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The Polynomials of a Graph

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ABSTRACT. In this paper, we are presented a formula for the polynomial of a graph. Our main result is the following formula:

$$\sum_{u \in V(G)} d_u^{\ k} = \sum_{j=1}^k a_{kj} \ S_G^{(j)}(1).$$

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1. INTRODUCTION

The graphs in this paper are connected and simple. Denote the vertex and edge sets of graph G by V(G) and E(G), respectively. For a simple graph G(p,q), we

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define the degree sequence of G as

$$S: d_1, d_2, \cdots, d_p$$

where $d_i = degv_i$, $1 \leq i \leq p$, and v_i 's are vertices of G. Suppose a_0 is number of vertices of degree 0, a_1 the number of vertices of degree 1, ..., and $a_{\Delta(G)}$ is number of number vertices of degree $\Delta(G)$, where $\Delta(G) = \max\{d_i\}$. The polynomial of G is defined as:

Definition 1.1. If $S : d_1, d_2, \dots, d_p$ is a degree sequence of graph G. Then the polynomial of graph G is

$$S_G(x) = \sum_{j=0}^{\Delta(G)} a_j x^j$$

Also a polynomial p(x) is said to be graphical if there exists a graph G such that $p(x) = S_G(x)$.

Example 1.2. Suppose G is defined by the following diagram:



Then the degree sequence of G is S : 0, 1, 1, 2, 3, 3 and $\Delta(G) = 3$. Thus the polynomial of G is

$$S_G(x) = \sum_{j=0}^3 a_j x^j$$

where $a_0 = 1, a_1 = 2, a_2 = 1$ and $a_3 = 2$. Hence we have

$$S_G(x) = 1x^0 + 2x + 1x^2 + 2x^3 = 1 + 2x + x^2 + 2x^3.$$

Remark 1.3. It is easy to see that

$$S_G(x) = \sum_{j=0}^{\Delta(G)} a_j x^j = \sum_{u \in V(G)} x^{d_u}$$

where d_u is the degree of u.

Corollary 1.4. If G(p,q) is a graph with p vertices and q edges, then we have:

(1)
$$S_G(1) = p$$
 (2) $\sum_{j=0}^{\Delta(G)} ja_j = 2q$ (3) $S'_G(1) = 2q = \sum_{u \in V(G)} d_u$

Suppose P_n, C_n, K_n denoted the path, cycle and complete graphs with exactly n vertices, respectively. Also a general k-regular graph is denoted by G_k . Then,

$$S_{P_n}(x) = 2x + (n-2)x^2$$
 $S_{C_n}(x) = nx^2$
 $S_{K_n}(x) = nx^{n-1}$ $S_{G_k}(x) = p x^k$

Definition 1.5. Let G_1 and G_2 be two graphs. If $V(G_1) \cap V(G_2) = \phi$. Then

- (1) $G_1 \cup G_2$ is a graph that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$
- (2) $G_1 \times G_2$ is a graph that $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $\{(u, v), (u', v')\} \in E(G_1 \times G_2)$ if and only if u = u' and $\{v, v'\} \in E(G_2)$ or v = v' and $\{u, u'\} \in E(G_1)$
- (3) $G_1 + G_2$ is a graph that $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{\{u, v\} \mid u \in V(G_1), v \in V(G_2)\}$

Example 1.6. Suppose G_1 and G_2 are two graphs such that their diagrams are as follows:



then the diagram graph $G_1 \times G_2$ and $G_1 + G_2$ as follows:



Theorem 1.7. If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two graphs, then the polynomial of graphs $G_1 \cup G_2$, $G_1 \times G_2$ and $G_1 + G_2$ are given by

(1)
$$S_{G_1 \cup G_2}(x) = S_{G_1}(x) + S_{G_2}(x)$$

(2) $S_{G_1 \times G_2}(x) = S_{G_1}(x) \cdot S_{G_2}(x)$
(3) $S_{G_1 + G_2}(x) = x^{p_2}S_{G_1}(x) + x^{p_1}S_{G_2}(x)$

Proof.

(1)
$$S_{G_1 \cup G_2}(x) = \sum_{u \in V(G_1 \cup G_2)} x^{d_u} = \sum_{u \in V(G_1)} x^{d_u} + \sum_{u \in V(G_2)} x^{d_u}$$
$$= S_{G_1}(x) + S_{G_2}(x)$$

(2)
$$S_{G_1 \times G_2}(x) = \sum_{u \in V(G_1 \times G_2)} x^{d_u} = \sum_{u = (u_1, u_2) \in V(G_1 \times G_2)} x^{d_u}$$
$$= \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} x^{d_{u_1} + d_{u_2}} = \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} x^{d_{u_1}} x^{d_{u_2}}$$
$$= \sum_{u_1 \in V(G_1)} x^{d_{u_1}} \cdot \sum_{u_2 \in V(G_2)} x^{d_{u_2}}$$
$$= S_{G_1}(x) \cdot S_{G_2}(x)$$

(3)
$$S_{G_1+G_2}(x) = \sum_{u \in V(G_1+G_2)} x^{d_u} = \sum_{u \in V(G_1)} x^{d_u+p_2} + \sum_{u \in V(G_2)} x^{d_u+p_1}$$
$$= x^{p_2} \sum_{u \in V(G_1)} x^{d_u} + x^{p_1} \sum_{u \in V(G_2)} x^{d_u}$$
$$= x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x)$$

Corollary 1.8. If $S_{G_1}(x)$ and $S_{G_2}(x)$ are graphical then

S_{G1}(x) ⋅ S_{G2}(x) is graphical and conversely.
 x^{p2}S_{G1}(x) + x^{p1}S_{G2}(x) is graphical and conversely.
 ∑_{u∈V(G1×G2)} d_u = 2 (p₁ q₂ + p₂ q₁)

(4)
$$\sum_{u \in V(G_1+G_2)} d_u = 2 (p_1 \ p_2 + q_1 + q_2)$$

Example 1.9. The polynomial $S_G(x) = 4x^2 + 4x^3 + x^4$ is graphical, because

$$S_G(x) = 4x^2 + 4x^3 + x^4 = (2x + x^2)^2$$

On the other hand, we have the following graph for the polynomial $S_{G_1}(x) = 2x + x^2$.



Hence the polynomial $S_G(x)$ is graphical, because $S_G(x) = S_{G_1}(x) \times S_{G_1}(x)$. Also its graph is as follows:



Example 1.10. The polynomial $S_G(x) = 3x^4 + 2x^3$ is graphical, because

$$S_G(x) = 3x^4 + 2x^3 = 2x^4 + x^4 + 2x^3 = x^3(2x) + x^2(x^2 + 2x)$$

On the other hand, we have the following graphs for the polynomials $S_{G_1}(x) = 2x$ and $S_{G_2}(x) = x^2 + 2x$, respectively:



Hence the polynomial $S_G(x)$ is graphical, because $S_G(x) = x^{p_2}S_{G_1}(x) + x^{p_1}S_{G_2}(x)$. Also its graph is as following:



Definition 1.11. Let G be a graph. The polynomial $H_G(x)$ is defined as follows:

$$H_G(x) = \sum_{\{u,v\}\in E(G)} x^{d_u + d_v}$$

Example 1.12. The polynomial $H_G(x) = x^3 + x^3 = 2x^3$ is the graph polynomial of the following graph:



Corollary 1.13. Let G(p,q) is a graph with p vertices and q edges. Then we have:

$$\begin{aligned} H_G(1) &= q & H'_G(1) = \sum_{\{u,v\} \in E(G)} d_u + d_v = \sum_{u \in V(G)} d_u^2 \\ H_{P_n}(x) &= 2x^3 + (n-3)x^4 & H_{C_n}(x) = nx^4 \\ H_{K_n}(x) &= \frac{n(n-1)}{2}x^{2n-2} & H_{G_k}(x) = qx^{2k} \end{aligned}$$

Theorem 1.14. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two graphs. Then

(1) $H_{G_1 \cup G_2}(x) = H_{G_1}(x) + H_{G_2}(x)$ (2) $H_{G_1 \times G_2}(x) = H_{G_1}(x) \cdot S_{G_2}(x^2) + H_{G_2}(x) \cdot S_{G_1}(x^2)$ (3) $H_{G_1+G_2}(x) = x^{2p_2} H_{G_1}(x) + x^{2p_1} H_{G_2}(x) + x^{p_1+p_2} S_{G_1}(x) \cdot S_{G_2}(x)$

Proof. (1) is trivial. To prove (2), we have:

$$H_{G_1 \times G_2}(x) = \sum_{\{u,v\} \in E(G_1 \times G_2)} x^{d_u + d_v}$$

$$= \sum_{u_1 = v_1} \sum_{\{u_2, v_2\} \in E(G_2)} x^{2d_{u_1} + d_{v_2} + d_{u_2}}$$

$$+ \sum_{\{u_1, v_1\} \in E(G_1)} \sum_{u_2 = v_2} x^{d_{u_1} + d_{v_1} + 2d_{u_2}}$$

$$= \sum_{\{u_2, v_2\} \in E(G_2)} x^{d_{u_2} + d_{v_2}} \sum_{u_1 \in V(G_1)} (x^2)^{d_{u_1}}$$

$$+ \sum_{\{u_1, v_1\} \in E(G_1)} x^{d_{u_1} + d_{v_1}} \sum_{u_2 \in V(G_2)} (x^2)^{d_{u_2}}$$

$$= H_{G_2}(x) S_{G_1}(x^2) + H_{G_1}(x) S_{G_2}(x^2)$$

$$H_{G_1+G_2}(x) = \sum_{\{u,v\}\in E(G_1+G_2)} x^{d_u+d_v}$$

= $\sum_{\{u,v\}\in E(G_1)} x^{d_u+d_v+2p_2} + \sum_{\{u,v\}\in E(G_2)} x^{d_u+d_v+2p_1}$
+ $\sum_{u\in V(G_1), v\in V(G_2)} x^{d_u+d_v+p_1+p_2}$
= $x^{2p_2} \sum_{\{u,v\}\in E(G_1)} x^{d_u+d_v} + x^{2p_1} \sum_{\{u,v\}\in E(G_2)} x^{d_u+d_v}$
+ $x^{p_1+p_2} \sum_{u\in V(G_1)} x^{d_u} \sum_{v\in V(G_2)} x^{d_v}$
= $x^{2p_2} H_{G_1}(x) + x^{2p_1} H_{G_2}(x) + x^{p_1+p_2} S_{G_1}(x) S_{G_2}(x)$

Example 1.15. Consider the following diagrams for graphs G_1 and G_2 :



then, we have:

$$H_{G_1}(x) = 2x^3$$
 $S_{G_1}(x) = 2x + x^2$
 $H_{G_2}(x) = 3x^4$ $S_{G_2}(x) = 3x^2$

Thus:

$$H_{G_1+G_2}(x) = x^6(2x^3) + x^6(3x^4) + x^6(2x+x^2)(3x^2)$$
$$= 2x^9 + 3x^{10} + 6x^9 + 3x^{10} = 8x^9 + 6x^{10}$$

Hence the diagram $G_1 + G_2$ is:



Corollary 1.16.

$$\sum_{u \in V(G_1 \times G_2)} d_u^2 = p_2 \sum_{u \in V(G_1)} d_u^2 + p_1 \sum_{u \in V(G_2)} d_u^2 + 8q_1q_2$$

Proof. We know that

$$H_{G_1 \times G_2}(x) = H_{G_1}(x)S_{G_2}(x^2) + H_{G_2}(x)S_{G_1}(x^2)$$

Hence,

$$H'_{G_1 \times G_2}(x) = H'_{G_1}(x)S_{G_2}(x^2) + 2xH_{G_1}(x)S'_{G_2}(x^2) + H'_{G_2}(x)S_{G_1}(x^2) + 2xH_{G_2}(x)S'_{G_1}(x^2)$$

Therefore

$$H'_{G_1 \times G_2}(1) = H'_{G_1}(1)S_{G_2}(1) + 2H_{G_1}(1)S'_{G_2}(1) + H'_{G_2}(1)S_{G_1}(1) + 2H_{G_2}(1)S'_{G_1}(1)$$

On the other hand, we know that $H_G(1) = q$, $H'_G(1) = \sum_{u \in V(G)} d_u^2$, $S_G(1) = p$ and $S'_G(1) = 2q$. Thus

$$\sum_{u \in V(G_2 \times G_1)} d_u^2 = p_2 \sum_{u \in V(G_1)} d_u^2 + 4q_1q_2 + p_1 \sum_{u \in V(G_2)} d_u^2 + 4q_1q_2$$
$$= p_2 \sum_{u \in V(G_1)} d_u^2 + p_1 \sum_{u \in V(G_2)} d_u^2 + 8q_1q_2$$

Definition 1.17. Let G be a graph. The polynomial $F_G(x)$ is defined as follows:

$$F_G(x) = \sum_{u \in V(G)} d_u x^{d_u}$$

Example 1.18. The polynomial of the graph G defined by the following graph is $H_G(x) = 2x + 2x^2$.



Corollary 1.19. We have:

$$\begin{split} F_G(1) &= S'_G(1) & F'_G(1) = H'_G(1) \\ F_{P_n}(x) &= 2x + 2(n-2)x^2 & F_{C_n}(x) = 2nx^2 \\ F_{K_n}(x) &= n(n-1)x^{n-1} & F_{G_k}(x) = kp \ x^k \end{split}$$

Theorem 1.20. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two graphs. Then

(1) $F_{G_1 \cup G_2}(x) = F_{G_1}(x) + F_{G_2}(x)$ (2) $F_{G_1 \times G_2}(x) = F_{G_1}(x) \cdot S_{G_2}(x) + F_{G_2}(x) \cdot S_{G_1}(x)$ (3) $F_{G_1 + G_2}(x) = x^{p_2} F_{G_1}(x) + p_2 x^{p_2} S_{G_1}(x) + x^{p_1} F_{G_2}(x) + p_1 x^{p_1} S_{G_2}(x)$

Proof. (1) is trivial. Prove (2), we have:

$$\begin{split} F_{G_1 \times G_2}(x) &= \sum_{u \in V(G_1 \times G_2)} d_u x^{d_u} \\ &= \sum_{(u_1, u_2) \in V(G_1) \times V(G_2)} (d_{u_1} + d_{u_2}) x^{d_{u_1} + d_{u_2}} \\ &= \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} (d_{u_1} + d_{u_2}) x^{d_{u_1} + d_{u_2}} \\ &= \sum_{u_2 \in V(G_2)} x^{d_{u_2}} \sum_{u_1 \in V(G_1)} d_{u_1} x^{d_{u_1}} + \sum_{u_1 \in V(G_1)} x^{d_{u_1}} \sum_{u_2 \in V(G_2)} d_{u_2} x^{d_{u_2}} \\ &= F_{G_1}(x) \cdot S_{G_2}(x) + F_{G_2}(x) \cdot S_{G_1}(x) \\ F_{G_1 + G_2}(x) &= \sum_{u \in V(G_1 + G_2)} d_u x^{d_u} \\ &= \sum_{u \in V(G_1)} (d_u + p_2) x^{d_u + p_2} + \sum_{u \in V(G_2)} (d_u + p_1) x^{d_u + p_1} \\ &= x^{p_2} \sum_{u \in V(G_1)} d_u x^{d_u} + p_2 x^{p_2} \sum_{u \in V(G_2)} x^{d_u} \end{split}$$

$$= x^{p_2} F_{G_1}(x) + p_2 x^{p_2} S_{G_1}(x) + x^{p_1} \sum_{u \in V(G_2)} x^{d_u}$$

Definition 1.21. Let G be a graph. The polynomial $W_G(x)$ is defined as following:

$$W_G(x) = \sum_{\{u,v\} \in E(G)} (d_u + d_v) x^{d_u + d_v}$$

Example 1.22. Consider the following diagram for the graph G. Then $W_G(x) = 3x^3 + 3x^3 = 6x^3$.



Corollary 1.23. We have:

$$W_{G}(1) = \sum_{\{u,v\} \in E(G)} d_{u} + d_{v} = \sum_{u \in V(G)} d_{u}^{2} \qquad W_{G}(1) = H'_{G}(1)$$
$$W_{P_{n}}(x) = 6x^{3} + 4(n-3)x^{4} \qquad W_{C_{n}}(x) = 4nx^{4}$$
$$W_{K_{n}}(x) = n(n-1)^{2}x^{2n-2} \qquad W_{G_{k}}(x) = 2kqx^{2k}$$

Theorem 1.24. Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two graphs. Then

- (1) $W_{G_1 \cup G_2}(x) = W_{G_1}(x) + W_{G_2}(x)$
- (2) $W_{G_1 \times G_2}(x) = 2F_{G_1}(x^2) \cdot H_{G_2}(x) + S_{G_1}(x^2) \cdot W_{G_2}(x) + 2F_{G_2}(x^2) \cdot H_{G_1}(x) + S_{G_2}(x^2) \cdot W_{G_1}(x)$
- (3) $W_{G_1+G_2}(x) = x^{2p_2} W_{G_1}(x) + 2p_2 x^{2p_2} H_{G_1}(x) + x^{2p_1} W_{G_2}(x) + 2p_1 x^{2p_1} H_{G_2}(x) + x^{p_1+p_2} F_{G_1 \times G_2}(x) + (p_1 + p_2) x^{p_1+p_2} S_{G_1 \times G_2}(x)$

Proof. (1) is trivial. To prove (2), we consider the following equation:

$$\begin{split} W_{G_1 \times G_2}(x) &= \sum_{\{u,v\} \in E(G_1 \times G_2)} (d_u + d_v) x^{d_u + d_v} \\ &= \sum_{u_1 = v_1} \sum_{\{u_2, v_2\} \in E(G_2)} (2d_{u_1} + d_{u_2} + d_{v_2}) x^{2d_{u_1} + d_{u_2} + d_{v_2}} \\ &+ \sum_{u_2 = v_2} \sum_{\{u_1, v_1\} \in E(G_1)} (2d_{u_2} + d_{u_1} + d_{v_1}) x^{2d_{u_2} + d_{u_1} + d_{v_1}} \\ &= 2 \sum_{u_1 \in V(G_1)} d_{u_1}(x^2)^{d_{u_1}} \sum_{\{u_2, v_2\} \in E(G_2)} x^{d_{u_2} + d_{v_2}} \\ &+ \sum_{u_1 \in V(G_1)} (x^2)^{d_{u_1}} \sum_{\{u_2, v_2\} \in E(G_2)} (d_{u_2} + d_{v_2}) x^{d_{u_2} + d_{v_2}} \\ &+ 2 \sum_{u_2 \in V(G_2)} d_{u_2}(x^2)^{d_{u_2}} \sum_{\{u_1, v_1\} \in E(G_1)} x^{d_{u_1} + d_{v_1}} \\ &+ \sum_{u_2 \in V(G_2)} (x^2)^{d_{u_2}} \sum_{\{u_1, v_1\} \in E(G_1)} (d_{u_1} + d_{v_1}) x^{d_{u_1} + d_{v_1}} \\ &= 2F_{G_1}(x^2) \cdot H_{G_2}(x) + S_{G_1}(x^2) \cdot W_{G_2}(x) \\ &+ 2F_{G_2}(x^2) \cdot H_{G_1}(x) + S_{G_2}(x^2) \cdot W_{G_1}(x) \end{split}$$

$$\begin{split} W_{G_1+G_2}(x) &= \sum_{\{u,v\}\in E(G_1+G_2)} (d_u+d_v) x^{d_u+d_v} \\ &= \sum_{\{u,v\}\in E(G_1)} (d_u+d_v+2p_2) x^{d_u+d_v+2p_2} \\ &+ \sum_{\{u,v\}\in E(G_2)} (d_u+d_v+2p_1) x^{d_u+d_v+2p_1} \\ &+ \sum_{u\in V(G_1), v\in V(G_2)} (d_u+d_v+p_1+p_2) x^{d_u+d_v+p_1+p_2} \\ &= x^{2p_2} \sum_{\{u,v\}\in E(G_1)} (d_u+d_v) x^{d_u+d_v} + 2p_2 x^{2p_2} \sum_{\{u,v\}\in E(G_1)} x^{d_u+d_v} \\ &+ x^{2p_1} \sum_{\{u,v\}\in E(G_2)} (d_u+d_v) x^{d_u+d_v} + 2p_1 x^{2p_1} \sum_{\{u,v\}\in E(G_2)} x^{d_u+d_v} \\ &+ x^{p_1+p_2} \sum_{u\in V(G_1), v\in V(G_2)} (d_u+d_v) x^{d_u+d_v} \\ &+ (p_1+p_2) x^{p_1+p_2} \sum_{u\in V(G_1), v\in V(G_2)} x^{d_u+d_v} \\ &= x^{2p_2} W_{G_1}(x) + 2p_2 x^{2p_2} H_{G_1}(x) + x^{2p_1} W_{G_2}(x) + 2p_1 x^{2p_1} H_{G_2}(x) \\ &+ x^{p_1+p_2} F_{G_1\times G_2}(x) + (p_1+p_2) x^{p_1+p_2} S_{G_1\times G_2}(x) \end{split}$$

In the end of this paper, we define a new triangle \mathcal{A} as follows:

that entry a_{ij} of triangle \mathcal{A} is:

$$a_{ij} = \begin{cases} 1 & j = 1 \text{ or } j = i \\ a_{(i-1)(j-1)} + j a_{(i-1)j} & 1 < j < i \end{cases}$$

Theorem 1.25. If G is a graph with the polynomial $S_G(x)$, then

$$\sum_{u \in V(G)} d_u^{\ k} = \sum_{j=1}^k a_{kj} \ S_G^{(j)}(1)$$

where $k \in \mathbb{N}$ and $a_{kj} \in \mathcal{A}$.

Example 1.26. Let G is a graph, such that its diagram is as following:



Hence the degree sequence and the polynomial $S_G(x)$ are "1, 1, 2" and $2x + x^2$, respectively. Thus for k = 3 we have:

$$\sum_{u \in V(G)} d_u{}^3 = 1^3 + 1^3 + 2^3 = 10$$

On the other hand, we have $S'_G(1) = 4$, $S''_G(1) = 2$, $S^{(3)}_G(1) = 0$, $a_{31} = 1$, $a_{32} = 3$ and $a_{33} = 1$. Therefore

$$\sum_{j=1}^{3} a_{3j} S_G^j(1) = 1 \times 4 + 3 \times 2 + 1 \times 0 = 10$$

Proof of Theorem 1.25. According to remark (1.3) $S_G(x) = \sum_{u \in V(G)} x^{d_u}$. Hence,

(1.1)
$$S'_G(x) = \sum_{u \in V(G)} d_u x^{d_u - 1}$$

therefore

$$S'_G(1) = \sum_{u \in V(G)} d_u$$

On the other hand, according to table of \mathcal{A} for k = 1, we have:

$$\sum_{j=1}^{1} a_{1j} S_G^{(j)}(1) = a_{11}S_G'(1) = S_G'(1)$$

From above relations, we obtain that the theorem (1.25) for k = 1 is true. Now from the relation (1.1), we have $xS'_G(x) = \sum_{u \in V(G)} d_u x^{d_u}$ then

(1.2)
$$S'_G(x) + x S''_G(x) = \sum_{u \in V(G)} d_u^2 x^{d_u - 1}$$

therefore

$$S'_{G}(1) + S''_{G}(1) = \sum_{u \in V(G)} d_{u}{}^{2}$$

On the other hand, according to table of \mathcal{A} for k = 2, we have:

$$\sum_{j=1}^{2} a_{2j} S_G^{(j)}(1) = a_{21} S_G'(1) + a_{22} S_G''(1) = S_G'(1) + S_G''(1)$$

From two relations before, we obtain that the theorem (1.25) for k = 2 is true. Similarly from the relation (1.2), we can prove the theorem (1.25) for k = 3. Therefore, if we continue the above process, then the proof is completed.

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