

**A topology on  $BCK$ -algebras via left and right stabilizers**

T. Roudabri\* and L. Torkzadeh

Department of Mathematics, Islamic Azad University of Kerman, Kerman,  
Iran

E-mail: T.Roodbarylor@yahoo.com

E-mail: lTorkzadeh@yahoo.com

ABSTRACT. In this paper, we use the left(right) stabilizers of a  $BCK$ -algebra  $(X, *, 0)$  and produce two basis for two different topologies. Then we show that the generated topological spaces by these basis are Baire, connected, locally connected and separable. Also we study the other properties of these topological spaces.

**Keywords:**  $BCK$ -algebra, basis topological, stabilizers, Baire space, connected space, locally connected and separable space.

**2000 Mathematics subject classification:** 06F35, 03G25, 08A05.

1. INTRODUCTION

The study of  $BCK$ -algebras was initiated by Y. Imai and K. Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. In 1997, Y. Hung and Z. Chen introduced the notions of right and left stabilizers of every subset of a  $BCK$ -algebra. In this note, we by considering the left(right) stabilizers of a  $BCK$ -algebra  $(X, *, 0)$ , construct two basis for two topologies on  $(X, *, 0)$ . Then we obtain some results as mentioned in the abstract. M. M. Zahedi defined hyper  $K(BCK)$ -algebras. Also, T. Roudbari and M. M. Zahedi defined simple hyper  $K(BCK)$ -algebras [2,4,7]. Similar to

---

\*Corresponding Author

them, we can define two topologies via left and right hyper  $K(BCK)$ -stabilizers in hyper  $K(BCK)$ -algebras.

## 2. PRELIMINARIES

We give herein the basic notions on  $BCK$ -algebras. For further information, we refer to the book [5]. By a  $BCK$ -algebra we mean an algebra  $(X, *, 0)$  of type  $(2,0)$  satisfying the following axioms: for every  $x, y, z \in X$ ,

- (i)  $((x * y) * (x * z)) * (z * y) = 0$ ,
- (ii)  $(x * (x * y)) * y = 0$ ,
- (iii)  $x * x = 0$ ,
- (iv)  $x * y = y * x = 0 \Rightarrow x = y$ ,
- (v)  $0 * x = 0$ .

We can define a partial ordering  $\leq$  by  $x \leq y$  if and only if  $x * y = 0$ . In a  $BCK$ -algebra  $X$ , the following hold: for all  $x, y, z \in X$ ,

- (a)  $x * 0 = x$ ,
- (b)  $x * y \leq x$ ,
- (c)  $(x * y) * z = (x * z) * y$ ,
- (d)  $x \leq y$  implies  $x * z \leq y * z$  and  $z * y \leq z * x$ ,
- (e)  $x * (x * (x * y)) = x * y$ .

A nonempty subset  $A$  of  $X$  is called an **ideal** of  $X$  if it satisfies

- (i)  $0 \in A$ ,
- (ii)  $(\forall x \in X)(\forall y \in A) (x * y \in A \Rightarrow x \in A)$ .

A **subalgebra** of  $X$  is a nonempty subset  $A$  of  $X$  such that  $x * y \in A$ , for all  $x, y \in A$ .

If there is an element  $1$  of  $X$  satisfying  $x \leq 1$ , for all  $x \in X$ , then the element  $1$  is called **unit** of  $X$ . A  $BCK$ -algebra with unit is called **bounded**.

**Definition 2.1.** [3] Let  $X$  be a  $BCK$ -algebra and  $A$  be a nonempty subset of  $X$ . Then the sets

$$A_l^* = \{x \in X \mid a * x = a, \forall a \in A\}$$

and

$$A_r^* = \{x \in X \mid x * a = x, \forall a \in A\}$$

are called the left and right stabilizers of  $A$ , respectively and the set  $A^* = A_l^* \cap A_r^*$  is called the stabilizer of  $A$ .

**Theorem 2.2.** [3] Let  $A$  be a nonempty subset of a  $BCK$ -algebra  $X$ . Then

- (i)  $A_l^*$  is an ideal of  $X$ .
- (ii)  $A_r^*$  is a subalgebra of  $X$ .

**Definition 2.3.** [1] Consider  $A$  as a nonempty set, a mapping  $\phi : P(A) \rightarrow P(A)$  is called a closure operator on  $A$ , if for all  $X, Y \in P(A)$  the following holds:

- (1)  $X \subseteq \phi(X)$ ,
- (2)  $\phi^2(X) = \phi(X)$ ,
- (3)  $X \subseteq Y$  implies  $\phi(X) \subseteq \phi(Y)$ .

Note that all definitions and notations on a given topological space  $(X, \tau)$  are stated from [6].

### 3. CLOSURE OPERATOR ON BCK-ALGEBRAS

In the sequel  $X$  is a BCK-algebra.

**Theorem 3.1.** Let  $A$  and  $B$  be two nonempty subsets of  $X$ . Then

- (i)  $0 \in A_l^* \cap A_r^*$ ,
- (ii)  $A \subseteq (A_l^*)_r^* \cap (A_r^*)_l^*$ ,
- (iii) If  $A \subseteq B$ , then  $B_l^* \subseteq A_l^*$  and  $B_r^* \subseteq A_r^*$ ,
- (iv)  $A_l^* = ((A_l^*)_r^*)_l^*$  and  $A_r^* = ((A_r^*)_l^*)_r^*$ ,
- (v)  $(A_j)_i^* = (A_j)_i^*$ .

**Proof.** (i) Since  $0 * x = 0$  and  $x * 0 = x$ , for all  $x \in X$ , then  $0 \in A_l^* \cap A_r^*$ .  
 (ii) Let  $a \in A$ . Then  $x * a = x, \forall x \in A_r^*$  and  $a * y = a, \forall y \in A_l^*$ . So  $a \in (A_r^*)_l^* \cap (A_l^*)_r^*$ .  
 (iii) Let  $x \in B_l^*$ . Then  $b * x = b, \forall b \in B$ . Since  $A \subseteq B$  and  $b * x = b, \forall b \in A$ . So  $x \in A_l^*$ . Similarly  $B_r^* \subseteq A_r^*$ .  
 (iv) By (ii) we get that  $A_l^* \subseteq ((A_l^*)_r^*)_l^*$  and  $A_r^* \subseteq ((A_r^*)_l^*)_r^*$ . Also by (ii) and (iii) we have  $((A_r^*)_l^*)_r^* \subseteq A_r^*$  and  $((A_l^*)_r^*)_l^* \subseteq A_l^*$ . Therefore  $A_l^* = ((A_l^*)_r^*)_l^*$  and  $A_r^* = ((A_r^*)_l^*)_r^*$ .  
 (iv) The proof is easy.

Note that we define  $\emptyset_l^* = \emptyset$  and  $\emptyset_r^* = \emptyset$ .

**Theorem 3.2.** The function  $\alpha : P(X) \rightarrow P(X)$ , where  $\alpha(D) = (D_l^*)_r^*$  is a closure operator on  $X$ .

**Proof.** By Theorem 3.1(ii),  $D \subseteq \alpha(D)$ , for all  $D \in P(X)$ . Also by Theorem 3.1(iv),  $\alpha(D) = (D_l^*)_r^* = (((D_l^*)_r^*)_l^*)_r^* = \alpha^2(D)$ , for all  $D \in P(X)$ . Let

$A \subseteq B$ . Then by Theorem 3.1(iii),  $\alpha(A) \subseteq \alpha(B)$ . Therefore  $\alpha$  is a closure operator on  $X$ .

**Theorem 3.3.** The function  $\gamma : P(X) \rightarrow P(X)$ , where  $\gamma(D) = (D_r^*)_1^*$  is a closure operator on  $X$ .

**Proof.** The proof is similar to the proof of Theorem 3.2.

**Theorem 3.4.** Consider the function  $\alpha$  given in Theorem 3.2. Then we can obtain that  $\beta = \{A \in P(X) | \alpha(A) = A\}$  is a basis for a topology on  $X$ .

**Proof.** It is easy to see that  $X_1^* = \{0\}$  and also  $\{0\}_r^* = X$ . Then  $\alpha(X) = X$  and so  $X \in \beta$ . Thus for all  $x \in X$  there is at least one element of  $\beta$  containing  $x$ . Let  $x \in A \cap B$ , for  $A, B \in \beta$ . Since  $\alpha$  is a closure operator, then we can obtain that  $\alpha(A \cap B) = A \cap B$ , i.e.  $A \cap B \in \beta$  containing  $x$ . Therefore  $\beta$  is a basis topology on  $X$ .

**Theorem 3.5.** Consider the function  $\gamma$  given in Theorem 3.3. Then  $\beta = \{A \in P(X) | \gamma(A) = A\}$  is a basis for a topology on  $X$ .

**Proof.** The proof is similar to the proof of Theorem 3.4.

Note that by Theorem 2.2 elements of  $\beta$  are subalgebras of  $X$  and elements of  $\beta$  are ideals of  $X$ .

We define the topologies  $\tau$  and  $\tau$  generated by basis  $\beta$  and  $\beta$ , respectively.

**Example 3.6.** Let  $X = \{0, a, b, c\}$  and  $*$  operation be given by the following table

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then  $(X, *, 0)$  is a BCK-algebra. We see that  $0 \in \alpha(A) (0 \in \gamma(A))$ , for all nonempty sub sets  $A$  of  $X$ , so if  $0 \notin A \subseteq X$ , we have  $A \notin \beta$  ( $\beta$ ). By some manipulations we get that  $\beta = \{\emptyset, X, \{0, b\}, \{0, c\}, \{0\}\}$  and  $\beta = \{\emptyset, X, \{0, a\}, \{0, c\}, \{0, a, c\}, \{0, a, b\}, \{0\}\}$ . Thus  $\tau = \{\emptyset, X, \{0\}, \{0, b\}, \{0, c\}, \{0, b, c\}\}$  and  $\tau = \{\emptyset, X, \{0\}, \{0, a\}, \{0, c\}, \{0, a, c\}, \{0, a, b\}\}$ . We see that in this example  $\{0, a, b\} \notin \beta$ , because  $\{0, a, b\}_1^* = \{0, b\}$  and  $\{0, b\}_r^* = \{0, c\}$  and

so  $\{0, c\} = (\{0, a, b\}_1^*)^* \neq \{0, a, b\}$ . Also since  $\tau \not\subseteq \tau$  and  $\tau \not\subseteq \tau$ , then  $\tau$  is not finer than  $\tau$  and also  $\tau$  is not finer than  $\tau$ .

**Theorem 3.7.**  $(X, \tau)((X, \tau))$  is a Hausdorff space if and only if  $X = \{0\}$ .

**Proof.** Since for any  $U \in \tau$ , we have  $0 \in U$ , so for any two arbitrary elements  $U, V$  of  $\tau$ , we have  $U \cap V \neq \emptyset$ . Thus  $(X, \tau)((X, \tau))$  is not Hausdorff. Conversely, let  $X = \{0\}$ . Then  $\tau = \{\emptyset, X\}$ . Thus it is clear that  $(X, \tau)$  is a Hausdorff space.

**Theorem 3.8.**  $(X, \tau)((X, \tau))$  is connected.

**Proof.** Since  $0 \in U$ , for any nonempty open set of  $X$ , then there are not nonempty open subsets  $U$  and  $V$  of  $X$  such that  $X = U \cup V$  and  $U \cap V = \emptyset$ . Thus  $(X, \tau)$  is connected space.

**Corollary 3.9.** Let  $U$  be a nonempty open subset of  $(X, \tau)((X, \tau))$ . Then  $U$  is a connected set of  $X$ .

**Proof.** It is similar to the proof of Theorem 3.8.

**Corollary 3.10.** Let  $U$  be a nonempty non-connected subset of  $(X, \tau)((X, \tau))$ . Then  $0 \in U$ .

**Proof.** It is straightforward.

**Corollary 3.11.** Let  $A \neq X$  and  $A \neq \emptyset$  be a closed subset of  $(X, \tau)((X, \tau))$ . Then  $A$  is a connected set of  $X$ .

**Proof.** Since  $A \neq X$  is a closed set of  $(X, \tau)((X, \tau))$ , then  $\emptyset \neq X - A$  is an open set of  $(X, \tau)((X, \tau))$ . By Theorem 3.1 we have  $0 \in X - A$ , therefore  $0 \notin A$ . Thus by Corollary 3.10, we get that  $A$  is a connected set of  $X$ .

Note that Corollaries 3.9 and 3.11 imply that all proper subsets of  $(X, \tau)((X, \tau))$  are connected, whenever they are closed or open.

**Theorem 3.12.** Let  $A$  be a subset of topological space  $(X, \tau)((X, \tau))$  and  $0 \in A$ . Then  $\overline{A} = X$ .

**Proof.** Let  $x \in X$ . If  $x = 0$ , then  $0 \in \overline{A}$ . Let  $x \neq 0$ , since  $0 \in U$ , for any nonempty open subset of  $X$ , then  $U \cap A \neq \emptyset$ , for any open set containing  $x$ . Therefore  $x \in \overline{A}$ .

By the above theorem we can get that the following corollary.

**Corollary 3.13.** Let  $U$  be a nonempty open subset of  $X$ . Then  $\overline{U} = X$ .

**Proof.** It is similar to a proof of Theorem 3.12.

**Open problem.** Is there any  $A \subseteq (X, \tau) ((X, \tau))$  such that  $\overline{A} = X$ , but  $0 \notin A$ .

**Theorem 3.14.** Let  $A$  be a nonempty subset of the topological space  $(X, \tau) ((X, \tau))$ . Then  $0 \in \overline{A}$  if and only if  $\overline{A} = X$ .

**Proof.** Let  $0 \in \overline{A}$ . Then  $0 \in C$ , for all closed subset  $C$  of  $X$  containing  $A$ . Since  $0$  is in any nonempty open subset of the topological space  $X$ , then the only closed subset of  $X$  containing  $0$  and  $A$  is  $X$ . So  $\overline{A} = X$ . The proof of the converse is clear.

**Lemma 3.15.**  $\{0\}$  is an open subset of the topological space  $(X, \tau) ((X, \tau))$ .

**Proof.** By Definition 3.1 we can get that  $\{0\}_1^* = X$  and  $X_r^* = \{0\}$ , then  $\{0\} \in \beta$ . Thus  $\{0\}$  is an open set of the topological space  $(X, \tau) ((X, \tau))$ .

**Theorem 3.16.**  $(X, \tau) ((X, \tau))$  is separable.

**Proof.** By Theorem 3.15 and Lemma 3.16 we get that  $\overline{\{0\}} = X$ . Then  $(X, \tau) ((X, \tau))$  is separable.

**Theorem 3.17.**  $(X, \tau) ((X, \tau))$  is locally connected.

**Proof.** Let  $x$  be an arbitrary element of  $X$  and  $U$  be an open set containing  $x$ . By Theorem 3.8, we get that  $U$  is connected and also containing  $x$ . Therefore  $(X, \tau)$  is locally connected.

**Open problem.** How are the exact characterization of the compact sets in these topological spaces?

**Convention 3.18.** Let  $(X, *, 0)$  be a totally ordered  $BCK$ -algebra and let  $\beta_0$  be the all sets of the following types:

- (i) All open intervals  $(a, b) = \{x \in X \mid a < x < b\}$ ,
- (ii) All intervals  $[0, b) = \{x \in X \mid 0 \leq x < b\}$ .
- (ii) All intervals  $(0, 1] = \{x \in X \mid 0 < x \leq 1\}$ , where  $1$  is unit of  $X$

As we can see similar to [4]  $\beta_0$  is a basis for a topology on  $X$ , which is called the order topology. The topology induced by  $\beta_0$  is denoted by  $\tau_0$ .

**Theorem 3.19.** Let  $(X, *, 0)$  be a totally ordered BCK-algebra. Then  $(X, \tau)$  is finer than  $(X, \tau_0)$ .

**Proof.** Let  $x \in A$  and  $A \in \beta$ . Then there is  $a \in A$  such that  $x * a = 0$ . We show that  $[0, a] \subseteq A = (A_\tau^*)^*$ . Let  $b \in [0, a]$ . Then  $b * a = 0$  and so  $b \in A$ , by Theorem 2.2(i). Hence  $a \in A$  implies that  $b \in A$ . Thus  $[0, a] \subseteq A$ . Also  $x \in [0, a] \in \beta_0$ . Therefore  $(X, \tau)$  is finer than  $(X, \tau_0)$ .

The following example shows that the condition " totally order " in Convention 3.18 is necessary.

**Example 3.20.** Let  $X = \{0, 1, 2, 3\}$  and  $*$  operation be given by the table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	3	3	0

Then  $(X, *, 0)$  is not a totally ordered BCK-algebra. We see that  $\beta_0$  is not a basis for a topology on  $X$ , because  $1 \in [0, 2) \cap [0, 3)$ , but there is not a  $M \in \tau_0$  such that  $1 \in [0, 2) \cap [0, 3)$ .

The following example shows that  $(X, \tau_0)$  may not finer than  $(X, \tau)$ .

**Example 3.21.** Let  $X = \{0, 1, 2, \dots\}$ . Define " $*$ " on  $X$  by

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{if } y \leq x, y \neq 0, \\ x & \text{if } y \leq x, y = 0 \end{cases}$$

Then  $(X, *, 0)$  is a non bounded BCK-algebra. By some manipulations we get that  $\beta = \{\emptyset, X, \{0\}\}$ . Consider  $A = [0, 2)$ . We see that  $1 \in A$ , but there is not any  $M \in \beta_0$  such that  $1 \in M \subseteq A$ .

The following example shows that  $(X, \tau_0)$  may not finer than  $(X, \tau)$ .

**Example 3.22.** Let  $X = [0, 1]$ . Define " $*$ " on  $X$  by

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise} \end{cases}$$

Then  $(X, *, 0)$  is a bounded *BCK*-algebra. By some manipulations we get that  $\beta = \{\emptyset, X, \{0\}\}$ . Consider  $A = [0, 1/2)$ . We see that  $1/3 \in A$ , but there is not any  $M \in \beta_0$  such that  $1/3 \in M \subseteq A$ .

**Open problem.** Is there a *BCK*-algebra  $(X, *, 0)$  such that  $(X, \tau)$  does not be finer than  $(X, \tau_0)$ ?

**Conclusion.** The paper has shown that  $\{0\}$  is an open subset of the topological spaces  $(X, \tau)$  and  $((X, \tau))$ . The authors have proved that topological spaces  $(X, \tau)$  and  $((X, \tau))$  are Bair, locally connected and separable. Finally they have shown that all proper subsets of  $(X, \tau)$  and  $((X, \tau))$  are connected, whenever they are closed or open.

#### REFERENCES

- [1] S. Burris, H. P. Sankappanavar, *A Course in universal algebra*, springer -verlag, New york. Heidelbery, Berlin, 1981.
- [2] M. Golmohammadian and M. M. Zahedi, *BCK-Algebras and Hyper BCK-Algebras Induced by a Deterministic Finite Automaton*, *Iranian Journal of Mathematical Sciences and Informatics*, 4 (1) (2009), 79-98.
- [3] Yisheng Hunge and Zhaomu Chen, Normal BCK-algebra, *Math. Japon*, (1997), 541-546.
- [4] T. Roudbari and M. M. Zahedi, Some result on simple hyper *K*-algebras, *Iranian Journal of Mathematical Sciences and Informatics*, 3 (2) (2008), 29-48.
- [5] J. Meng and Y. B. Jun, *BCK-algebras*, Kyungmoon Sa Co, Korea, 1994.
- [6] J. R. Munkres, *Topology a Prst course*, Prentice-Hall, 1975.
- [7] M. M. Zahedi, A Review on Hyper *K*-Algebra, *Iranian Journal of Mathematical Sciences and Informatics*, 1 (1) (2006), 55-112.