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The Polynomials of a Graph

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Abstract. In this paper, we are presented a formula for the polynomial of a graph. Our main result is the following formula:

$$\sum_{u\in V(G)} {d_u}^k = \sum_{j=1}^{k} a_{kj}\ S_G^{(j)}(1).$$

Keywords: Graph, polynomial, graphical sequence.

2000 Mathematics subject classification: P54E40; 54E35; 54H25.

1. Introduction

The graphs in this paper are connected and simple. Denote the vertex and edge sets of graph G by $V(G)$ and $E(G)$, respectively. For a simple graph $G(p, q)$, we

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define the degree sequence of G as

$$S : d_1, d_2, \cdots, d_p$$

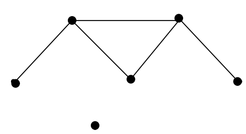
where $d_i = degv_i$, $1 \leq i \leq p$, and v_i's are vertices of G. Suppose a_0 is number of vertices of degree 0, a_1 the number of vertices of degree 1, ..., and $a_{\Delta(G)}$ is number of number vertices of degree $\Delta(G)$, where $\Delta(G) = \max\{d_i\}$. The polynomial of G is defined as:

Definition 1.1. If $S : d_1, d_2, \cdots, d_p$ is a degree sequence of graph G. Then the polynomial of graph G is

$$S_G(x) = \sum_{j=0}^{\Delta(G)} a_j x^j$$

Also a polynomial $p(x)$ is said to be graphical if there exists a graph G such that $p(x) = S_G(x)$.

Example 1.2. Suppose G is defined by the following diagram:



Then the degree sequence of G is $S : 0, 1, 1, 2, 3, 3$ and $\Delta(G) = 3$. Thus the polynomial of G is

$$S_G(x) = \sum_{j=0}^{3} a_j x^j$$

where $a_0 = 1, a_1 = 2, a_2 = 1$ and $a_3 = 2$. Hence we have

$$S_G(x) = 1x^0 + 2x + 1x^2 + 2x^3 = 1 + 2x + x^2 + 2x^3.$$

Remark 1.3. It is easy to see that

$$S_G(x) = \sum_{j=0}^{\Delta(G)} a_j x^j = \sum_{u \in V(G)} x^{d_u}$$

where d_u is the degree of u.

Corollary 1.4. *If $G(p, q)$ is a graph with p vertices and q edges, then we have:*

(1) $S_G(1) = p$ (2) $\sum_{j=0}^{\Delta(G)} j a_j = 2q$ (3) $S'_G(1) = 2q = \sum_{u \in V(G)} d_u$

Suppose P_n, C_n, K_n denoted the path, cycle and complete graphs with exactly n vertices, respectively. Also a general k-regular graph is denoted by G_k. Then,

$$S_{P_n}(x) = 2x + (n-2)x^2 \qquad S_{C_n}(x) = nx^2$$
$$S_{K_n}(x) = nx^{n-1} \qquad S_{G_k}(x) = p\ x^k$$

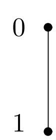
Definition 1.5. Let G_1 and G_2 be two graphs. If $V(G_1) \cap V(G_2) = \phi$. Then

(1) $G_1 \cup G_2$ is a graph that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

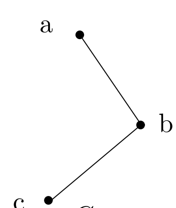
(2) $G_1 \times G_2$ is a graph that $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $\{(u,v),(u',v')\} \in E(G_1 \times G_2)$ if and only if $u = u'$ and $\{v,v'\} \in E(G_2)$ or $v = v'$ and $\{u,u'\} \in E(G_1)$

(3) $G_1 + G_2$ is a graph that $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{\{u,v\} \mid u \in V(G_1), v \in V(G_2)\}$

Example 1.6. Suppose G_1 and G_2 are two graphs such that their diagrams are as follows:

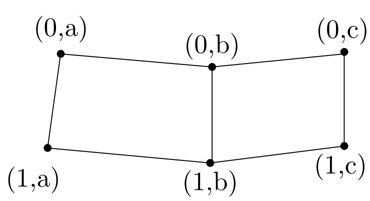


G_1

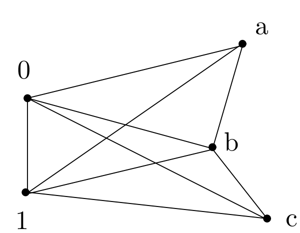


G_2

then the diagram graph $G_1 \times G_2$ and $G_1 + G_2$ as follows:



$G_1 \times G_2$



$G_1 + G_2$

Theorem 1.7. *If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two graphs, then the polynomial of graphs $G_1 \cup G_2$, $G_1 \times G_2$ and $G_1 + G_2$ are given by*

(1) $S_{G_1 \cup G_2}(x) = S_{G_1}(x) + S_{G_2}(x)$

(2) $S_{G_1 \times G_2}(x) = S_{G_1}(x) \, . \, S_{G_2}(x)$

(3) $S_{G_1 + G_2}(x) = x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x)$

Proof.

(1)
$$\begin{aligned} S_{G_1 \cup G_2}(x) &= \sum_{u \in V(G_1 \cup G_2)} x^{d_u} = \sum_{u \in V(G_1)} x^{d_u} + \sum_{u \in V(G_2)} x^{d_u} \\ &= S_{G_1}(x) + S_{G_2}(x) \end{aligned}$$

(2)
$$\begin{aligned} S_{G_1 \times G_2}(x) &= \sum_{u \in V(G_1 \times G_2)} x^{d_u} = \sum_{u=(u_1,u_2) \in V(G_1 \times G_2)} x^{d_u} \\ &= \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} x^{d_{u_1} + d_{u_2}} = \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} x^{d_{u_1}} x^{d_{u_2}} \\ &= \sum_{u_1 \in V(G_1)} x^{d_{u_1}} \cdot \sum_{u_2 \in V(G_2)} x^{d_{u_2}} \\ &= S_{G_1}(x) \cdot S_{G_2}(x) \end{aligned}$$

(3)
$$\begin{aligned} S_{G_1 + G_2}(x) &= \sum_{u \in V(G_1 + G_2)} x^{d_u} = \sum_{u \in V(G_1)} x^{d_u + p_2} + \sum_{u \in V(G_2)} x^{d_u + p_1} \\ &= x^{p_2} \sum_{u \in V(G_1)} x^{d_u} + x^{p_1} \sum_{u \in V(G_2)} x^{d_u} \\ &= x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x) \end{aligned}$$

□

Corollary 1.8. *If $S_{G_1}(x)$ and $S_{G_2}(x)$ are graphical then*

(1) *$S_{G_1}(x) \cdot S_{G_2}(x)$ is graphical and conversely.*

(2) *$x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x)$ is graphical and conversely.*

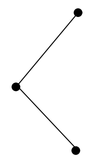
(3) $\displaystyle\sum_{u \in V(G_1 \times G_2)} d_u = 2 \, (p_1 \, q_2 + p_2 \, q_1)$

(4) $\displaystyle\sum_{u \in V(G_1 + G_2)} d_u = 2 \, (p_1 \, p_2 + q_1 + q_2)$

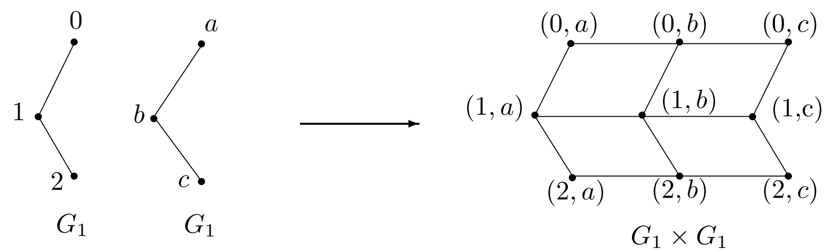
Example 1.9. The polynomial $S_G(x) = 4x^2 + 4x^3 + x^4$ is graphical, because

$$S_G(x) = 4x^2 + 4x^3 + x^4 = (2x + x^2)^2$$

On the other hand, we have the following graph for the polynomial $S_{G_1}(x) = 2x + x^2$.



Hence the polynomial $S_G(x)$ is graphical, because $S_G(x) = S_{G_1}(x) \times S_{G_1}(x)$. Also its graph is as follows:



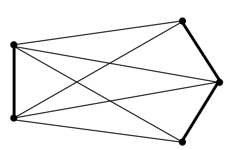
Example 1.10. The polynomial $S_G(x) = 3x^4 + 2x^3$ is graphical, because

$$S_G(x) = 3x^4 + 2x^3 = 2x^4 + x^4 + 2x^3 = x^3(2x) + x^2(x^2 + 2x)$$

On the other hand, we have the following graphs for the polynomials $S_{G_1}(x) = 2x$ and $S_{G_2}(x) = x^2 + 2x$, respectively:

G_1 G_2

Hence the polynomial $S_G(x)$ is graphical, because $S_G(x) = x^{p_2} S_{G_1}(x) + x^{p_1} S_{G_2}(x)$. Also its graph is as following:

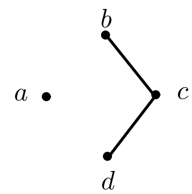


$G_1 + G_2$

Definition 1.11. Let G be a graph. The polynomial $H_G(x)$ is defined as follows:

$$H_G(x) = \sum_{\{u,v\}\in E(G)} x^{d_u+d_v}$$

Example 1.12. The polynomial $H_G(x) = x^3 + x^3 = 2x^3$ is the graph polynomial of the following graph:



Corollary 1.13. *Let $G(p, q)$ is a graph with p vertices and q edges. Then we have:*

$$H_G(1) = q \qquad H'_G(1) = \sum_{\{u,v\}\in E(G)} d_u + d_v = \sum_{u\in V(G)} {d_u}^2$$

$$H_{P_n}(x) = 2x^3 + (n-3)x^4 \qquad H_{C_n}(x) = nx^4$$

$$H_{K_n}(x) = \tfrac{n(n-1)}{2}x^{2n-2} \qquad H_{G_k}(x) = qx^{2k}$$

Theorem 1.14. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two graphs. Then*

(1) $H_{G_1\cup G_2}(x) = H_{G_1}(x) + H_{G_2}(x)$

(2) $H_{G_1\times G_2}(x) = H_{G_1}(x)\ .\ S_{G_2}(x^2) + H_{G_2}(x)\ .\ S_{G_1}(x^2)$

(3) $H_{G_1+G_2}(x) = x^{2p_2}H_{G_1}(x) + x^{2p_1}H_{G_2}(x) + x^{p_1+p_2}S_{G_1}(x)\ .\ S_{G_2}(x)$

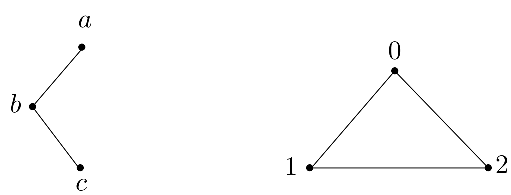
Proof. (1) is trivial. To prove (2), we have:

$$\begin{aligned}
H_{G_1\times G_2}(x) &= \sum_{\{u,v\}\in E(G_1\times G_2)} x^{d_u+d_v} \\
&= \sum_{u_1=v_1}\ \sum_{\{u_2,v_2\}\in E(G_2)} x^{2d_{u_1}+d_{v_2}+d_{u_2}} \\
&\quad + \sum_{\{u_1,v_1\}\in E(G_1)}\ \sum_{u_2=v_2} x^{d_{u_1}+d_{v_1}+2d_{u_2}} \\
&= \sum_{\{u_2,v_2\}\in E(G_2)} x^{d_{u_2}+d_{v_2}} \sum_{u_1\in V(G_1)} (x^2)^{d_{u_1}} \\
&\quad + \sum_{\{u_1,v_1\}\in E(G_1)} x^{d_{u_1}+d_{v_1}} \sum_{u_2\in V(G_2)} (x^2)^{d_{u_2}} \\
&= H_{G_2}(x)S_{G_1}(x^2) + H_{G_1}(x)S_{G_2}(x^2)
\end{aligned}$$

$$\begin{aligned}
H_{G_1+G_2}(x) &= \sum_{\{u,v\}\in E(G_1+G_2)} x^{d_u+d_v} \\
&= \sum_{\{u,v\}\in E(G_1)} x^{d_u+d_v+2p_2} + \sum_{\{u,v\}\in E(G_2)} x^{d_u+d_v+2p_1} \\
&\quad + \sum_{u\in V(G_1),v\in V(G_2)} x^{d_u+d_v+p_1+p_2} \\
&= x^{2p_2} \sum_{\{u,v\}\in E(G_1)} x^{d_u+d_v} + x^{2p_1} \sum_{\{u,v\}\in E(G_2)} x^{d_u+d_v} \\
&\quad + x^{p_1+p_2} \sum_{u\in V(G_1)} x^{d_u} \sum_{v\in V(G_2)} x^{d_v} \\
&= x^{2p_2} H_{G_1}(x) + x^{2p_1} H_{G_2}(x) + x^{p_1+p_2} S_{G_1}(x) S_{G_2}(x)
\end{aligned}$$

□

Example 1.15. Consider the following diagrams for graphs G_1 and G_2:



G_1 G_2

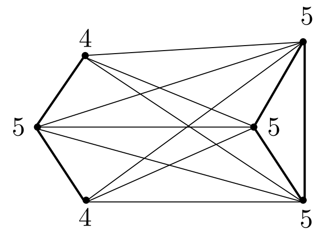
then, we have:

$$H_{G_1}(x) = 2x^3 \qquad S_{G_1}(x) = 2x + x^2$$
$$H_{G_2}(x) = 3x^4 \qquad S_{G_2}(x) = 3x^2$$

Thus:

$$\begin{aligned}
H_{G_1+G_2}(x) &= x^6(2x^3) + x^6(3x^4) + x^6(2x+x^2)(3x^2) \\
&= 2x^9 + 3x^{10} + 6x^9 + 3x^{10} = 8x^9 + 6x^{10}
\end{aligned}$$

Hence the diagram $G_1 + G_2$ is:



Corollary 1.16.

$$\sum_{u\in V(G_1\times G_2)} {d_u}^2 = p_2 \sum_{u\in V(G_1)} {d_u}^2 + p_1 \sum_{u\in V(G_2)} {d_u}^2 + 8q_1q_2$$

Proof. We know that

$$H_{G_1\times G_2}(x) = H_{G_1}(x)S_{G_2}(x^2) + H_{G_2}(x)S_{G_1}(x^2)$$

Hence,

$$\begin{aligned} H'_{G_1\times G_2}(x) &= H'_{G_1}(x)S_{G_2}(x^2) + 2xH_{G_1}(x)S'_{G_2}(x^2) \\ &\quad + H'_{G_2}(x)S_{G_1}(x^2) + 2xH_{G_2}(x)S'_{G_1}(x^2) \end{aligned}$$

Therefore

$$\begin{aligned} H'_{G_1\times G_2}(1) &= H'_{G_1}(1)S_{G_2}(1) + 2H_{G_1}(1)S'_{G_2}(1) \\ &\quad + H'_{G_2}(1)S_{G_1}(1) + 2H_{G_2}(1)S'_{G_1}(1) \end{aligned}$$

On the other hand, we know that $H_G(1) = q$, $H'_G(1) = \sum_{u\in V(G)} {d_u}^2$, $S_G(1) = p$ and $S'_G(1) = 2q$. Thus

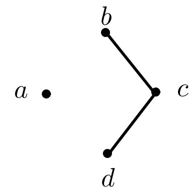
$$\begin{aligned} \sum_{u\in V(G_2\times G_1)} {d_u}^2 &= p_2 \sum_{u\in V(G_1)} {d_u}^2 + 4q_1q_2 + p_1 \sum_{u\in V(G_2)} {d_u}^2 + 4q_1q_2 \\ &= p_2 \sum_{u\in V(G_1)} {d_u}^2 + p_1 \sum_{u\in V(G_2)} {d_u}^2 + 8q_1q_2 \end{aligned}$$

□

Definition 1.17. Let G be a graph. The polynomial $F_G(x)$ is defined as follows:

$$F_G(x) = \sum_{u\in V(G)} d_u x^{d_u}$$

Example 1.18. The polynomial of the graph G defined by the following graph is $H_G(x) = 2x + 2x^2$.



Corollary 1.19. *We have:*

$$F_G(1) = S'_G(1) \qquad F'_G(1) = H'_G(1)$$
$$F_{P_n}(x) = 2x + 2(n-2)x^2 \qquad F_{C_n}(x) = 2nx^2$$
$$F_{K_n}(x) = n(n-1)x^{n-1} \qquad F_{G_k}(x) = kp\ x^k$$

Theorem 1.20. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two graphs. Then*

(1) $F_{G_1 \cup G_2}(x) = F_{G_1}(x) + F_{G_2}(x)$

(2) $F_{G_1 \times G_2}(x) = F_{G_1}(x)\ .\ S_{G_2}(x) + F_{G_2}(x)\ .\ S_{G_1}(x)$

(3) $F_{G_1 + G_2}(x) = x^{p_2} F_{G_1}(x) + p_2 x^{p_2} S_{G_1}(x) + x^{p_1} F_{G_2}(x) + p_1 x^{p_1} S_{G_2}(x)$

Proof. (1) is trivial. Prove (2), we have:

$$\begin{aligned}
F_{G_1 \times G_2}(x) &= \sum_{u \in V(G_1 \times G_2)} d_u x^{d_u} \\
&= \sum_{(u_1,u_2) \in V(G_1) \times V(G_2)} (d_{u_1} + d_{u_2}) x^{d_{u_1} + d_{u_2}} \\
&= \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_2)} (d_{u_1} + d_{u_2}) x^{d_{u_1} + d_{u_2}} \\
&= \sum_{u_2 \in V(G_2)} x^{d_{u_2}} \sum_{u_1 \in V(G_1)} d_{u_1} x^{d_{u_1}} + \sum_{u_1 \in V(G_1)} x^{d_{u_1}} \sum_{u_2 \in V(G_2)} d_{u_2} x^{d_{u_2}} \\
&= F_{G_1}(x).S_{G_2}(x) + F_{G_2}(x).S_{G_1}(x)
\end{aligned}$$

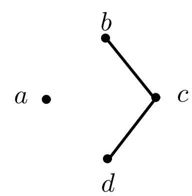
$$\begin{aligned}
F_{G_1 + G_2}(x) &= \sum_{u \in V(G_1 + G_2)} d_u x^{d_u} \\
&= \sum_{u \in V(G_1)} (d_u + p_2) x^{d_u + p_2} + \sum_{u \in V(G_2)} (d_u + p_1) x^{d_u + p_1} \\
&= x^{p_2} \sum_{u \in V(G_1)} d_u x^{d_u} + p_2 x^{p_2} \sum_{u \in V(G_1)} x^{d_u} \\
&\quad + x^{p_1} \sum_{u \in V(G_2)} d_u x^{d_u} + p_1 x^{p_1} \sum_{u \in V(G_2)} x^{d_u} \\
&= x^{p_2} F_{G_1}(x) + p_2 x^{p_2} S_{G_1}(x) + x^{p_1} F_{G_2}(x) + p_1 x^{p_1} S_{G_2}(x)
\end{aligned}$$

□

Definition 1.21. Let G be a graph. The polynomial $W_G(x)$ is defined as following:

$$W_G(x) = \sum_{\{u,v\}\in E(G)} (d_u + d_v)x^{d_u+d_v}$$

Example 1.22. Consider the following diagram for the graph G. Then $W_G(x) = 3x^3 + 3x^3 = 6x^3$.



Corollary 1.23. *We have:*

$$W_G(1) = \sum_{\{u,v\}\in E(G)} d_u + d_v = \sum_{u\in V(G)} {d_u}^2 \qquad W_G(1) = H'_G(1)$$

$$W_{P_n}(x) = 6x^3 + 4(n-3)x^4 \qquad W_{C_n}(x) = 4nx^4$$

$$W_{K_n}(x) = n(n-1)^2x^{2n-2} \qquad W_{G_k}(x) = 2kqx^{2k}$$

Theorem 1.24. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two graphs. Then*

(1) $W_{G_1\cup G_2}(x) = W_{G_1}(x) + W_{G_2}(x)$

(2) $W_{G_1\times G_2}(x) = 2F_{G_1}(x^2).H_{G_2}(x) + S_{G_1}(x^2).W_{G_2}(x) + 2F_{G_2}(x^2).H_{G_1}(x) + S_{G_2}(x^2).W_{G_1}(x)$

(3) $W_{G_1+G_2}(x) = x^{2p_2}W_{G_1}(x) + 2p_2x^{2p_2}H_{G_1}(x) + x^{2p_1}W_{G_2}(x) + 2p_1x^{2p_1}H_{G_2}(x) + x^{p_1+p_2}F_{G_1\times G_2}(x) + (p_1+p_2)x^{p_1+p_2}S_{G_1\times G_2}(x)$

Proof. (1) is trivial. To prove (2), we consider the following equation:

$$\begin{aligned}
W_{G_1\times G_2}(x) &= \sum_{\{u,v\}\in E(G_1\times G_2)} (d_u+d_v)x^{d_u+d_v}\\
&= \sum_{u_1=v_1}\sum_{\{u_2,v_2\}\in E(G_2)} (2d_{u_1}+d_{u_2}+d_{v_2})x^{2d_{u_1}+d_{u_2}+d_{v_2}}\\
&\quad+\sum_{u_2=v_2}\sum_{\{u_1,v_1\}\in E(G_1)} (2d_{u_2}+d_{u_1}+d_{v_1})x^{2d_{u_2}+d_{u_1}+d_{v_1}}\\
&= 2\sum_{u_1\in V(G_1)} d_{u_1}(x^2)^{d_{u_1}} \sum_{\{u_2,v_2\}\in E(G_2)} x^{d_{u_2}+d_{v_2}}\\
&\quad+\sum_{u_1\in V(G_1)} (x^2)^{d_{u_1}} \sum_{\{u_2,v_2\}\in E(G_2)} (d_{u_2}+d_{v_2})x^{d_{u_2}+d_{v_2}}\\
&\quad+2\sum_{u_2\in V(G_2)} d_{u_2}(x^2)^{d_{u_2}} \sum_{\{u_1,v_1\}\in E(G_1)} x^{d_{u_1}+d_{v_1}}\\
&\quad+\sum_{u_2\in V(G_2)} (x^2)^{d_{u_2}} \sum_{\{u_1,v_1\}\in E(G_1)} (d_{u_1}+d_{v_1})x^{d_{u_1}+d_{v_1}}\\
&= 2F_{G_1}(x^2).H_{G_2}(x)+S_{G_1}(x^2).W_{G_2}(x)\\
&\quad+2F_{G_2}(x^2).H_{G_1}(x)+S_{G_2}(x^2).W_{G_1}(x)
\end{aligned}$$

$$\begin{aligned}
W_{G_1+G_2}(x) &= \sum_{\{u,v\}\in E(G_1+G_2)} (d_u+d_v)x^{d_u+d_v}\\
&= \sum_{\{u,v\}\in E(G_1)} (d_u+d_v+2p_2)x^{d_u+d_v+2p_2}\\
&\quad+\sum_{\{u,v\}\in E(G_2)} (d_u+d_v+2p_1)x^{d_u+d_v+2p_1}\\
&\quad+\sum_{u\in V(G_1),v\in V(G_2)} (d_u+d_v+p_1+p_2)x^{d_u+d_v+p_1+p_2}\\
&= x^{2p_2}\sum_{\{u,v\}\in E(G_1)} (d_u+d_v)x^{d_u+d_v}+2p_2x^{2p_2}\sum_{\{u,v\}\in E(G_1)} x^{d_u+d_v}\\
&\quad+x^{2p_1}\sum_{\{u,v\}\in E(G_2)} (d_u+d_v)x^{d_u+d_v}+2p_1x^{2p_1}\sum_{\{u,v\}\in E(G_2)} x^{d_u+d_v}\\
&\quad+x^{p_1+p_2}\sum_{u\in V(G_1),v\in V(G_2)} (d_u+d_v)x^{d_u+d_v}\\
&\quad+(p_1+p_2)x^{p_1+p_2}\sum_{u\in V(G_1),v\in V(G_2)} x^{d_u+d_v}\\
&= x^{2p_2}W_{G_1}(x)+2p_2x^{2p_2}H_{G_1}(x)+x^{2p_1}W_{G_2}(x)+2p_1x^{2p_1}H_{G_2}(x)\\
&\quad+x^{p_1+p_2}F_{G_1\times G_2}(x)+(p_1+p_2)x^{p_1+p_2}S_{G_1\times G_2}(x)
\end{aligned}$$

□

In the end of this paper, we define a new triangle $\mathcal{A}$ as follows:

$$\mathcal{A} = \begin{matrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 3 & 1 & & & & \\ 1 & 7 & 6 & 1 & & & \\ 1 & 15 & 25 & 10 & 1 & & \\ 1 & 31 & 90 & 65 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

that entry a_{ij} of triangle $\mathcal{A}$ is:

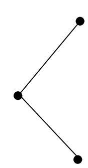
$$a_{ij} = \begin{cases} 1 & j = 1 \ \ or \ \ j = i \\ a_{(i-1)(j-1)} + j \ a_{(i-1)j} & 1 < j < i \end{cases}$$

Theorem 1.25. *If G is a graph with the polynomial $S_G(x)$, then*

$$\sum_{u \in V(G)} {d_u}^k = \sum_{j=1}^{k} a_{kj} \ S_G^{(j)}(1)$$

where $k \in \mathbb{N}$ and $a_{kj} \in \mathcal{A}$.

Example 1.26. Let G is a graph, such that its diagram is as following:



Hence the degree sequence and the polynomial $S_G(x)$ are "1, 1, 2" and $2x + x^2$, respectively. Thus for $k = 3$ we have:

$$\sum_{u \in V(G)} {d_u}^3 = 1^3 + 1^3 + 2^3 = 10$$

On the other hand, we have $S'_G(1) = 4$, $S''_G(1) = 2$, $S_G^{(3)}(1) = 0$, $a_{31} = 1$, $a_{32} = 3$ and $a_{33} = 1$. Therefore

$$\sum_{j=1}^{3} a_{3j} \ S_G^j(1) = 1 \times 4 + 3 \times 2 + 1 \times 0 = 10$$

Proof of Theorem 1.25. According to remark (1.3) $S_G(x) = \sum_{u\in V(G)} x^{d_u}$. Hence,

$$S'_G(x) = \sum_{u\in V(G)} d_u x^{d_u-1} \tag{1.1}$$

therefore

$$S'_G(1) = \sum_{u\in V(G)} d_u$$

On the other hand, according to table of $\mathcal{A}$ for $k=1$, we have:

$$\sum_{j=1}^{1} a_{1j}\ S_G^{(j)}(1) = a_{11}S'_G(1) = S'_G(1)$$

From above relations, we obtain that the theorem (1.25) for $k=1$ is true. Now from the relation (1.1), we have $xS'_G(x) = \sum_{u\in V(G)} d_u x^{d_u}$ then

$$S'_G(x) + xS''_G(x) = \sum_{u\in V(G)} {d_u}^2\ x^{d_u-1} \tag{1.2}$$

therefore

$$S'_G(1) + S''_G(1) = \sum_{u\in V(G)} {d_u}^2$$

On the other hand, according to table of $\mathcal{A}$ for $k=2$, we have:

$$\sum_{j=1}^{2} a_{2j} S_G^{(j)}(1) = a_{21}S'_G(1) + a_{22}S''_G(1) = S'_G(1) + S''_G(1)$$

From two relations before, we obtain that the theorem (1.25) for $k=2$ is true. Similarly from the relation (1.2), we can prove the theorem (1.25) for $k=3$. Therefore, if we continue the above process, then the proof is completed.

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