Iranian Journal of Mathematical Sciences and Informatics
Vol. 4, No. 2 (2009), pp. 25-31

A K-Theoritic Approach to Some C^*-Algebras

B. Tabatabaie Shourijeh* and M. Farjood

Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71454, Iran

E-mail: tabataba@math.susc.ac.ir

Abstract. In this paper we look at the K-theory of a specific C^*-algebra closely related to the irrational rotation algebra. Also it is shown that any automorphism of a C^*-algebra A induces group automorphisms of $K_0(A)$ amd $K_1(A)$ in an obvious way. An interesting problem for any C^*-algebra A is to find out whether, given an automorphism of $K_0(A)$ and an automorphism of $K_1(A)$, we can lift them to an automorphism of A or $M_n(A)$ for some positive integer n.

Keywords: C^*-algebra, K-theory, Exact group, Crossed Product, and Representation.

2000 Mathematics subject classification: 46L05.

1. Introduction

We look at a particular C^*-algebra and attempt to calculate its K and Ext groups.

Let T^2 be the 2-tours and let $C(T^2)$ have standared generators u and w. We can think of $C(T^2)$ as acting by multiplication on the Hilbert space $L^2(T^2, dt)$ where dt is the Lebesgue measure on T^2.

Let α be the automorphism of $C(T^2)$ given by

$$\alpha(u) = \lambda u; \qquad \alpha(w) = uw,$$

*Corresponding Author

Received 15 April 2009; Accepted 24 September 2009

where $\lambda = e^{2\pi i\alpha}$ for some irrational number α (as in the case of the irrational rotation algebra this double meaning for α will cause no confusion). If we regard T^2 as $R/Z \times R/Z$, then α is induced by the homeomorphism

$$(s,t) \to (s+\alpha, s+t).$$

For any irrational α we define

$$B_\alpha = C(T^2) \times_\alpha Z.$$

Regard $C(T^2)$ as a subalgebra of B_α and let the action of α be implemented by the unitary v in B_α, so that $\alpha(f) = v^* f v$ for f in $C(T^2)$.

Now α is induced by a minimal homeomorphism on T^2 so B_α is simple. Hence, in the same way as for the irrational rotation algebra, the standard covariant representation of B_α on $L^2(T^2)$ is faithful and also irreducible. There is a unique, normalized trace τ on B_α which is also faithful. τ acts on finite sums of the form $\sum f_i v^i$ (f_i in $C(T^2)$) via the formula

$$\tau(\sum f_i v^i) = \int_{T^2} f_o dt.$$

τ then extends by continuity to the whole of B_α.

B_α can also be thought of as the crossed product of the irrational rotation algebra A_α (generators u and v with $uv = \lambda vu$) by the automorphism ϑ where

$$\vartheta(u) = u; \quad \vartheta(v) = vu.$$

In our original definition of B_α the subalgebra generated by u and v is isomorphic to A_α ([6, Proposition 1.3]) and conjugation by w is the same as the action of ϑ on A_α. The simplicity of B_α ensures the two possible definitions do indeed give the same algebra.

We now consider the K-theory of B_α. We use the six term exact sequence of Pimsner and Voiculescu ([1, 2.4]) in two different ways. Considering B_α as $C(T^2) \times_\alpha Z$ we obtain,

$$\begin{array}{ccccc} K_0(C(T^2)) & \xrightarrow{1-\alpha_*^{-1}} & K_0(C(T^2)) & \xrightarrow{i_*} & K_0(B_\alpha) \\ \uparrow & & & & \downarrow \\ K_1(B_\alpha) & \xleftarrow{i_*} & K_1(C(T^2)) & \xleftarrow{1-\alpha_*^{-1}} & K_1(C(T^2)) \end{array}$$

which gives

$$\begin{array}{ccccc} Z^2 & \xrightarrow{0} & Z^2 & \longrightarrow & K_0(B_\alpha) \\ \uparrow & & & & \downarrow \\ K_1(B_\alpha) & \longleftarrow & Z^2 & \longleftarrow & Z^2 \end{array}$$

where the bottom right hand map takes $[u]_1$ to 0 an $[w]_1$ to $-[u]_1$. Hence,

$$K_0(B_\alpha) \approx Z^3; \;\; K_1(B_\alpha) \approx Z^3.$$

$\mathbf{K_0(B_\alpha)}$. Two generators, $[1]_0$ and $k = [E]_0 - [1]_0$ come from $K_0(C(T^2))$. It is easy enough to construct E explicity in $M_2(C(T^2))$ by using the six term exact sequence of K-groups associated with the quotient sequence of topological spaces

$$T \hookrightarrow T^2 \longrightarrow Susp(T),$$

where $Susp(X) = X \times [0,1]/X \times \{0\} \cup X \times \{1\}$ for X a topological space. The inclusion $T \subset T^2$ is as one of the factors of $T \times T = T^2$. Alternatively, we can start with the non-trivial element of $K_0(C(S^2))$ constructed in [10, 8.5] and use the continuous map : $T^2 \to S^2$ (think of T^2 as a square with opposite edges identified, then identify all edges to a point).

From the way that E is constructed we see that $\tau(E) = 1$, so τ doesn't distinguish between $k = [E]_0 - [1]_0$ and zero.

To obtain the third generator for $K_0(B_\alpha)$ we use the second crossed product structure to obtain a six term exact loop

$$\begin{array}{ccccc} K_0(A_\alpha) & \stackrel{1-v_*^{-1}}{\longrightarrow} & K_0(A_\alpha) & \longrightarrow & K_0(B_\alpha) \\ \uparrow & & & & \downarrow \\ K_1(B_\alpha) & \longleftarrow & K_1(A_\alpha) & \stackrel{1-\vartheta_*^{-1}}{\longleftarrow} & K_1(A_\alpha) \end{array}$$

which gives

$$\begin{array}{ccccc} Z^2 & \stackrel{0}{\longrightarrow} & Z^2 & \longrightarrow & K_0(B_\alpha) \\ \uparrow & & & & \downarrow \\ K_1(B_\alpha) & \longleftarrow & Z^2 & \longleftarrow & Z^2 \end{array}$$

where the bottom right hand map takes $[u]_1$ to 0 and $[v]_1$ to $-[u]_1$. The diagram gives us a generator $[p]_0$ where p is a projection in A_α and has trace α there. τ restricts to the unique normalized trace on A_α so $\tau(p) = \alpha$, and since the members of $K_0(B_\alpha)$ given so far have integer trace, this is indeed the missing generator.

The order on $\mathbf{K_0(B_\alpha)}$. The subgroup generated by $[1]_0$ and $[p]_0$ ordered in the same way as $K_0(A_\alpha)$ via τ. To see how k fits in, note that there is an automorphism of B_α given by

$$u \to \lambda u^*; \quad v \to v^*; \quad w \to w.$$

This automorphism restricts to an automorphism of $C(T^2)$ which reverses the orientation of T^2, thus sending $[E]_0$ to $[2]_0 - [E]_0$ and k to $-k$. Hence both k and $-k$ are positive in $K_0(B_\alpha)$ and so k plays no part in the ordering. Thus the ordering is,

$1[1]_0 + mk + n[p]_0 \geq 0$ if and only if $1 + n\alpha \geq 0$ where $1, m, n$ are integers.

$\mathbf{K_1(B_\alpha)}$. The two exact loops immediately give $[v]_1$ and $[w]_1$ as two of the generators. The third is obtained from the second exact loop using [1, Lemma 1.2]. That is, it is represented by a unitary

$$z = 1_n - F + Fx(W^* \otimes 1_n)F \text{ in } B_\alpha \otimes M_n$$

where x and F in $A_\alpha \otimes M_n$ and F is a projection. It is always possible to choose an x so that z is unitary. In this case F has trace α, so by [6, Cor. 2.5] F is unitarily equivalent in $M_n(A_\alpha)$ to a projection p in A_α of trace α. Thus if we can choose an x in $M_n(A_\alpha)$ to make z unitary then we can do it all in A_α so that z is unitary in B_α.

We can give a fairly concrete example of an x and therefore of a z as follows. For z to be unitary we must have

$$pxw^*pwx^*p = p.$$

Now w^*pw is a projection in A_α of trace α so, again by [6, Cor. 2.5], it is unitarily equivalent in A_α to p. Let x be the unitary that implements this equivalence, that is, $xw^*pwx^* = p$, and it is finished.

To sum up, the three generators, $[v]_1, [w]_1$ and $[z]_1$ of $K_1(B_\alpha)$ all have representatives in B_α.

Ext(B$_\alpha$). The Universal Coefficient Theorem already tells us that $Ext(B_\alpha) \simeq Z^3$ and $Ext_0(B_\alpha) \simeq Z^3$. To find a concrete example of an extension of B_α we apply the Pimsner-Voiculescu six term exact sequence for Ext to $A_\alpha \times_\vartheta Z$. We get

$$\begin{array}{ccccc} Ext(B_\alpha) & \longrightarrow & Ext(A_\alpha) & \overset{1-(\vartheta-1)^*}{\longrightarrow} & Ext(A_\alpha) \\ \uparrow & & & & \downarrow \\ Ext_0(A_\alpha) & \overset{0}{\longleftarrow} & Ext_0(A_\alpha) & \longleftarrow & Ext_0(B_\alpha) \end{array}$$

which gives

$$\begin{array}{ccccc} Ext(B_\alpha) & \longrightarrow & Z^2 & \longrightarrow & Z^2 \\ \uparrow & & & & \downarrow \\ Z^2 & \overset{0}{\longleftarrow} & Z^2 & \longleftarrow & Ext_0(B_\alpha). \end{array}$$

Let S be the forward unilateral shift operator on $H = L^2(N)$ and let M be the multiplication operator by $(1, \lambda, \lambda^2, \cdots)$. Then the map $\rho_1 : A_\alpha \to Q$ given by

$$\rho_1(u) = \pi(M); \quad \rho_1(v) = \pi(S)$$

is an extension of A_α. The other generator of $Ext(A_\alpha)$ is ρ_2 where

$$\rho_2(u) = \pi(S); \quad \rho_2(v) = \pi(M^*),$$

(see [5]). The top right hand map on our exact loop takes ρ_1 to "$-\rho_2$" and ρ_2 to zero. Thus our first generator for $Ext(B_\alpha)$ comes from ρ_1. Indeed, if we let N be the operator on H which is multiplication by $(1, \lambda, \lambda^3, \cdots, \lambda^{\frac{1}{2}j(j-1)}, \cdots)$ then the map $\sigma_1 : B_\alpha \to Q$ given by

$$\sigma_1(u) = \pi(M); \quad \sigma_1(v) = \pi(S); \quad \sigma_1(w) = \pi(N)$$

extends ρ_1 and is the required generator of $Ext(B_\alpha)$.

Of the three generators for $Ext(B_\alpha), \sigma_1$ appears to be the only one with an easy formula that can be written down.

Before we close this section, we consider the following example which plays an important role in the next section.

Example 1.1. We apply [1, 2.4] to the irrational rotation C^*-algebra, A_α (see [8], [9]) we know that $K_0(C(T)) \simeq Z$ and $K_1(C(T)) \simeq Z$ (It should be noted that it is possible to write $C(T)$ as $\mathbf{C} \times_{Id} Z$ and use [1, 2.4]). The generator for $K_1(C(T))$ is u, the identity map: $T \to \mathbf{C}$ where T regarded as being embedded in $\mathbf{C}$ as the set of complex numbers of modules one. It is clear that $[\alpha(u)]_1 = [u]_1$ so

$$Id_* - (\alpha(-1))_*$$

is the zero map on $K_1(C(T))$. It is also the zero map on $K_0(C(T))$ so we obtain the exact loop,

$$\begin{array}{ccccc} Z & \xrightarrow{0} & Z & \longrightarrow & K_0(A_\alpha) \\ \uparrow & & & & \downarrow \\ K_0(A_\alpha) & \longleftarrow & Z & \xleftarrow{0} & Z \end{array}$$

which splits into two exact sequences. Thus $K_0(A_\alpha) \simeq Z^2$ and $K_1(A_\alpha) \simeq Z^2$.

It appears, at first, that K-theory can not tell us anything about whether the A_α are isomorphic for various values of α.

However, [1, 2.4] tells us nothing about the order structure on K_0 and it is this which was used by M. Pimsner, D. Voiculescu [2] and by M. Rieffel [7] to distinguish between the A_α.

It is important to note that any trace τ on a C^*-algebra A induces an order-preserving group homomorphism

$$\tau_* : K_0(A) \to R$$

defined by,

$$\tau_*([P]_0) = \tau(P)$$

for each $[P]_0$ in the positive part of $K_0(A)$. In [2], [4] M. Pimsner and D. Voiculescu showed that A_α can be embedded in an AF-algebra whose K_0-group is known to be Z^2, ordered as $Z + \alpha Z$. Thus the range of τ_* on projections in A_α lies within $Z + \alpha Z$ (They later proved this result in a simpler way in [1] by looking in detail at the six term exact sequence for crossed products [1, 2.4]). In [7] M. A. Rieffel gave a method of constructing projections in A_α with a given trace β, one for each value β in $(Z + \alpha Z) \cap [0, 1]$. Projections of this form will be known as Rieffel projections.

Putting these together we have that

$$\{\tau(p) : p \text{ in } P(A_\alpha)\} = (Z + \alpha Z) \cap [0, 1].$$

As an immediate consequence of this we have the following theorem.

Theorem 1.2. [1, Corollary 2]. Let α and β be two irrational numbers in $[0, 1]$. Then $A_\alpha \simeq A_\beta$ if and only if $\alpha = \beta$ or $\alpha = 1 - \beta$.

2. Automorphism of the Irrational Rotation Algebra

We start with $K_0(A_\alpha)$. It is clear that for a group automorphism to lift in the way described above it must preserve the order structure. We know (see Example 1.1) that $K_0(A_\alpha) \simeq Z^2$ ordered as $Z + \alpha Z \subseteq R$. This is a total ordering so the only order-preserving automorphism is the identity map which lifts to any automorphism of $M_n(A_\alpha)$. The interesting part is to see which automorphisms of $K_1(A_\alpha)$ lift.

Theorem 1.2. An automorphism of $K_1(A_\alpha)$ lifts to an automorphism of $M_n(A_\alpha)$ for some positive integer n if and only if it lifts to an automorphism of A_α.

Proof. It suffices to prove the "only if" implication. Let the automorphism of $K_1(A_\alpha)$ be ρ and suppose that ρ lifts to an automorphism v of $M_n(A_\alpha)$. Denote by e_j (where $1 \leq j \leq n$) the projection in $M_n(A_\alpha)$ which has 1 in $j^{th.}$ place down the diagonal and zero's every where else. Then, extending the trace τ on A_α to a trace τ on $M_n(A_\alpha)$, we see that $v(e_1)$ is a projection in $M_n(A_\alpha)$ of trace 1. Thus by [6, Corollary 2.5] there is a unitary u_1 in $M_n(A_\alpha)$ with

$$u_1 v(e_1) u_1^* = e_1$$

writting $v_1 = Ad(u_1)\ o\ v$ we see that v_1 is also a lifting of ρ. Now $(1_n - e_1)M_n(A_\alpha)(1_n - e_1)$ is isomorphic to $M_{n-1}(A_\alpha)$ and is fixed by v_1. Thus we can find a unitary u_2 in $M_n(A_\alpha)$ such that $v_2 = Ad(u_2)\ o\ v_1$ fixes e_1 and e_2 obviously v_2 is a lifting of ρ. Proceeding in this way we finally obtain v_n which can be written in the form $\varphi \otimes Id$. Where we think of $M_n(A_\alpha)$ as $A_\alpha \otimes M_n$. Then φ is also a lifting of ρ and it is finished.

We regard $K_1(A_\alpha) \simeq Z^2$ as integer-valued column vector with generators $[u]_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $[v]_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Aut $(K_1(A_\alpha)) \simeq GL(2, Z)$ and we can try to lift $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ directly as

$$v : u \to u^a v^b; \quad v \to u^c v^d.$$

Note that any automorphism of A_α is determined by effect on u and v.

References

[1] M. Pimsner and D. Voiculescu, Exact sequences for K-groups and Ext-groups of certain crossed product C^*-algebras, J. Operator Theory , 4 (1980), 93-118.

[2] M. Pimsner and D. Voiculescu, embedding the irrational rotation C^*-algebras into an AF-algebra, J. Operator Theory , 4 (1980), 201-210.

[3] M. Pimsner, A class of C^*-algebras Generalizing both Cuntz-Kriger Algebra and Crossed Products by Z, Fields Inst. Commun. (D. Voiculescu, ed.), Vol. 12, 1997.

[4] D. Voiculescu, Dynamical approximation entropies and topological entropy in operator algebras, Commun. Math. Phys. , 170 (1995), 249-281.

[5] S. Popa and M. A. Rieffel, The ext-groups of the C^*-algebras associated with irrational rotations, J. Operator Theory , 3 (2) (1980), 271-274.

[6] M. A. Rieffel, The cancellation theorem for projective modules over irrational rotation C^*-algebras, Proc. London Math. Soc. , 47 (1983), 285-302.

[7] M. A. Rieffel, C^*-algebras associated with irrational rotation, PaciÞc J. of Maths. , 93 (2) (1981), 415-429.

[8] B. Tabatabaie Shourijeh, The irrational rotation C^*-algebras, Proc. of the $26^{th.}$ A. I. M. C., Kerman, Iran , (1995), 393-395.

[9] B. Tabatabaie Shourijeh, Topological Action of a unital inverse semigroup , $16^{th.}$ Seminar of Mathematical Analysis and its application 4-5 Feb., Ferdowsi University of Mashhad, Iran, 2007.

[10] J. L. Taylor, Banach algebras and Topology, Algebras in Analysis , J. H. Williamson, Academic Press, London, 1975.