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A study of Nilpotent groups through right transversals

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Abstract. In group theory nilpotency of a group has a great importance. In this paper we have studied some concept of nilpotency through right transversals. We have also studied prime power groups and frattini subgroups through right transversals.

Keywords: Nilpotent group, Right transversal, Frattini subgroup, C-groupoid.

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1. Introduction:

An extension of a group H is a group G, which contains H as a normal subgroup. We shall call a group G a general extension of a subgroup H if G contains H as a subgroup (not necessarily as a normal subgroup). A pair (G, H), where H is a subgroup of a group G, we will be called a **general extension**. We have a category *E,* whose objects are general extensions and a morphism from (G_1, H_1) to (G_2, H_2) is a group homomorphism ϕ from G_1 to G_2 such that ϕ $(\mathrm{H}_1) \subseteq \mathrm{H}_2$. The category E will be called the **category of general extensions**

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Definition 1.1: A groupoid (S, o) (a nonempty set S with a binary operation o) is called a **right quasigroup** if$\forall\, x,\, y\, \in$ S, the equation X$ox\, = y$ where X is unknown in the equation, has a unique solution. If $\exists\; e\, \in$ Ssuch that $eox\, = xoe\, = x\, \forall\, x \in$ S, then (S, o) is called a right quasigroup with identity.

Throughout the Paper, a right quasigroup will always be assumed to be with identity.

Let G be a group and H a subgroup of G. A **right transversal** (right coset representative system) to H in G is a subset S of G obtained by selecting one and only one member from each right coset of G mod H.

Throughout the Paper, a right transversal will be assumed to contain the identity of the group.

Let S be a right transversal to H in G. Let $x,\, y\, \in$ S. Define $\{xoy\}$ = S ∩ Hxy. Then z, where $\{z\}$ = S ∩ Hyx^{-1} is the unique solution of the equation X$ox\, = y$, where X is unknown in the equation. Therefore (S, o) is a right quasigroup with identity. This right quasigroup structure on S is said to be the right quasigroup structure induced by the group G on the right transversal S.

2. C-groupoid and right quasi group

Definition 2.1: A quadruple (S, H, σ, f), where S is a groupoid with identity e, H a group which acts on S from right through a given action θ, σ a map from S to H^H (the set of all maps from H to H) and f a map from S×S to H, is called a **c-groupoid** if it satisfies the following conditions:

C_1) $xoy\, = y \Rightarrow x = e$

C_2) $\forall\, x \in\, S\, \exists\, x^{'}\, \in\, S$ such that $x^{'}o\, x\, = e$

C_3) $\forall\, x\, \in\, S$, let $\sigma_x$denote the image $\sigma\, (x)$ of x under the map σ. Then $\sigma_e\, = \mathrm{I_H}$, the identity map on H.

C_4) $f\,(x,\, e)\, = \, f\,(e,\, x)\, = \, 1$, the identity of H

C_5) $\sigma_x\,(h_1\, h_2)\, = \, \sigma_x\,(h_1)\,\sigma_{x\theta h_1}\,(\,h_2)$

C_6) $(xoy)oz\, = \, x\theta\, f(y,z)\, o\,(yoz)$

C_7) $(xoy)\theta\; h\, = \, x\theta\, \sigma_y(h)oy\, \theta\; hC_7)\; (xoy)\theta\; h\, = \, x\theta\, \sigma_y(h)oy\, \theta\; h$

C_8) $f\,(x,y)f(xoy,z)\, = \, \sigma_x(f(y,z))f(x\theta\, f(y,z), yoz)$

C_9) $f(x,y)\sigma_{xoy}(h\,)\, = \, \sigma_x(\sigma_y(y))f(x\theta\, \sigma_y(h),\, y\theta\; h)$

where $x,\, y\, \in$ S and $h_1, h_2\, \in$ H

Remark 2.2 From the above definition it follows that if (S, H, σ, f) is a c-groupoid, then

(i) $\sigma_x(1) = \, 1$ for all $x \in$ S.

(ii) $e\theta\; h\, = \, e$ for all $h \in$ H.

Example 2.3: Let S be a right transversal to a subgroup H of a group G. Let $x,\, y \in$ S and $h\, \in$H. Then $x.\;\, y\, = f(x,\, y)\, x \circ\, y$ for some$f(x,\, y) \in$ H and $x \circ\, y$ $\in$ S. Also $x.h = \sigma_{\,x}(h)\, x\theta\; h$ for some $\sigma_{\,x}(h) \in$ H and $x\theta\; h\in$ S. This gives us a map f: S×S → H and a map σ : S → $\mathrm{H^H}$ defined by $f\;((x,\, y)) = f\;(x,\, y)$

and$\sigma(x)(h) = \sigma_x(h)$. Then (S, H,$\sigma$, f) is a c-groupoid. Conversely we have the following result.

Theorem 2.4 Given a c-groupoid (S, H, σ, f) there is a group G which contains H as a subgroup and S as a right transversal of H in G such that the corresponding c-groupoid is (S, H, σ, f). [Ref. 8]

The pair (G, H) where G is the group described in the above Theorem, will be termed as the general extension associated to the c-groupoid (S, H, σ, f).

3. Right Quasigroups and General Extensions

Let S be a set and Sym(S), the symmetric group on S (for the product in Sym(S) we adopt the convention $(r.s)(x) = s(r(x))$; r, $s \in$Sym(S) and $x \in$S.

Now, let (S, o) be a right quasigroup with identity e. Let y, $z \in$ S. Define a map $f^s(y, z)$from S to S as follows: for$x \in$S, $f^s(y, z)(x)$ is the unique solution of the equation X o (y o z) = (x o y) o z. It can be checked that $f^s(y, z) \in$ Sym(S).

Definition 3.1: The subgroup $G_s$of Sym(S) generated by the subset $\{f^s(y, z) : y, z \in S\}$of Sym(S) will be called the **group torsion** (or **associator**) of S.

Remark 3.2 Since$(e \circ y) \circ z = e \circ (y \circ z)$, for all $y, z \in S$, G_s is a subgroup of Sym(S$\backslash\{e\}$) also. Clearly, we have

Proposition 3.3: The group torsion $G_s$of a right quasigroup S is trivial if and only if S is a group.

Proposition 3.4 A subgroup H of a group G is normal if and only if group torsion of every transversal to H in G is trivial.

Proposition 3.5 Let S be a right quasigroup with identity e. Then, there exists a group G^Scontaining S as a right transversal such that if there is any group G containing S as a right transversal, then there is a unique homomorphism from G to G^s which is identity on S. **Proof:** By Theorems 2.5 and 2.10 it is sufficient to construct a c-groupoid (S, H^S, σ^S, f^S), where S is the given right quasigroup such that given any c-groupoid (S, H, σ, f), where S is again the given right quasigroup, there exists a unique c-homomorphism (p, q, g)from (S, H, σ, f) to (S, H^S, σ^S, f^S)such that $p = I_s$, the identity map on S. Take H^s = Sym(S $-$ $\{e\}$), the symmetric group on S $-$ $\{e\}$. We extend the natural action of H^Son S $-$ $\{e\}$ to the action $\theta^S$of H^S on S defining $e\,\theta^S h = e, \forall h \in H^S$. Already, we have a map $f^S : S \times S \rightarrow H^S$ defined by $f^S((x, y)) = f^S(x, y)$. Next we define a map $\sigma^S_y : S \rightarrow S$ by the equation

$$\sigma^S_y(h)(x) \circ (y\,\theta^S h) = (x \circ y)\,\theta^S h$$

It can be seen that$\sigma^S_y(h) \in \mathrm{Sym}(S - \{e\}) = H^S$. Finally, it can be checked that (S, H^S, σ^S, f^S) is a c-groupoid, which satisfies the required property. #

The group G^S described in the theorem 3.5 will be termed the **general extension** of Sym(S $-$ $\{e\}$) by the right quasigroup S.

4. Nilpotent Group

A group G is said to be **nilpotent** if Ln(G) = {e} or equivalently Zn(G) = G for some n. A group G is said to be **nilpotent of class n** if Ln(G) = {e} but Ln-1(G) $\neq$ {e} or equivalently Zn(G) = G but Zn-1(G) $\neq$ G.

Proposition 4.1: Let G be a nilpotent group and H a subgroup of G then for every transversal S to H in G there exits $x \in S-\{e\}$such that $x\theta\ h = x\ \forall h \in H$. In particular $x\theta\ f\,(y,z) = x\ \ \forall\ y,z \in S$ and $G_S \subseteq Sym\,(S-\{e,x\})\ and\, G_S \subset N_{G_S S}\,(G_S)$.

Proof: Let G be a nilpotent group, take H be any subgroup of G then $H \subset N_G\,(H)$
$\exists\ g \in N_G\,(H) - H$ such that $gHg^{-1} = H$ i.e.; $\exists g \in G - H$ such that $ghg^{-1} \in H$ i.e.
$ghg^{-1} = h'$ for all $h \in H$ and h' is an element of H.

Since $g \in G = HS \Rightarrow g = h_1 x$ where $h_1 \in H,\ x \in S-\{e\}$

Since $g \notin H$ so that $(h_1x)\,h\,(h_1x)^{-1} = h'$

$$h_1xhx^{-1}h_1^{-1} = h' \text{ or } xhx^{-1} = h'' \text{ for some } h'' \in H$$

$$\Rightarrow xh = h''x$$

$$\Rightarrow x\theta\ h = x\ \ \forall h \in H$$

i.e. $\exists x \in S-\{e\}$ such that $x\theta\ h = x$
In particular $x\theta\ f\,(y,z) = x\ \ \forall y,z \in S$

Since $f\,(S\times S) \subseteq H$

If q be the permutation representation of $f\,(S\times S)$ on S defined by $q\,(f\,(y,z)) = f^S\,(y,z)$ then $\exists$ a homomorphism $\psi : f\,(S\times S)\,S \to G_S S$ defined by
$\psi\,(hx) = q\,(h)\,x$.Then we see that

$$x\theta^S f^S\,(y,z) = x\ \ \forall y,z \in S$$
$$i.e.\ f^S\,(y,z)\,(x) = x\ \ \ \forall y,z \in S$$

so that $f^S\,(y,z) \in Sym\,(S-\{e,x\})\ \ \forall y,z \in S$

$$\Rightarrow G_S \subseteq Sym\,(S-\{e,x\})$$

Now

$xf^{S}(y,z)\,x^{-1}$
$= {}^{S}_{x}\ f^{S}(y,z)\ \ x\ {}^{S}f^{S}(y,z)\,x^{-1}$
$= {}^{S}_{x}\ f^{S}(y,z)\ \ x\ \ f^{S}\ \ x',x\ \ {}^{-1}x'$
$= {}^{S}_{x}\ f^{S}(y,z)\ \ {}^{S}_{x}\ f^{S}\ \ x',x\ \ {}^{-1}f^{S}\ \ x\ {}^{S}\ \ f^{S}\ \ x',x^{-1}\ \ {}^{-1},x'\ \ x\ {}^{S}\ \ f^{S}\ \ x',x\ \ {}^{-1}o\,x'$
$= {}^{S}_{x}\ f^{S}(y,z)\ \ {}^{S}_{x}\ \ f^{S}\ \ x',x\ \ {}^{-1}f^{S}\ \ x,x'\ \ xox'$
$= {}^{S}_{x}\ f^{S}(y,z)\ \ {}^{S}_{x}\ \ f^{S}\ \ x',x\ \ {}^{-1}\ \ f^{S}\ \ x',x\ \ \in G_S$

(R.Lal, Journal of algebra-1996, Transversals in group-3.6)

$$G_S \subset N_{G_S S}\left(G_S\right).$$

Proposition 4.2: Let G be a group of orderp^{m+1}, where p is a prime.
(i) If G has a nilpotent class$c > 1$, then all transversals of $Z_{c-1}\left(G\right)$ in G are isomorphic with trivial group torsion and their order is at least p^2.
(ii)For every divisor of$|G|$, G has a proper transversal of that order.

Proof: (i) To prove this it is sufficient to prove that ${}^{G}\!/\!_{Z_{c-1}(G)}$ is not cyclic (i.e. not prime cyclic). Suppose $\left|{}^{G}\!/\!_{Z_{c-1}(G)}\right| = p$

Then $Z\left({}^{G}\!/\!_{Z_{c-1}(G)}\right) = {}^{G}\!/\!_{Z_{c-1}(G)}$ $\quad ---(1)$

Consider ${}^{G}\!/\!_{Z_{c-2}(G)} \Big/ {}^{Z_{c-1}(G)}\!/\!_{Z_{c-2}(G)} \approx {}^{G}\!/\!_{Z_{c-1}(G)}$ $\quad ---(2)$

But ${}^{Z_{c-1}(G)}\!/\!_{Z_{c-2}(G)} = Z\left({}^{G}\!/\!_{Z_{c-2}(G)}\right)$ $\quad ---(3)$

$\Rightarrow {}^{G}\!/\!_{Z_{c-2}(G)} \Big/ Z\left({}^{G}\!/\!_{Z_{c-2}(G)}\right)$ is cyclic $\Rightarrow {}^{G}\!/\!_{Z_{c-2}(G)}$ is abelian such that ${}^{Z_{c-1}(G)}\!/\!_{Z_{c-2}(G)} = Z\left({}^{G}\!/\!_{Z_{c-2}(G)}\right) = {}^{G}\!/\!_{Z_{c-2}(G)} \Rightarrow Z_{c-1}\left(G\right) = G$, a contradiction

$$\left|{}^{G}\!/\!_{Z_{c-1}(G)}\right| \geq p^2.$$

(ii) since $Z_{c-1} \trianglelefteq G$. Hence the result. ($ii$) Since for every divisor of$|G|$, G has a subgroup of that order. Hence if $|G| = dd_1$ Since d_1 divides $|G|$, G has a subgroup H of order d_1 so that $[G:H] = d \Rightarrow G$ has a proper transversal of order d.

Definition 4.3: Let G be a group. If G has no maximal subgroups, then define the **Frattini subgroup** of G denoted by Φ(G) to be G itself. If G has maximal subgroups then the Frattini subgroup Φ(G) of G is defined to be the intersection of all maximal subgroups of G.

(Wielandt)4.4: Let G be a finite group. Then G is nilpotent$\Leftrightarrow G' \subseteq frat\left(G\right)$.

Proposition 4.5: Let G be a finite group. Then $G' \subseteq frat\left(G\right)$ if and only if each right transversal of $frat\left(G\right)$is contained in some right transversal of G', where G' is the commutator subgroup of G.

Proof: $G' \subseteq frat\left(G\right)$, thus

$$\begin{aligned} &G'g_1 = G'g_2 \\ &\Rightarrow g_1 g_2^{-1} \in G' \\ &\Rightarrow g_1 g_2^{-1} \in frat\left(G\right) \\ &\Rightarrow frat\left(G\right) g_1 = frat\left(G\right) g_2 \end{aligned}$$

This shows that $frat\,(G)\,g_1 \neq frat\,(G)\,g_2 \Rightarrow G'g_1 \neq G'g_2$. Let S be any right transversal of$frat\,(G)$. Let$1 \neq g \in S$, then $\Rightarrow frat\,(G)\,g \neq frat\,(G) \Rightarrow G'g \neq G'$. $\Rightarrow g \in S_1$ for some right transversal ofG'. Since $g_1 \neq g_2 \in S \;\; \Rightarrow frat\,(G)\,g_1 \neq frat\,(G)\,g_2$

$$\Rightarrow G'g_1 \neq G'g_2$$

$\Rightarrow$we have a right transversal S_1 of G'containing g_1 and g_2. This shows that every right transversal of $frat\,(G)$is contained in some right transversal ofG'. Conversely, let every right transversal of $frat\,(G)$is contained in some right transversal of G'. From this it immediately follows that

$$frat\,(G)\,g_1 \neq frat\,(G)\,g_2 \Rightarrow G'g_1 \neq G'g_2$$

i.e. $G'g_1 = G'g_2 \Rightarrow frat\,(G)\,g_1 = frat\,(G)\,g_2$ Let $g \in G' \Rightarrow G'g = G' \Rightarrow frat\,(G)\,g = frat\,(G)$

$$\Rightarrow g \in frat\,(G)$$
$$\Rightarrow G' \subseteq frat\,(G)\,.$$

From this result it immediately follows that:

Proposition 4.6: Let G be a finite group. G is nilpotent $\Leftrightarrow$every right transversal of $frat\,(G)$is contained in some right transversal of G'.

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