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ON QUASI UNIVERSAL COVERS FOR GROUPS ACTING ON TREES WITH INVERSIONS

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ABSTRACT. In this paper we show that if G is a group acting on a tree X with inversions and if (T;Y) is a fundamental domain for the action of G on X, then there exist a group \tilde{G} and a tree \tilde{X} induced by (T;Y) such that \tilde{G} acts on \tilde{X} with inversions, G is isomorphic to \tilde{G} , and X is isomorphic to \tilde{X} . The pair $(\tilde{G};\tilde{X})$ is called the quasi universal cover of (G;X) induced by the (T;Y).

Keywords: Groups acting on trees with inversions, Fundamental domains, Quasi universal cover, Isomorphic trees.

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1. INTRODUCTION

The structure of groups acting on trees without inversions known as Bass-Serre theory obtained in [6], and the action with inversions obtained by Mahmud in [5]. Let G is a group acting on a tree X without inversions, T be a maximal tree of the quotient graph Y for the action of G on X, and $\tilde{G} = \pi(G, Y, T)$ be the fundamental group of the graph of groups associated with Y relative T as defined in [6, p 42]. Various trees \tilde{X} were constructed on which \tilde{G} acts on \tilde{X} without inversions, G is isomorphic to \tilde{G} , and X is isomorphic to \tilde{X} . For more details we refer the readers to [1, p 419], or [2, p 205], or [6, p 55]. In this case $(\tilde{G}; \tilde{X})$ is called the universal cover of (G; X). In this paper we generalize such result to groups acting on trees with inversions as follows. Let G is a group acting on a tree X with inversions, (T; Y) be a fundamental domain for the action of G on X, and $\tilde{G} = \pi(T; Y)$ be the fundamental group of (T; Y) defined

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later. Then there exists a tree denoted $\tilde{X} = (T, Y)$ such that \tilde{G} acts on \tilde{X} with inversions, G is isomorphic to \tilde{G} , and X is isomorphic to \tilde{X} . The pair $(\tilde{G}; \tilde{X})$ is called the quasi universal cover of (G; X) relative to (T; Y).

We begin by giving preliminary definitions. By a graphX we understand a pair of disjoint sets V(X) and E(X) with V(X) non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$, and $\eta : E(X) \to E(X)$ satisfying the conditions that $\eta \partial_0 = \partial_1, \eta \partial_1 = \partial_0$, and η is an involution fixing some elements of E(X). For simplicity, if $e \in E(X)$, we write $\partial_0(e) = o(e)$, $\partial_1(e) = t(e)$, and $\eta(e) = \bar{e}$. This implies that $o(\bar{e}) = t(e)$, $t(\bar{e}) = o(e)$, and $\bar{e} = e$ on which the case $\bar{e} = e$ is allowed. We call the elements of V(X) vertices and those of E(X) edges. For $e \in E(X)$, we call o(e) the initial of e, t(e) the terminal of e, and \bar{e} the inverse of e. If A is a set of edges of X, define \bar{A} to be the set of inverses of the edges of A. That is, $\bar{A} = \{\bar{y} : y \in A\}$.

There are obvious definitions of subgraphs, trees, morphisms of graphs and Aut(X), the set of all automorphisms of the graph X which is a group under the composition of morphisms of graphs. For more details we refer the readers to Serre [6], or to Mahmud [5]. We say that a group G acts on a graph X, if there is a group homomorphism $\phi : G \to Aut(X)$. If $x \in X$ (vertex or edge) and $g \in G$, we write g(x) for $(\phi(g))(x)$. If $y \in E(X)$ and $g \in G$, then g(o(y)) = o(g(y)), g(t(y)) = t(g(y)), and $g(\bar{y}) = \overline{g(y)}$. The case $g(y) = \bar{y}$ for some $g \in G$ and some $y \in E(X)$ may occur. That is, G acts on X with inversions.

We have the following notations related to the action of the group G on the graph X.

(1) If $x \in X$ (vertex or edge), we define $G(x) = \{g(x) : g \in G\}$, and this set is called the orbit of x. (2) If $x, y \in X$, we define $G(x \to y) = \{g \in G : g(x) = y\}$ and $G(x \to x) = G_x$, the stabilizer of x. Thus $G(x \to y) \neq \emptyset$ if and only if x and y are in the same orbit. It is clear that if $v \in V(X)$, $y \in E(X)$, and $u \in \{o(y), t(y)\}$, then $G(v, y) = \emptyset$, $G_{\bar{y}} = G_y$, and $G_y \leq G_u$.

2. Structure of groups acting on trees with inversions

The aim of this section is to establish various notational conventions and results that we shall use throughout the paper.

Let G be a group acting on a tree X with inversions. Let T and Y be two subtrees of $X, T \subseteq Y$ satisfying the conditions that T contains exactly one vertex from each vertex orbit, and each edge of Y has at least one end in T and Y contains exactly one edge y from each edge orbit such that $G(y \to \bar{y}) = \emptyset$, and exactly one pair x and \bar{x} from each edge orbit such that $G(x \to \bar{x}) \neq \emptyset$. The pair (T; Y) is called a *fundamental domain* for the action of G on X. It is clear that the structure of Y implies that if e_1 and e_2 are two edges of Y such that e_1 and e_2 are in tha same G-edge orbit, then $e_1 = e_2$, or $e_1 = \bar{e}_2$. For the existence of T and Y we refer the readers to [3].

For the rest of this section G, X, T and Y will be as above. We have the following notations.

(i) Let +Y and -Y be the sets defined as follows. $+Y = \{y \in E(Y) : o(y) \in V(T), t(y) \notin V(T), G(y \to \bar{y}) = \emptyset\}$, and $-Y = \{x \in E(Y) : o(x) \in V(T), t(x) \notin V(T), G(x \to \bar{x}) \neq \emptyset\}$. It is clear that $Y = E(T) \cup +Y \cup +\overline{Y} \cup -Y \cup -\overline{Y}$. (ii) For each vertex v of X, let v^* be the unique vertex of T such that $G(v^* \to v) \neq \emptyset$. That is, v and v^* are in the same vertex orbit.

(iii) For each edge e of $E(T) \cup +Y \cup -Y$ define [e] be an element be an arbitrary element of $G(t(e) \to (t(e)^*)$. That is, $[e]((t(e))^*) = t(e)$ to be chosen as follows. [e] = 1 if $e \in E(T)$, and $e = \overline{e}$ if $e \in -Y$.

It is clear that $[e]^{-1}G_e[e]$ is a subgroup of $G_{(t(e))^*}$, and if $e \in -Y$, then $[e]^2 \in G_e$.

Proposition 2.1. G is generated by the elements [e] and by the generators of G_v , where e runs over the edges of Y and v runs over the vertices of T.

Proof. By Theorem 5.1 of [5].

3. QUASI UNIVERSAL COVERS FOR GROUPS ACTING ON TREES WITH INVERSIONS

Throughout this section G will be a group acting on a tree X with inversions, and (T; Y) be a fundamental domain for the action G on X. In [4], Mahmood introduced the concept of a subfundamental domain $(T_1; Y_1)$ for the action of G on X, and defined it is fundamental group $\pi(T_1; Y_1)$, and then showed that there exists a tree denoted $(T_1; Y_1)$ on which $\pi(T_1; Y_1)$ acts with inversions. In this section we take T_1 and Y_1 of Definition 4.1 of [4] to be $T_1 = T$ and $Y_1 = Y$, and $\tilde{G}_v = G_v$, $\tilde{G}_y = [y]^{-1}G_y[y]$ such that $\phi_y : [y]^{-1}G_y[y] \to G_y$ is given by $\phi_y(g) = [y]g[y]^{-1}$ and $\phi_y(g) = [y]g[y]^{-1}$ if $G(y,\bar{y}) \neq \emptyset$ for any vertex v of T and any edge y of Y. Then by Proposition 5.2 of [4] implies that the group $\pi(T, Y)$ has the presentation

$$\begin{split} \pi(T,Y) &= \langle G_v, t_y, t_x | \operatorname{rel} G_v, G_m = G_{\bar{m}}, \\ & t_y.[y]^{-1} G_y[y].t_y^{-1} = G_y, t_x.G_x.t_x^{-1} = G_x, t_x^2 = [y]^2 \rangle \end{split}$$

where $v \in V(T)$, $m \in E(T)$, $y \in +Y$, and $x \in -Y$.

The notations of the presentation of $\pi(T, Y)$ are defined as follows.

(i) $\langle G_v | rel G_v \rangle$ is any presentation of G_v .

(ii) $G_m = G_{\bar{m}}$ is the set of relations w(g) = w'(g), where w(g) and w'(g) are words in the generating symbols of $G_{t(m)}$ and $G_{o(m)}$ respectively of value g, where g is an element in the set of the generators of G_m .

(iii) $t_y \cdot [y]^{-1} G_y[y] \cdot t_y^{-1} = G_y$ is the set of relations $t_y w([y]^{-1}g[y])$ $t_y^{-1} = w(g)$, where $w([y]^{-1}g[y])$ and w(g) are words in the sets of generating symbols of $G_{(t(y))^*}$ and $G_{o(y)}$ of values $[y]^{-1}g[y]$ and g respectively, where g is an element in the set of the generators of G_y .

(iv) $t_x \cdot G_x \cdot t_x^{-1} = G_x$ is the set of relations $t_x w(g) t_x^{-1} = w'(g)$, where w(g) and w'(g) are words in the set of generating symbols of $G_{o(x)}$ of values g and $[x] g [x]^{-1}$ respectively, where g is an element in the set of the generators of G_x . (v) $t_x^2 = [x]^2$ is the relation $x^2 = w([x]^2)$, where $w([x]^2)$ is a word in the set of the generating symbols of $G_{o(x)}$ of value $[x]^2$.

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By Theorem 7.14 of [4], we have the tree (T, Y) defined as follows. $V((T, Y)) = \{[g, v] : g \in \pi(T, Y), v \in V(T)\}$, and $E((T, Y)) = \{[g, y] : g \in \pi(T, Y), y \in E(T) \cup +Y \cup +Y \cup -Y\}$, where [g, v] is the ordered pair (gG_v, v) and [g, y] is the ordered pair (gG_y, y) . (Note that if $g \in G_v$, or $g \in G_y$, then [g, v] = [1, v], and [g, y] = [1, y]). Define the ends and the inverse of the edge [g, y] of (T, Y) to be as follows. $o([g, y]) = [g, (o(y))^*]$, $t([g, y]) = [gt_y, (t(y))^*]$, and

$$\overline{[g,y]} = \begin{cases} [gt_y, \bar{y}]ify \in E(T) \cup +Y \cup \overline{+Y} \\ [gt_y, y]ify \in -Y \end{cases}$$

Proposition 7.4 of [4], implies that $\pi(T, Y)$ acts on (T, Y) with inversions as follows. If $f \in \pi(T, Y)$, $[g, v] \in V((T, Y))$, and $[g, v] \in E((T, Y))$, then f[g, v] = [fg, v], and f[g, y] = [fg, y].

We note that Corollary 7.5 of [4], implies that if $y \in -Y$, then $\pi(T, Y)$ inverts all edges [g, y] in (T, Y).

For example, the element t_y takes the edge [1, y] into its inverse $[t_y, y]$, because $\overline{[1, y]} = [t_y, y] = t_y[1, y]$.

Now we show that $\pi(T, Y)$ is isomorphic to G and $\pi(T, Y)$ is isomorphic to X. First we start by the following definitions and propositions.

Definition 3.1. Define the mapping $\theta : \pi(T, Y) \to G$ by the identity mapping on G_v and by the mapping $t_y \to [y], t_x \to [x]$, where $v \in V(T), y \in +Y$, and $x \in -Y$.

Proposition 3.2. θ is an onto homomorphism.

Proof. It is clear that the images [y] and [x] of y and x respectively under the given mapping $t_y \to [y], t_x \to [x]$, where $y \in +Y$, and $x \in -Y$ satisfy the defining relations $t_y.[y]^{-1}G_y[y].t_y^{-1} = G_y, t_x.G_x.t_x^{-1} = G_x$, and $t_x^2 = [x]^2$ of $\pi(T,Y)$. So, by Dyck's Theorem [2, Th.14. p.19] the given mapping defines the given homomorphism $\theta : \pi(T,Y) \to G$. Since by Proposition 2.1, G is generated by G_v and by [y] and [x], where $v \in V(T), y \in +Y$, and $x \in -Y$, therefore θ is an onto homomorphism. This completes the proof.

 $\begin{array}{l} \textbf{Definition 3.3. Define } \sigma: \overbrace{(T,Y)} \to X \text{ by } \sigma([g,v]) = (\theta(g))(v) \text{ , and} \\ \sigma([g,y] = \left\{ \begin{array}{l} (\theta(g))(y) \ if \ o(y) \in V(T) \\ (\theta(g))y \ if \ o(y) \notin V(T) \\ \text{where } v \in V(T), \text{ and } y \in E(T) \cup +Y \cup \overline{+Y} \cup -Y \right\}. \end{array} \right. \end{array}$

Proposition 3.4. σ is an onto morphism.

Proof. It is clear that σ maps vertices to vertices and edges to edges. If [f, u] and [g, v] are two vertices of (T, Y) such that [f, u] = [g, v], then u = v and $fG_v = gG_v$.

Then $f = gh, h \in G_v$, and

$$\begin{aligned} \sigma([f, u]) &= \sigma([gh, v]) \\ &= (\theta(gh))(v) \\ &= (\theta(g)\theta(h))(v) \text{ because } \theta \text{ is a homomorphism} \\ &= \theta(g)(\theta(h))(v) \\ &= \theta(g)(\sigma[h, v]) \\ &= \theta(g)(\sigma[1, v]) \text{ because} h \in G_v \\ &= \theta(g)(\theta(1))(v) \\ &= \theta(g)(\theta(1))(v) \\ &= \theta(g)(v) \\ &= \sigma([g, v]). \end{aligned}$$

Similarly, if [f, x] and [g, y] are two edges of (T, Y) such that [f, x] = [g, y], then $\sigma[f, x] = \sigma[g, y]$.

This implies that σ is a well-defined mapping.

Now let [g, y] be an edge of (T, Y). We need to prove the following. (i) $\sigma(o[g, y]) = o(\sigma[g, y])$, (ii) $\sigma(t[g, y]) = t(\sigma[g, y])$, and (iii) $\sigma([g, y]) = \overline{\sigma[g, y]}$. Now $\sigma(o([g, y])) = \sigma([g, (o(y))^*])$ $= (\theta(g))(o(y))^*$ $= \{ \begin{array}{l} (\theta(g))(o(y)) & \text{if } o(y) \in V(T) \\ (\theta(g))[y](o(y)) & \text{if } o(y) \notin V(T) \\ (\theta(g))[y](o(y)) & \text{if } o(y) \notin V(T) \end{array} \right.$

$$= \begin{cases} o((\theta(g))(y)) & \text{if } o(y) \in V(T) \\ o((\theta(g))y) & \text{if } o(y) \notin V(T) \end{cases}$$

$$= o(\sigma[g,y]).$$

$$\begin{split} \sigma(t([g,y])) &= \sigma([gt_y,(t(y))^*]) \\ &= (\theta(g)t_y)((t(y))^*) \\ &= (\theta(g)[y])((t(y))^*) \\ &= \begin{cases} (\theta(g))[y][\bar{y}](t(y)) & if \ o(y) \notin V(T) \\ (\theta(g))[y](t(y)) & if \ o(y) \notin V(T) \\ &= \begin{cases} t((\theta(g))(y) & if \ o(y) \notin V(T) \\ t((\theta(g))y & if \ o(y) \notin V(T) \\ &= t(\sigma[g,y]), \end{cases} \end{split}$$

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$$\begin{split} \sigma(\overline{[g,y]}) &= \begin{cases} [gt_y,\bar{y}] & if \ y \in E(T) \cup +Y \cup \overline{+Y} \\ [gt_y,y] & if \ y \in -Y \end{cases} \\ &= \begin{cases} (\theta(gt_y))\bar{y} & if \ y \in E(T) \cup +Y \\ (\theta(gt_y))(\bar{y}) & if \ y \in E(T) \cup \overline{+Y} \\ (\theta(gt_y))(y) & if \ y \in -Y \end{cases} \\ &= \begin{cases} (\theta(g[y]))\bar{y} & if \ y \in E(T) \cup +Y \\ (\theta(g[y]))(\bar{y}) & if \ y \in E(T) \cup \overline{+Y} \\ (\theta(g[y]))(y) & if \ y \in E(T) \cup \overline{+Y} \\ (\theta(g[y]))(y) & if \ y \in E(T) \cup \overline{+Y} \\ (\theta(g[y]))(\bar{y}) & if \ y \in E(T) \cup \overline{+Y} \\ (\theta(g[y]))(\bar{y}) & if \ y \in E(T) \cup \overline{+Y} \\ (\theta(g[y]))(\bar{y}) & if \ y \in -Y \end{cases} \\ &= \begin{cases} (\overline{\theta(g)})(\bar{y}) & if \ y \in E(T) \cup \overline{+Y} \\ (\theta(g))(\bar{y}) & if \ y \in -Y \end{cases} \\ &= \begin{cases} (\overline{\theta(g)})(y) & if \ y \in E(T) \cup \overline{+Y} \\ (\overline{\theta(g)})(y) & if \ y \in E(T) \cup \overline{+Y} \\ (\overline{\theta(g)})(y) & if \ y \in -Y \end{cases} \\ &= \\ \hline \overline{\sigma[g, y]}. \end{split}$$

Thus σ is a well-defined morphism. Since θ is onto, therefore σ is onto. This completes the proof.

The following concept is needed in order to show that $\sigma: (T, Y) \to X$ is an isomorphism.

If Γ_1 and Γ_2 are two graphs, and $f : \Gamma_1 \to \Gamma_2$ is a morphism, then locally injective if for every two edges e_1 and e_2 of Γ_1 such that $o(e_1) = o(e_2)$, and $f(e_1) = f(e_2)$, then $e_1 = e_2$.

The following proposition is essential to prove the main theorem of this section.

Proposition 3.5. σ is locally injective.

Proof. Let $[a_1, e_1]$ and $[a_2, e_2]$ be two edges of (\overline{T}, Y) such that $o[a_1, e_1] = o[a_2, e_2]$ and $\sigma[a_1, e_2] = \sigma[a_2, e_2]$. We need to show that $e_1 = e_2$, and $a_1^{-1}a_2 \in G_{e_1}$. It is clear that $(o(e_1))^* = (o(e_2))^*$, $a_1^{-1}a_2 \in G_{(o(e_1))^*}$, and $\theta(a_1^{-1}a_2)(e_1) = e_2$. This implies that e_1 and e_2 are in the same G-edge orbit on X. Since e_1 and e_2 are in Y, therefore the properties of Y imply that $e_1 = e_2$, or $e_1 = \overline{e_2}$. If $e_1 = e_2$, then it is clear that $\theta(a_1^{-1}a_2) \in G_{e_1}$. Since $G_{e_1} \leq G_{(o(e_1))^*}$, and θ is the identity on $G_{(o(e_1))^*}$, therefore θ is the identity on G_{e_1} . This implies that $a_1^{-1}a_2 \in G_{e_1}$. Consequently $[a_1, e_1] = [a_2, e_2]$.

 $a_1^{-1}a_2 \in G_{e_1}$. Consequently $[a_1, e_1] = [a_2, e_2]$. If $e_1 = \overline{e_2}$, then $G(e_1, \overline{e_1}) \neq \emptyset$. This implies that e_2 is in $\overline{-Y_1}$. This contradicts the fact that the edges of (T, Y) are of the forms [g, e], where $g \in \pi(T, Y)$, and $y \in E(T) \cup +Y \cup \overline{+Y} \cup -Y$. This completes the proof.

For the proof of the following lemma we refer the readers to [6, Lemma 5, p 39].

Lemma 3.6. If Γ_1 is a connected graph, Γ_2 is a tree, and $f : \Gamma_1 \to \Gamma_2$ is locally injective, then f is injective.

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Now we state the main result of this section.

Theorem 3.7. Let G, X, T, Y, σ , and θ be as above such that X is a tree. Then $(1) \sigma : (T, Y) \to X$ is an isomorphism. $(2) \theta : \pi(T, Y) \to G$ is an isomorphism.

Proof. (i) By Proposition 3.4, σ is an onto morphism, and by Proposition 3.5, σ is locally injective. Since X is a tree, therefore by Lemma 3.6. σ is an isomorphism.

(ii) Suppose that $h \in ker(\theta)$, and v be any vertex of T. Then $\sigma[1, v] = (\theta(1))(v) = v$, because θ is a homomorphism, and $\sigma[h, v] = (\theta(h))(v) = v$, since $h \in ker(\theta)$. Then we have $\sigma[h, v] = \sigma[1, v]$. Since σ is an isomorphism, therefore [h, v] = [1, v]. This implies that $h \in G_v$. Since θ restricted to G_v is an isomorphism, therefore h = 1. Since by Proposition 3.2, θ is an onto homomorphism, therefore θ is an isomorphism. This completes the proof.

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