^{q'} \geq \int_0^\tau (\tau-t)^\beta z^q(t)dt,
\end{aligned} \tag{3.3}$$

where $\Lambda := \frac{k^{q'/q}}{q'}$. Hence

$$
\begin{aligned}
&-(\sum_{i=1}^{k} z^{(i-1)}(0))\tau^{\beta} - \sum_{i=1}^{k}\sum_{p=2}^{i} \tau^{\beta-p+1} c_{\beta,p-1} z^{(i-p)}(0) + \Lambda \sum_{i=1}^{k} \frac{\tau^{\beta-iq'+1}}{\beta-iq'+1} c_{\beta,i}^{q'} \\
&\geq (1-\frac{1}{q}) \int_0^{\tau} (\tau-t)^{\beta} z^q(t)dt
\end{aligned}
\tag{3.4}
$$

Now, if $\sum_{i=1}^{k} z^{(i-1)}(0) > 0$ and $\tau \to \infty$, the left-hand side of (3.4) will be negative but the right-hand side is positive, which is a contradiction. Therefore, we have proved the following Theorem.

Theorem 3.1. *The ordinary differential inequality* $(P1)$ *has no positive smooth global solution if* $\sum_{i=1}^{k} z^{(i-1)}(0) > 0$.

Remark 3.2. Consider the following equation

$$\sum_{i=1}^{k} a_i z^{(i)} \geq cz^q.$$

Similar to the proof of the last Theorem, one can show this inequality has no global solution if $\sum_{i=1}^{k} a_i z^{(i-1)}(0) > 0$, $a_i \geq o$ and $\sum_{i=1}^{k} a_i > 0$.

We have the following theorem about nonglobal solution of (P_2).

Theorem 3.3. *If* $z(0) > \sqrt[p-1]{\frac{\lambda}{c}}, z'(0) \geq 0, \dots, z^{(k-1)}(0) \geq 0$, *then the solution of* $(P2)$ *is not global.*

Proof. Suppose that z is a global solution for (P_2). At first, we are going to prove that $z^{(k)}(t) > 0$ for all $t > 0$.

Let $f(t) = cz^p(t) - \lambda z(t)$. Since $z(0) > \sqrt[p-1]{\frac{\lambda}{c}}$, we have

$$f(0) = cz^p(0) - \lambda z(0) = z(0)(cz^{p-1}(0) - \lambda) > 0.$$

Thus, there exists $\delta > 0$ such that for $0 < t < \delta$, we have $f(t) > 0$. Hence for $0 < t < \delta$, we have

$$z^{(k)}(t) \geq cz^p(t) - \lambda z(t) = f(t) > 0.$$

Now, let $t_0 > 0$ be the first point such that $z^{(k)}(t_0) = 0$. In the interval $[0,t_0]$, the function $z^{(k)}$ is a nonnegative function. Thus $z^{(k-1)}$ is an increasing function in $[0,t_0]$.But $z^{(k-1)}(0) \geq 0$ implies that $z^{(k-1)}$ is a nonnegative function in $[0,t_0]$. By induction, we conclude that z' is a nonnegative function in $[0,t_0]$, so z is an increasing function in $[0,t_0]$. Thus, $z(t_0) \geq z(0)$. But the function $g(x) = cx^p - \lambda x$ for $x > \sqrt[p-1]{\frac{\lambda}{cp}}$ is an increasing function. Therefore, $z(t_0) \geq z(0) > \sqrt[p-1]{\frac{\lambda}{c}} \geq \sqrt[p-1]{\frac{\lambda}{pc}}$, yields

$$z^{(k)}(t_0) \geq cz^p(t_0) - \lambda z(t_0) \geq cz^p(0) - \lambda z(0) > 0,$$

which is a contradiction. Therefore $z^{(k)}(t) > 0$ for all $t > 0$. Thus for a fix number $\varepsilon > 0$, $z^{(k-1)}(\varepsilon) > z^{(k-1)}(0) \geq 0$. By integrating over $[\varepsilon, t]$, we obtain

$$z^{(k-2)}(t) > (t-\varepsilon)z^{(k-1)}(\varepsilon) + z^{(k-1)}(\varepsilon).$$

Hence $z^{(k-2)}(t) \to +\infty$ as $t \to +\infty$.We obtain inductively, z is an increasing function and $z \to +\infty$ as $t \to +\infty$. Thus, there exists $T > 0$ such that $z(T) \geq \sqrt[p-1]{\frac{2\lambda}{c}}$ or

$$cz^p(T) - \lambda z(T) > \frac{c}{2}z^p(T).$$

Hence, for $t > T$ we must have $cz^p(t) - \lambda z(t) \geq \frac{c}{2}z^p(t)$ or

$$z^{(k)} > \frac{c}{2}z^p, \quad (t > T). \tag{3.5}$$

Now, by considering Remark 3.2 and $z^{(k-1)}(T) > 0$, the inequality (3.5) has no global solution, which is a contradiction. □

Eventually, we have the following result for (P_3).

Theorem 3.4. *If $z(0) > \sqrt[p-1]{\lambda/c}$, $z'(0) \geq 0, z''(0) \geq 0, \ldots, z^{(k-1)}(0) > 0$, then the solution of the problem (P_3) is not global.*

Proof. We claim that $z^{(k-1)}$ is positive. Let $t_0 > 0$ be the first point that $z^{(k-1)}(t_0) = 0$. Hence $z^{(k-1)}$ is positive on $(0, t_0)$ and consequently $z^{(k)}(t_0) \leq 0$. Therefore

$$\lambda z(t_0) \geq z^{(k)}(t_0) + z^{(k-1)}(t_0) + \lambda z(t_0) \geq cz^q(t_0). \tag{3.6}$$

By using the same argument in the last theorem,we can show $z(t)$ is an increasing function on $[0, t_0]$. Thus $z(t_0) > z(0) > \sqrt[p-1]{\frac{\lambda}{c}}$ or

$$cz^q(t_0) > \lambda z(t_0),$$

which, contradicts (3.6). Hence $z^{(k-1)}(t) > 0$ for every $t > 0$. Similar to the proof of the last theorem, there exists $T > 0$ such that $cz^p - \lambda z > \frac{c}{2}z^p$ for all $t > T$. Thus

$$z^{(k)}(t) + z^{(k-1)}(t) \geq \frac{c}{2}z^p(t), (t > T) \tag{3.7}$$

Now, by considering Remark 3.2, the solution of the inequality (3.7) is not global. □

Remark 3.5. For the ordinary differential inequalities's (P_1), (P_2) and (P_3), we have proved that the solutions under some assumption on initial data, cannot be global. Moreover, we can prove that these solutions blow-up in finite time. Here, by blow-up we mean that if $T^* < \infty$ is the maximal existence time, then the solution blows-up whenever

$$\|z^k\|_{L^1(0,T^*)} = +\infty.$$

In order to prove this claim, one can suppose that, $\int_0^{T^*} z^{(k)}(t)dt < \infty$. Then $z, z', \ldots, z^{(k-1)}$ bear some limits which are denoted by $z(T^*), z'(T^*), \ldots, z^{(k-1)}(T^*)$ as t tend to T^*_-. Then the solution of the inequalities could be extended to the right hand side of T^* by a local solution of the following O.D.I. for $(P1), (P2)$ and $(P3)$, respectively

$$\begin{cases} w^{(k)}(t) + \cdots + w'(t) \geq cw^q(t) \\ w(T^*) = z(T^*), w'(T^*) = z'(T^*), \ldots, w^{(k-1)}(T^*) = z^{(k-1)}(T^*). \end{cases}$$

$$\begin{cases} w^{(k)}(t)+\lambda w(t)\geq cw^{q}(t) \\ w(T^{*})=z(T^{*}),w'(T^{*})=z'(T^{*}),\dots,w^{k-1}(T^{*})=z^{(k-1)}(T^{*}). \end{cases}$$

$$\begin{cases} w^{(k)}(t)+w^{(k-1)}(t)+\lambda w(t)\geq cw^{q}(t) \\ w(T^{*})=z(T^{*}),w'(T^{*})=z'(T^{*}),\dots,w^{(k-1)}(T^{*})=z^{(k-1)}(T^{*}). \end{cases}$$

4. Application

In this section, by using the results of the last section,we investigate nonglobal solution for some higher-order evolution equations.Consider the following equations,in a bounded domain $\Omega\subset\mathbb{R}^{N}$.

$$\frac{\partial^{k}u}{\partial t^{k}}+\cdots+u_{t}-\triangle u=\lambda u+u^{q}, \tag{4.1}$$

$$\frac{\partial^{k-1}u}{\partial t^{k-1}}+\cdots+u_{t}-\triangle u+\lambda u=\int_{0}^{t}u^{q}, \tag{4.2}$$

$$\frac{\partial^{k}u}{\partial t^{k}}+\frac{\partial^{k-1}u}{\partial t^{k-1}}-\triangle u=a|\bigtriangledown u|^{p}+bu^{q} \tag{4.3}$$

$$\frac{\partial^{k}u}{\partial t^{k}}-\triangle u=a|\bigtriangledown u|^{p}+bu^{q}, \tag{4.4}$$

$$\frac{\partial^{k}u}{\partial t^{k}}+\cdots+u_{t}-\triangle u=a|\bigtriangledown u|^{p}+\lambda u, \tag{4.5}$$

$$\frac{\partial^{k}u}{\partial t^{k}}+\cdots+u_{t}-\triangle u=a|\bigtriangledown u|^{p}+M, \tag{4.6}$$

where a,b,λ and M are nonnegative constants. In this section, we also assume that

$$u\in\mathcal{C}^{k}((0,T];L_{2}(\Omega))\cap\mathcal{C}((0,T];W_{0}^{1,r}(\Omega)),u\geq 0, \tag{4.7}$$

where $r=max\{p,q\}$. Let λ_{1} be the lowest eigenvalue of the operator $-\triangle$ in $H_{0}^{1}(\Omega)$ and φ_{1} be the corresponding positive eigenfunction with

$$\int_{\Omega}\varphi_{1}dx=1.$$

At first, we consider the equation (4.1). Multiplying this equation by φ_{1} and integrating over Ω, yields

$$\frac{\partial^{k}}{\partial t^{k}}\int_{\Omega}u\varphi_{1}dx+\cdots+\frac{\partial}{\partial t}\int_{\Omega}u\varphi_{1}dx+\lambda_{1}\int_{\Omega}u\varphi_{1}dx=\lambda\int_{\Omega}u\varphi_{1}dx+\int_{\Omega}\varphi_{1}u^{q}dx. \tag{4.8}$$

By setting $z(t)=\int_{\Omega}u\varphi_{1}dx$, and using the Holder's inequality, we get

$$z^{(k)}+z^{(k-1)}+\cdots+z'+\lambda_{1}z\geq z^{q}+\lambda z.$$

Now, if $\lambda\geq\lambda_{1}$, then

$$z^{(k)}+z^{(k-1)}+\cdots+z'\geq z^{q},$$

Thus, by Theorem 3.1 and Remark 3.5 if $\sum_{i=1}^{k} z(0)^{(i-1)} > 0$, then the solution of the above inequality must blow-up in finite time.

Now, we consider the equation (4.2). If we differentiate this equation with respect to t, we get

$$\frac{\partial^k u}{\partial t^k} + \cdots + u_{tt} - \triangle u_t + \lambda u_t = u^q.$$

Let $z(t)$ be as above. By using a similar argument , we obtain

$$z^{(k)} + \cdots + z'' + \lambda_1 z' + \lambda z' \geq z^q. \tag{4.9}$$

Hence, if $z(0) > 0$ then

$$z^{(k)}(0) + \cdots + z''(0) + (\lambda + \lambda_1) z'(0) > 0. \tag{4.10}$$

But, for the equation (4.2), we must have $z^{(k-1)}(0) + \cdots + z'(0) + (\lambda + \lambda_1) z(0) = 0$. Indeed, By multiplying the equation (4.2) with φ_1 and letting $t \to 0$, this equality must be satisfied. Hence, by considering the last equality and (4.10), one can easily deduce that there exists a $\varepsilon > 0$ such that

$$z^{(k-1)}(\varepsilon) + \cdots + z'(\varepsilon) + (\lambda + \lambda_1) z(\varepsilon) > 0.$$

Therefore, if $\lambda + \lambda_1 > 0$ then by Theorem 3.1 and Remark 3.5 the solution of the inequality (4.9) blows-up in finite time.

Remark 4.1. Notice that for the equation (4.2) when $k = 1$, Souplet in [17] showed that if $\lambda > \lambda_1$ and $z(0) > 0$ then the solution blows-up in finite time. But in the last argument, we can see that if $z(0) > 0$ then this result remains valid even $\lambda > -\lambda_1$ and moreover for every $k \geq 1$.

For the equations (4.3),(4.4),(4.5) and (4.6) the following lemma is useful.

Lemma 4.2. *Let u and φ_1 be as above. If $p > 2$, there exists a constant $c(\Omega, \varphi_1) > 0$, such that*

$$c(\Omega, \varphi_1)\left(\int_\Omega u \varphi_1 dx\right)^p \leq \int_\Omega |\bigtriangledown u|^p \varphi_1 dx.$$

Proof. For u and φ_1 we can write,

$$\begin{aligned}
\int_\Omega |u| \varphi_1 dx &\leq \|\varphi_1\|_\infty \int_\Omega |u| dx \\
&\leq c(\Omega) \|\varphi_1\|_\infty \int_\Omega |\bigtriangledown u| dx \quad \text{(Poincare's inequality)} \\
&= c(\Omega) \|\varphi_1\|_\infty \int_\Omega |\bigtriangledown u| \varphi^{\frac{1}{p}} \varphi_1^{-\frac{1}{p}} dx \\
&\leq c(\Omega) \|\varphi_1\|_\infty \left(\int |\bigtriangledown u| \varphi_1 dx\right)^{\frac{1}{p}} \left(\int \varphi_1^{-\frac{p'}{p}} dx\right)^{\frac{1}{p'}}.
\end{aligned}$$

Thus

$$c(\Omega, \varphi_1)\left(\int_\Omega u \varphi_1 dx\right)^p \leq \int_\Omega |\bigtriangledown u|^p \varphi_1 dx.$$

This completes the proof.□

Remark 4.3. In Lemma 4.2 ,we assume that $\int_\Omega \varphi_1^{\frac{-p'}{p}} dx$, is finite. In fact, if φ_1 is the first eigenfunction of $-\triangle$ operator in $H_0^1(\Omega)$, then $\varphi_1 \in W^{2,2}(\Omega) \cap W_0^{1,\infty}(\Omega)$. Moreover the author in [16, Lemma 5.1] has proved that

$$\int_\Omega \varphi_1^{-\alpha}(x)dx = c(\alpha,\Omega) < \infty \quad \forall \alpha \in (0,1).$$

Now, we consider the equation (4.3). By using the same notation as above the equation (4.3) can be written

$$z^{(k)} + z^{(k-1)} + \lambda z = a\int_\Omega |\bigtriangledown u|^p \varphi_1 dx + b\int_\Omega u^q \varphi_1 dx. \tag{4.11}$$

Now, according the above lemma and (4.11), by considering $a,b \geq 0$ and $p > 2$,we obtain

$$z^{(k)} + z^{(k-1)} + \lambda z \geq ac(\Omega,\varphi_1)z^p + bz^q. \tag{4.12}$$

Thus, by Theorem 3.4 and Remark 3.5, if $a+b > 0$ then the solution of the inequality (4.12) blows-up in finite time whenever $z(0)$ is large enough and $z'(0) \geq 0, \cdots, z^{(k-1)}(0) > 0$.

For the equation (4.4), exactly similar to the argument of the equation (4.3), one can show that if $z(0)$ is large enough and $z'(0) \geq 0, \cdots, z^{(k-1)}(0) > 0$ then the function z blows-up in finite time, too.
For the equation (4.5), by employing Lemma 4.2 ,we obtain

$$z^{(k)} + \cdots + z' + \lambda_1 z \geq ac(\Omega,\varphi_1)z^q + \lambda z. \tag{4.13}$$

Thus by $(P1)$ for $\lambda \geq \lambda_1$, the solution of this inequality must blow-up whenever $\sum_{i=1}^{k} z^{(i-1)}(0) > 0$.

For the equation (4.6), we have the following Theorem.

Theorem 4.4. *Let $p > 2$. If M is large enough and $\sum_{i=1}^{k} z(0)^{(i-1)} > 0$, then the equation (4.6) cannot admit a global solution.*

Proof. By using Green's theorem ,we get

$$z^{(k)} + \cdots + z' + \int_\Omega \bigtriangledown u.\bigtriangledown \varphi_1 dx = a\int_\Omega |\bigtriangledown u|^p \varphi_1 dx + M\int_\Omega \varphi_1 dx, \tag{4.14}$$

By considering the Hölder's inequality and Young's inequality for the last term on the left-hand side, we can write

(4.15)

$$\begin{aligned}
\int_\Omega \bigtriangledown u.\bigtriangledown \varphi_1 dx &\leq \int_\Omega |\bigtriangledown u|\varphi_1^{\frac{1}{p}} |\bigtriangledown \varphi_1| \varphi_1^{-\frac{1}{p}} dx \\
&\leq \left(\int_\Omega |\bigtriangledown u|^p \varphi_1 dx\right)^{\frac{1}{p}} \left(\int_\Omega |\bigtriangledown \varphi_1|^{p'} \varphi_1^{-\frac{p'}{p}} dx\right)^{\frac{1}{p'}}, \text{(Hölder's inequality)} \\
&\leq \frac{a}{2}\int_\Omega |\bigtriangledown u|^p \varphi_1 dx + c_1 \int_\Omega |\bigtriangledown \varphi_1|^{p'} \varphi_1^{-\frac{p'}{p}} dx, \text{ (Young's inequality)},
\end{aligned}$$

for some positive constants c_1. Now, if $M > \frac{c_1 \int_\Omega |\nabla \varphi_1|^{p'} \varphi_1^{-\frac{p'}{p}} dx}{\int_\Omega \varphi_1 dx}$, then from (4.14) and (4.15), we get

$$z^{(k)} + \cdots + z' \geq \frac{a}{2} \int_\Omega |\nabla u|^p \varphi_1 dx. \tag{4.16}$$

Now,by using Lemma 4.2, we obtain

$$z^{(k)} + \cdots + z' \geq \frac{a}{2} c(\Omega, \varphi_1) z^p. \tag{4.17}$$

By considering Theorem 3.1 and Remark 3.5, the solution of the equation (4.6) blows-up in finite time whenever $\sum_{i=1}^{k} z^{(i-1)}(0) > 0.\Box$

Remark 4.5. Notice that in the last Theorem, we assumed Ω is bounded. Here, we should mention for some unbounded domains this theorem remains valid, too. In fact in [10], we showed that one can substitute Ω by an unbounded domain, Ω', such that the Poincare's inequality is valid in $H_0^1(\Omega')$ and there exist some positive functions $\phi \in W_0^{1,\infty}(\Omega')$ such that for $\delta = \frac{1}{p-1}$, we have

$$\int_\Omega \frac{1}{\phi(x)^\delta} dx < \infty.$$

For this domain Ω' the conclusion of Theorem 4.3 remains valid. For existence of such a domain Ω' ,the function ϕ and the rest of details one can see [10].

5. A NONLINEAR WAVE EQUATION

In this section, we consider the following nonlinear wave problem

$$\begin{cases} u_t + u_{tt} - \triangle u = u^p & Q := \Omega \times (0, T], \\ u(0,x) = u_0, & x \in \Omega, \\ u_t(0,x) = u_{0t} & x \in \Omega, \\ u(t,x) = 0 & x \in \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$. For this problem, at first by using the Generalized Convexity method, we show that if the initial data satisfy the following condition

$$\int_\Omega |\nabla u(0)|^2 dx + \int_\Omega |u_t(0)|^2 dx - \frac{2}{p+1} \int_\Omega u^{p+1}(0) dx \leq 0, \tag{5.2}$$

then the nonnegative classic solution blows-up in finite time. After that, we will investigate what happens for the solution if the condition (5.2) does not hold. In fact, by using the Galerkin's method we show that if the condition (5.2) does not hold, i.e.

$$\int |\nabla u(0)|^2 dx + \int |u_t(0)|^2 dx - \frac{2}{p+1} \int u^{p+1}(0) dx > 0,$$

and the left hand side of the last inequality is small, and the following inequality holds

$$\int |\nabla u(0)|^2 dx - \int u^{p+1}(0) dx > 0,$$

then the weak solution of the problem (5.1) is global.

In this step we define the global classic and global weak solution for the problem (5.1).

Definition 5.1. By a global classic solution for the problem (5.1), we mean a function $u \in \mathcal{C}^2((0,\infty);W_0^{2,p}(\Omega))$ such that

$$u_{tt} + u_t - \triangle u = u^p,$$

for every $x \in \Omega$ and $t > 0$.

Definition 5.2. By a weak solution in $\Omega \times [0,T]$ for the problem (5.1), we mean a function $u \in \mathcal{C}((0,T);W_0^{1,p}(\Omega))$ and $u_t, u_{tt} \in L^2(0,T;L^2(\Omega))$, such that

$$\int_0^T \int_\Omega [(u_t + u_{tt})\zeta + \bigtriangledown u. \bigtriangledown \zeta] dt dx = \int_0^T \int_\Omega u^p \zeta dt dx, \tag{5.3}$$

for every test function $\zeta \in L^2(0,T;H_0^1(\Omega))$.

Moreover, we say that the weak solution u is global whenever u is a weak solution for problem (5.1) in $\Omega \times [0,T]$, for every $T > 0$.

The following theorem shows all of our results in this section.

Theorem 5.3. *Consider the problem (5.1).*

a) Let u be a classic nonnegative solution of the problem (5.1) and the initial data satisfy the condition (5.2), then u blows-up in finite time in $L^2(\Omega)$.

b) Let $T > 0$*. If* $u(0) \geq 0$ *and* $u_t(0)$ *satisfy the following conditions*

$$\int |\bigtriangledown u(0)|^2 dx - \int u^{p+1}(0) dx > 0, \tag{5.4}$$

and

$$0 < \int |\bigtriangledown u(0)|^2 dx + \int |u_t(0)|^2 dx - \frac{2}{p+1} \int u^{p+1}(0) dx < \delta, \tag{5.5}$$

where $\delta := \inf_{0 \not\equiv u \in H_0^1(\Omega)} \sup_{\lambda \geq 0} H(\lambda u) > 0$ *(* $\delta > 0$ *will be proved in Lemma 5.5), then for* $2 < p+1 < 2^*$ *(where* $2^* = \frac{2N}{N-2}$ *for* $N > 2$ *and* $2^* = \infty$ *for* $N \leq 2$*), there exists a nonnegative solution u for the problem (5.1) such that* $u \in L^\infty(0,T;H_0^1(\Omega))$ *and* $u' \in L^2(0,T;L^2(\Omega))$ *and* $u'' \in L^2(0,T;H^{-1}(\Omega))$.

In order to prove part (a) of the theorem we need the following lemma.

Lemma 5.4. *If a function*

$$F(t) \in \mathcal{C}^2,\ F(t) \geq 0,$$

satisfies the inequality

$$F''(t)F(t) - (1+\alpha)F'(t)^2 \geq -2C_1F(t)F'(t) - C_2F(t)^2,$$

for some real numbers $\alpha > 0$ *and* $C_1, C_2 \geq 0$*, then we have*

a) If $F(0) > 0, F'(0) > -\gamma_2\alpha^{-1}F(0), C_1 + C_2 > 0$*, where*

$$\gamma_2 = -C_1 - \sqrt{C_1^2 + \alpha C_2},$$

then, for $\gamma_1 = -C_1 + \sqrt{C_1^2 + \alpha C_2}$*, there exists a positive real number*

$$t_1 < \frac{1}{2\sqrt{C_1^2 + C_2^2}} \ln \frac{\gamma_1 F(0) + \alpha F'(0)}{\gamma_2 F(0) + \alpha F'(0)}$$

such that $F(t) \to +\infty$ *as* $t \to t_1$.

b) If $F(0) > 0$, $F'(0) > 0$ *and* $C_1 = C_2 = 0$*, then there exists a positive real number* $t_1 < \frac{F(0)}{\alpha F'(0)}$ *such that* $F(t) \to +\infty$ *as* $t \to t_1$.

Part (a) of the lemma is taken from [11] which is due to O.A. Ladyzhenshaya and V.K. Kalantarov. Part (b) of the lemma is introduced by H.A. Levine in [14].

The most crucial point in the application of this lemma is to find a functional that represents the disipation on the boundary and satisfies the conditions of the lemma. This method is known as the "Generalized convexity" method.
Proof of part (a) of Theorem 5.3. Let $F(t) := \int_\Omega u^2(t,x)dx$. Then

$$F'(t) = 2\int_\Omega uu_t dx,$$

and

$$F''(t) = 2(\int_\Omega u_t^2 dx + \int uu_{tt} dx).$$

It follows that,

$$\begin{aligned} FF'' - (\alpha+1)F'^2 &= 4(\alpha+1)A^2 + 2F(t)[\int uu_{tt}dx \\ &+ \int uu_t dx - (2\alpha+1)\int u_t^2 dx] \\ &-2F(t)\int uu_t. \end{aligned} \tag{5.6}$$

In this step we show that

$$H(t) := \int uu_{tt}dx + \int uu_t dx - (2\alpha+1)\int u_t^2 dx \geq 0.$$

Notice that we have $u_{tt} = \triangle u + u^p - u_t$. Thus

$$\begin{aligned} H(t) &= \int u \triangle u dx + \int u^{p+1}dx - \int uu_t dx + \int uu_t dx - (2\alpha+1)\int u_t^2 dx \\ &= -\int |\triangle u|^2 dx + \int u^{p+1}dx - (2\alpha+1)\int u_t^2 dx. \end{aligned} \tag{5.7}$$

Hence

$$\begin{aligned} H'(t) &= -\tfrac{d}{dt}\int |\triangle u|^2 dx + (p+1)\int uu_t u^p dx - (2\alpha+1)\int u_t u_{tt} dx \\ &= -\tfrac{d}{dt}\int |\triangle u|^2 dx + (p+1)\int u_t u^p dx \\ &-2(2\alpha+1)[\int u_t u^p dx - \int u_t^2 dx \int u_t \triangle u] \\ &= 2\alpha \tfrac{d}{dt}\int |\triangle u|^2 dx + \tfrac{p-4\alpha-1}{p+1}\tfrac{d}{dt}\int u_{p+1} + 2(2\alpha+1)\int u_t^2. \end{aligned} \tag{5.8}$$

Now, (5.7) and (5.8) imply that

$$H(t) - H(0) \geq -2\alpha \int |\bigtriangledown u(0)|^2 dx - \frac{(p-4\alpha-1)}{p+1}\int u^{p+1}(0)dx.$$

Therefore

$$H(t) \geq -(2\alpha+1)[\int |\bigtriangledown u(0)|^2 dx + \int |u_t(0)|^2 dx - \frac{2}{p+1}\int u^{p+1}(0)dx] \geq 0. \tag{5.9}$$

Now, from (5.6) and (5.9) we obtain

$$F''(t)F(t)-(\alpha+1)F'(t)^2 \geq -F(t)F'(t)$$

Therefore the hypotheses of Lemma 5.4 are satisfied with $C_1 = 1/2$ and $C_2 = 0$. Hence from the conclusion of the lemma 5.4 the proof of the theorem is completed.□

In order to prove part (b) of the theorem we need the following lemmas.

Lemma 5.5. *Suppose that* $u \in H_0^1(\Omega)$ *and* $2 < p+1 < 2^*$ *(where* $2^* = \frac{2N}{N-2}$ *for* $N > 2$ *and* $2^* = \infty$ *for* $N \leq 2$*). Put*

$$H(u) = \frac{1}{2}\int_\Omega |\bigtriangledown u|^2 dx - \frac{1}{p+1}\int_\Omega \phi(u)dx,$$

in which $\phi(u) = \{0 \text{ if } u < 0, u^{p+1} \text{ if } u \geq 0\}$.

Then we have

$$\delta = \inf_{0\not\equiv u\in H_0^1(\Omega)} \sup_{\lambda\geq 0} H(\lambda u) > 0.$$

Proof. Evidently, we have

$$H(\lambda u) = \frac{\lambda^2}{2}\int |\bigtriangledown u|^2 dx - \frac{\lambda^{p+1}}{p+1}\int \phi(u)dx. \tag{5.10}$$

On the other hand by Poincaré's inequality we have

$$(\int_\Omega \phi(u)dx)^{\frac{1}{p+1}} \leq c(\int |\bigtriangledown u|^2 dx)^{\frac{1}{2}}. \tag{5.11}$$

Hence

$$\begin{aligned}\sup_{\lambda\geq 0} H(\lambda u) &= H((\frac{\int_\Omega |\bigtriangledown u|^2 dx}{\int_\Omega \phi(u)dx})^{\frac{1}{p-1}} u) \\ &\geq (\frac{1}{2}-\frac{1}{p+1})\frac{(\int_\Omega |\bigtriangledown u|^2 dx)^{\frac{p+1}{p-1}}}{(\int_\Omega \phi(u)dx)^{\frac{2}{p-1}}} \geq (\frac{1}{2}-\frac{1}{p+1})c^{2\frac{(p+1)}{1-p}} > 0.\square\end{aligned}$$

Now, we introduce the stable set

$$W := \{u | u \in H_0^1(\Omega),\ 0 \leq H(\lambda u) < \delta, \lambda \in [0,1]\}.$$

Lemma 5.6. *We have*

$$W = W_* \cup \{0\},$$

where

$$W* = \{u | u \in H_0^1(\Omega),\ \int_\Omega |\bigtriangledown u|^2 dx > \int_\Omega \phi(u)dx,\ H(u) < \delta\}.$$

Proof. Suppose that $u \in W$, $u \not\equiv 0$, then we have

$$\sup_{\lambda\geq 0} H(\lambda u) = H\left((\frac{\int_\Omega |\bigtriangledown u|^2 dx}{\int_\Omega \phi(u)dx})^{\frac{1}{p-1}} u\right) \geq \delta,$$

and, hence

$$\left(\frac{\int_\Omega |\bigtriangledown u|^2 dx}{\int_\Omega \phi(u)dx}\right)^{\frac{1}{p-1}} > 1,$$

which implies that $u \in W_*$.

Now, let $u \in W_*$. Then, since $\frac{\partial}{\partial\lambda}H(\lambda u) > 0$ for $0 < \lambda \leq 1$ and $H(0) = 0$, we have

$$\sup_{0\leq\lambda\leq 1} H(\lambda u) = H(u) < \delta.\square$$

Proof of part (b) of Theorem 5.3. In order to prove this part, we employ the Galerkin's method. Let $\{w_k\}_{k=1,2,...}$ be a complete system of function in $H_0^1(\Omega)$ such that

$$\{w_k\}_{k=1}^{\infty} \text{ is an ortogonal basis of } H_0^1(\Omega),$$

and

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Omega).$$

Notice that the conditions (5.4) and (5.5) yield that $u_0 \in W$. let u_{0m} be a sequence such that

$$\int_\Omega |\bigtriangledown u_{0m}|^2 dx > \int_\Omega \phi(u_{0m})dx \quad \text{and} \quad H(u_{0m}) < \frac{\delta}{2} - \frac{1}{2}\|u_0'\|^2_{L^2(\Omega)}, \tag{5.12}$$

and

$$u_{0m} := \sum_{k=1}^{m} \alpha_{km} w_k \to u_o \quad \text{in} \quad W_0^{1,p}(\Omega).$$

For a positive integer m, we write

$$u_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k,$$

where the function $d_m^k(t)$ are determined by the following system of ordinary differential equations,

$$\begin{aligned} &d_m^k(0) = \alpha_{km} \quad (k = 1, \ldots, m), \\ &d_m^k(0)' = (u_{0t}, w_k) \quad (k = 1, \ldots, m) \end{aligned} \tag{5.13}$$

and

$$(u_m'', w_k) + (u_m', w_k) + (\bigtriangledown u_m, \bigtriangledown w_k) = (\varphi(u_m), w_k) \quad (0 \leq t \leq T, k = 1, \ldots, m), \tag{5.14}$$

for $\varphi(u) = \begin{cases} 0 & u < 0, \\ u^p & u \geq 0. \end{cases}$

Our plan hereafter is to let $m \to \infty$. In order to do this, we will need some uniform estimates on m.

If we multiply equation (5.14) by $d_m^k(t)'$ and sum it for $k = 1, \ldots, m$, we get

$$(u_m'', u_m') + (u_m', u_m') + (\bigtriangledown u_m, \bigtriangledown u_m') = (\varphi(u_m), u_m'), \tag{5.15}$$

for $0 \leq t \leq T$. Observe that $(u_m'', u_m') = \frac{d}{dt}(\frac{1}{2}\|u_m'\|^2_{L^2(\Omega)})$ and

$$(\bigtriangledown u_m, \bigtriangledown u_m') = \frac{d}{dt}(\frac{1}{2}\|\bigtriangledown u_m\|^2_{L^2(\Omega)}), \ (\varphi(u_m), u_m') = \frac{d}{dt}(\frac{1}{p+1}\|\phi(u_m)\|_{L^1(\Omega)}). \tag{5.16}$$

Hence

$$\frac{d}{dt}(\frac{1}{2}\|u_m'\|^2_{L^2(\Omega)}) + \|u_m'\|^2_{L^2(\Omega)} + \frac{d}{dt}(\frac{1}{2}\|\bigtriangledown u_m\|^2_{L^2(\Omega)}) = \frac{d}{dt}(\frac{1}{p+1}\|\phi(u_m)\|_{L^1(\Omega)}), \tag{5.17}$$

Thus
(5.18)
$$\frac{1}{2}\|u'_m\|^2_{L^2(\Omega)}+\int_0^t\|u'_m\|_{L^2(\Omega)}ds+\frac{1}{2}\|\bigtriangledown u_m\|^2_{L^2(\Omega)}=$$
$$\frac{1}{p+1}(\|\phi(u_m)\|_{L^1(\Omega)}-\|\phi(u_m(0))\|_{L^1(\Omega)})+\frac{1}{2}(\|u'_m(0)\|^2_{L^2(\Omega)}+\|\bigtriangledown u_m(0)\|^2_{L^2(\Omega)}).$$
Then, (5.18) yields

$$\frac{1}{2}\|u'_m\|^2_{L^2(\Omega)}+\int_0^t\|u'\|^2_{L^2(\Omega)}ds+H(u_m(t))=\frac{1}{2}\|u'_m(0)\|^2_{L^2(\Omega)}+H(u_m(0)) \tag{5.19}$$

In this step, we will show that

$$u_m(t)\in W,\ \forall t\geq 0 \tag{5.20}$$

Suppose that (5.20) does not hold. Let t^* be the smallest time for which $u_m(t^*)\notin W$. Since $u_m(0)\in W$ then $t^*>0$. Thus in virtue of $u_m(t)$, we see that $u_m(t^*)\in\partial W$. Hence from Lemma 5.6 we have

$$H(u_m(t^*))=\delta \tag{5.21}$$

or

$$\int_\Omega|\bigtriangledown u_m(t^*)|^2dx-\int_\Omega\phi(u_m(t^*))dx=0 \tag{5.22}$$

which contradicts the equality (5.19). Indeed, if (5.21) holds then according to (5.19), we get

$$\frac{1}{2}\|u'_m\|^2_{L^2(\Omega)}+\int_0^t\|u'\|^2_{L^2(\Omega)}+\delta<\frac{1}{2}\|u'_m(0)\|^2_{L^2(\Omega)}+\delta-\frac{1}{2}\|u'(0)\|^2_{L^2(\Omega)}$$

which is a contradiction, and if (5.22) holds, then

$$H(u_m(t^*))=H\left((\frac{\int_\Omega|\bigtriangledown u_m(t^*)|^2dx}{\int_\Omega\phi(u_m(t^*))dx})^{\frac{1}{p-1}}u_m(t^*)\right)\geq\delta$$

which is again a contradiction.

Now, by using Lemma 5.6 we can write

$$\begin{aligned}&\frac{1}{2}\|u'_m\|^2_{L^2(\Omega)}+\int_0^t\|u'\|_{L^2(\Omega)}ds+(\frac{1}{2}-\frac{1}{p+1})\|\bigtriangledown u_m\|^2_{L^2(\Omega)}\leq\\&\frac{1}{2}\|u'_m(0)\|^2_{L^2(\Omega)}+\frac{1}{2}\|\bigtriangledown u_m(0)\|^2_{L^2(\Omega)}-\frac{1}{p+1}\|\phi(u_m(0))\|_{L^1(\Omega)}\end{aligned} \tag{5.23}$$

Let $j_m(0):=\frac{1}{2}\|u'_m(0)\|^2_{L^2(\Omega)}+\frac{1}{2}\|\bigtriangledown u_m(0)\|^2_{L^2(\Omega)}-\frac{1}{p+1}\|\phi(u_m(0))\|_{L^1(\Omega)}$. Then (5.23) yields

$$\sup_{0\leq t\leq T}(\|u'_m\|^2_{L^2(\Omega)}+\|u_m\|^2_{H^1_0(\Omega)})\leq cj_m(0)<c\delta \tag{5.24}$$

for some constant $c>0$. Now, we will show that $\|u''_m\|_{L^2(0,T;H^{-1}(\Omega))}$ is uniformly bounded.

Let $v\in H^1_0(\Omega)$, $\|v\|_{H^1_0(\Omega)}\leq 1$. We can write $v=v_1+v_2$, where $v_1\in Span\{w_k\}_{k=1}^\infty$ and $(v_2,w_k)=0$, $k=1,\dots,m$. Notice that $\|v_1\|_{H^1_0(\Omega)}\leq 1$. Thus (5.14) implies

$$(u''_m,v)=(u''_m,v_1)=(\varphi(u_m),v_1)-(u'_m,v_1)-(\bigtriangledown u_m,\bigtriangledown v_1).$$

Then, from Lemma 5.6 and the last equality, we get

$$|(u''_m, v)| \leq c_1$$

for some positive constant c_1. Consequently

$$\int_0^T \|u''_m\|^2_{H^{-1}(\Omega)} dt \leq c_1 T. \tag{5.25}$$

Now, from (5.24) and (5.25), we get

$$\begin{aligned} &\|u_m\|_{L^\infty(0,T;H^1_0(\Omega))} \leq c \\ &\|u'_m\|_{L^2(0,T;L^2(\Omega))} \leq c \\ &\|u''_m\|_{L^2(0,T;H^{-1}(\Omega))} \leq c \end{aligned}$$

From Aubin's compactness theorem, we see that there exist a function u and a subsequence $\{u_{m_L}\}$ of $\{u_m\}$ such that

$$\begin{aligned} &u_{m_L} \to u, \quad \text{weakly in } L^\infty(0,T;H^1_0(\Omega)) \\ &u'_{m_L} \to u', \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \\ &u''_{m_L} \to u'', \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)) \end{aligned}$$

are fulfilled. Finally, if we let $m \to \infty$ in (5.14) and using the above fact,we will see u is a solution of the problem (5.1) in $[0,T]$.□

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