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A NOTE VIA DIAGONALITY OF THE 2×2 BHATTACHARYYA MATRICES

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Abstract. In this paper, we consider characterizations based on the Bhattacharyya matrices. We characterize, under certain constraint, distributions such as normal, compound poisson and gamma via the diagonality of the 2×2 Bhattacharyya matrix.

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1. Introduction

A lower bound for the variance of an estimator is a fundamental quantities in the estimation theory. The exact inequality giving the lower bound for the variance was established under some mild conditions by C. R. Rao (1945) and Cramer (1946). The Rao-Cramer inequality states that the variance of an unbiased estimator T of $\tau(\theta)$, satisfies

$$Var_\theta(T(\underline{X})) \geq \frac{\left(\tau^{(1)}(\theta)\right)^2}{E_\theta\{-\frac{\partial^2 \ln f(\underline{X}|\theta)}{\partial\theta^2}\}}, \tag{1.1}$$

where $\tau^{(1)}(\theta)$ is the derivative of τ w.r.t. θ. The denominator of (1.1) is called the Fisher information. An important inequality which followed the Rao-Cramer inequality is that of A. Bhattacharyya (1946,1947,1948). The Bhattacharyya inequality achieves a greater lower bound for the variance of an unbiased estimator of a parametric function, and it becomes sharper and sharper as the order of the Bhattacharyya matrix increases. Shanbhag (1972,1979) characterized " NEF-QVF " (natural exponential family with quadratic variance function) via diagonality of the Bhattacharyya matrix, and also showed that for this family, the Bhattacharyya matrix of any order exists and is diagonal. Alharbi (1994), D. Pommeret (1996), Alharbi, Shanbhag and Thabane (1997) and Mohtashami (2001) obtained results related to Bhattacharyya bound and Bhattacharyya matrices.

In this paper we deal with certain characterizations based on the diagonality of the Bhattacharyya matrices and also, deal with the diagonality of the Bhattacharyya matrices for compound Poisson and gamma families and for normal family with location parameter as a function of scale parameter and vice versa.

2. Bhattacharyya Inequality and Bhattacharyya Matrices

The Bhattacharyya inequality involves the covariance matrix of the random vector

$$\frac{1}{f(\underline{X}|\theta)}[f^{(1)}(\underline{X}|\theta), f^{(2)}(\underline{X}|\theta), ..., f^{(n)}(\underline{X}|\theta)],$$

where $f^{(j)}(.|\theta)$ is the j^{th} derivative of the probability density function $f(.|\theta)$ w.r.t. the parameter θ. The covariance matrix of the above random vector is referred to as the $n \times n$ Bhattacharyya matrix.

The Bhattacharyya inequality states that

$$V_\theta(T(\underline{X})) \geq \xi_\theta^t J^{-1} \xi_\theta, \tag{2.1}$$

where t is the notation for transpose and $\xi_\theta = (\tau^{(1)}(\theta), \tau^{(2)}(\theta), ..., \tau^{(k)}(\theta))^t$, $\tau(\theta) = E_\theta(T(\underline{X}))$ and $\tau^{(j)} = \frac{\partial^j E_\theta(T(\underline{X}))}{\partial \theta^j}$ for $j = 1, 2, ..., k$, and J^{-1} is the inverse of the Bhattacharyya matrix $J = (J_{rs})$ such that $J_{rs} = Cov\{\frac{f^{(r)}(X|\theta)}{f(X|\theta)}, \frac{f^{(s)}(X|\theta)}{f(X|\theta)}\}$ on noting $E_\theta(\frac{f^{(r)}(X|\theta)}{f(X|\theta)}) = 0, \quad r, s = 1, 2, ..., n$. If we substitute $k = 1$ in (2.1), it reduces to the Rao-Cramer inequality.

Let X be a non-degenerate r.v. distributed according to distribution with density :

$$f(x|\theta) = \frac{\exp\{xg(\theta)\}}{\beta(g(\theta))}\psi(x), \; x \in R \tag{2.2}$$

where $\theta \in \Theta$, Θ is an open interval, g is thrice differentiable and the support of the random variable is free of θ. For this family Shanbhag (1972,1979) proved that Bhattacharyya Matrix of any order is diagonal if and only if

$$\begin{cases} E_\theta(X) = c_{11} + c_{21}\theta \\ E_\theta(X^2) = c_{12} + c_{22}\theta + c_{23}\theta^2. \end{cases} \tag{2.3}$$

(2.3) implies that $V_\theta(X) = c_{13} + c_{23}\theta + c_{33}\theta^2$ and $g'(\theta) = \frac{c_{21}}{c_{13}+c_{23}\theta+c_{33}\theta^2}$, where c_{ij} $i, j = 1, 2, 3$, are constants. Shanbhag characterized the normal, gamma, binomial, negative binomial, Poisson and Meixner (hyperbolic) distributions as in the case of Laha and Lukacs (1960) via (2.3). The six distributions involved here are referred to by Morris (1982,1983) as NEF-QVF. Blight & Rao (1974) considered Bhattacharyya bounds for an unbiased estimator of a parametric function for an exponential family and illustrated via exponential and negative binomial distributions, that the Bhattacharyya bound actually converges to the variance of the best unbiased estimator for members of the Meixner class. More recently, Fosam & Shanbhag (1996) extended the Laha-Luckas characterization result, to have a charaterization based on a cubic regression property, subsuming, the Letac-Mora (1990) characterization of the natural exponential families (NEF). The distributions that are characterized in this case are inverse Gaussian, normal, gamma, Meixner hypergeometric, Poisson, negative binomial, binomial, Abel, Takacs, strict arcsine, Ressel, and large arcsine distributions.

Remark 2.1. We have

$$\begin{aligned} J_{ir} &= E_\theta\{\frac{f^{(r)}}{f}\sum_{l=0}^{i} d_l^{(i)}(\theta)X^l\} \\ &= \sum_{l=0}^{i} d_l^{(i)}(\theta)\frac{d^r E_\theta(X^l)}{d\theta^r}, \end{aligned}$$

where $d_l^{(i)}(\theta)$ is the coefficient of X^l in $\frac{f^{(i)}}{f}$. This yields that $E_\theta(X)$ is linear in θ, the assertion that $J_{ir} = J_{ri} = 0$ for $r > (i-1)(k-1)+1$; this latter result holds for the Letac-Mora family of distributions with $k = 3$ and the Shanbhag-Morris family with $k = 2$.

Remark 2.2. If we consider the structure of the 4×4 Bhattacharyya matrix in NEF with cubic variance, then we can easily find the inverse of the Bhattacharyya matrix and a good approximation for the variance of the best unbiased estimator by the Bhattacharyya bound in view of Blight & Rao (1974).

3. Results due to Diagonality of the Bhattacharyya Matrices

Characterization via the diagonality of a 2×2 Bhattacharyya matrix yields a wider class of distributions than that given in Shanbhag (1972,1979) via the diagonality of a 3×3 Bhattacharyya matrix. Shanbhag and Kapoor (1993) showed that a large class of distributions possess the property that the 2×2 Bhattacharyya matrix is diagonal. Now, we study the structure of the 2×2 Bhattacharyya matrices relative to exponential families.

Theorem 3.1. *Let X be a non-degenerate r.v. distributed according to (2.2), then a 2×2 Bhattacharyya matrix is diagonal if and only if*

$$M_\theta(t) = \exp\{c_1\{\Phi[g^{-1}(g(\theta)+t)] - \Phi(\theta)\} + c_2 t\}, \tag{3.1}$$

where $M_\theta(t)$ is the m.g.f. (moment generating function) of the r.v. X_θ and $\Phi(\theta) = \int_a^\theta tg'(t)dt$.

Proof. We know from Shanbhag (1972) or otherwise that $J_{12} = 0$ if and only if $E_\theta(X)$ is linear in θ, which, in turn, is equivalent to

$$\ln \beta(g(\theta)) = c_1 \int_a^{g^{-1}(g(\theta))} tg'(t)dt + c_2 g(\theta), \tag{3.2}$$

with $c_1,\ c_2,\ a$ as real constant. Clearly, (3.2) is equivalent to (3.1).

Remark 3.2. The above theorem also follows as a corollary to Theorem 1 of Alharbi, Shanbhag and Thabane (1997) on taking $c = 0$ in it and noting that

$$\int_{g(\theta)}^{g(\theta)+t} g^{-1}(z)dz = \int_\theta^{g^{-1}(g(\theta)+t)} yg'(y)dy.$$

Remark 3.3. Let $M(.)$ be the moment generating function of a non-degenerate distribution F with non-degenerate domain of definition which is not necessarily an open interval, then $M_\theta(t) = \frac{M(g(\theta)+t)}{M(g(\theta))}$ is the moment generating function of an exponential family $\frac{e^{g(\theta)x}}{M(g(\theta))}dF(x)$ with g as an appropriate function. In this case, we shall call the family $F-$generated exponential family. If we are interested only in an exponential family with parameter space as an open interval, we could consider a subfamily of the exponential family meeting the requirement.

Remark 3.4. Shanbhag and Kapoor (1993) considered an exponential family $dF_\theta(x) = \frac{e^{g(\theta)x}}{\beta(g(\theta))}d\mu(x)$ where $\beta(g(\theta)) = M(g(\theta) - c)$ such that M as the m.g.f. of a non-degenerate distribution, with an open interval, containing the origin, as its domain of definition, and μ is such that $e^{cx}d\mu(x)$ is the distribution corresponding to M, where g with an appropriate domain of definition is such that $kg^{-1}(t+c) = \frac{M'(t)}{M(t)}$, for some real c and k and each t. Following Shanbhag (1972), it is easily seen that the 2×2 Bhattacharyya matrix of this exponential family is diagonal. Indeed $M_\theta(t)$ of Theorem 3.1 (i.e. of the form of the m.g.f. of F_θ) can be shown to be of this form. Let $d \in \Theta$, where Θ is the parameter space. M_θ of (3.1) for $\theta = d$ reduces to

$$M_d(t) = \exp\{c_1\{\Phi[g^{-1}(g(d) + t)] - \Phi(d)\} + c_2 t\}.$$

Hence $\frac{M_d'(t)}{M_d(t)} = c_2 + c_1 g^{-1}(g(d) + t)$, which, in turn implies that the exponential family characterized by (3.1) is of the form of Shanbhag and Kapoor (1993) with their $M(t) = e^{-c_2 t}M_d(t),\ k = c_1$ and $c = g(d)$.

Corollary 3.5. *If $g(\theta) = k\theta^\beta,\ \beta \neq 0$ in Theorem 3.1, then via diagonality of the 2×2 Bhattacharyya matrix, the assertion of the theorem holds with M_θ of the following form :*

$$M_\theta(t) = \begin{cases} \exp\{c_2 t + kc_1\frac{\beta}{\beta+1}\{(\frac{k\theta^\beta+t}{k})^{\frac{\beta+1}{\beta}} - \theta^{\beta+1}\}\} & \text{if } \beta \neq -1 \\ \exp\{c_2 t\}(\frac{k}{k+\theta t})^{-kc_1} & \text{if } \beta = -1. \end{cases} \tag{3.3}$$

Remark 3.6. On noting the properties of extreme stable law, we have the following result related to Corollary 3.5:

- If ($\beta < -1$), ($\beta \geq 1$), ($\beta \in (-1, 0)$) and ($\beta = -1$), then, the m.g.f. $M_\theta(t)$ in (3.3) corresponds to the extreme stable law with exponent $\frac{\beta+1}{\beta} \in (0, 1)$-generated exponential family or the conjugate family within a shift, the extreme stable law with exponent $\frac{\beta+1}{\beta} \in (1, 2]$-generated exponential families or the conjugate family within a shift, compound gamma family or its conjugate family within a shift and gamma family or its conjugate family within a shift respectively.
- For the family (3.3), $J_{12} = J_{13} = 0$ and $J_{23} = kc_1\beta^2(1 - \beta^2)\theta^{\beta-4}$. Hence, for $\beta = 1$ and $\beta = -1$ the $n \times n$ ($n \geq 3$) Bhattacharyya matrix is diagonal (i.e. when F_θ is normal and gamma families respectively). $\beta = 4$ implies that J_{23} does not depend on the parameter θ and so, the off-diagonal elements of the 3×3 Bhattacharyya matrix are independent of the parameter θ.

4. Conclusion

The structure of the Bhattacharyya matrix relative to exponential families are discussed. Also, some well-known families are characterized via diagonality of 2×2 Bhattacharyya matrix that is belong to a wider class than the class of families that the Bhattacharyya matrices of any order is diagonal.

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