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Qualitative Properties and Existence of Solutions for a Generalized Fisher-like Equation

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Abstract. This paper is devoted to the study of an eigenvalue second order differential equation, supplied with homogenous Dirichlet conditions and set on the real line. In the linear case, the equation arises in the study of a reaction-diffusion system involved in disease propagation throughout a given population. Under some relations upon the real parameters and coefficients, we present some existence and nonexistence results. We use a variational method and fixed point arguments.

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1. Motivation and introduction

The spread of an infectious disease within a population confined in a region $\Omega \subset \mathbb{R}^n$ may be represented by two components, one of infectives and the

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other one of susceptibles. If u and v denote respectively the spatial densities of infectives and susceptibles in the position x and at the time t, then the system reads

$$(1) \qquad \begin{cases} u_t - \Delta u + \lambda u = uvg(v) \\ v_t - \Delta v = -uvg(v), \end{cases}$$

with initial conditions

$$u(x,0) = u_0(x); \quad v(x,0) = v_0(x),$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega,$$

where $(x,t) \in \Omega \times [0,+\infty[$.

The nonlinear reaction term g is assumed to be a regular positive function. The special case $g = g_0 \in \mathbb{R}_*^+$ corresponds to the so-called Kermack-McKendric model; here g_0 is the contact rate. The positive constant λ stands for the removal rate. We refer to Baily, Britton and Murray (see [1, 4, 14]) for more details on the biological meaning of the model. In [14], p. 652, various interpretations of the removal rate λ are given.

In one dimensional space, the spatial spread of an epidemic wave is described by traveling front solutions propagating with a speed $c \in \mathbb{R}_*^+$, that is particular solutions of the form $u(x,t) = \hat{u}(x+ct)$ and $v(x,t) = \hat{v}(x+ct)$; here, we have assumed a wave to propagate from right to left. Substituting these expressions into (1), we get the following system of differential equations where $' = \frac{d}{d\xi}$ denotes the derivation with respect to the new variable $\xi = x + ct$ and where we have omitted the hats for simplicity of notations:

$$(2) \qquad \begin{cases} -u'' + cu' + \lambda u = uvg(v) \\ -v'' + cv' = -uvg(v). \end{cases}$$

In reality, c is a new parameter while λ is viewed as an unknown of the problem. It may be justified by physical motivations; a discussion along the same line can be found in [5, 6, 7] where a model from combustion theory was considered. Existence and nonexistence results for a boundary value problem associated with System (2) are provided in [5, 6]. Arguments using the Leray-Schauder degree are used. The case $\lambda = 0$ arises also in combustion theory and leads to the classical Fisher-Kolmogorov model equation. The resulting equation is extensively studied in the literature [2, 12, 13, 15, 16]. Sanchez [15] discussed existence of solutions to the non-autonomous equation $u'' + cu' + a(t)f(u) = 0$ on the positive half-line subject to the boundary $u(0) = u(+\infty) = 0$, with positive function f and a non-degenerate coefficient a. The autonomous case is considered in [16] where a shooting approach is used. Again in the positive half-line, Malaguti [12] proved existence of bounded solutions to the variable-coefficient Fisher Equation $u'' - c(x)u' + b(x)g(u) = 0$ where the nonlinearity

g has compact support and finitely many zeros. Using comparison-type arguments, the more general equation $u'' - \beta(c,x,u)u' + g(x,u) = 0$ is treated in [13] on the real line with the boundary conditions $u(-\infty) = 0,\ u(+\infty) = 1$. When the reaction term is degenerate, a phase plane analysis is undertaken in [2] for the equation $u'' + cu' + f(u,u';c) = 0$ with certain restrictions on the behavior of f. We notice that none of these works considered the perturbed differential operator $-u'' + cu' + \lambda u$. The difficulty is that the perturbation term of order 0, namely λu, modifies the monotonicity of every solution u and hence its a priori estimates and qualitative properties.

In [6], a system stem from combustion theory with $\lambda = 0$ and no balance reaction term, namely with $-uv$ in both equations, is investigated. The method of upper and lower solution is used to prove existence of positive solutions; the solutions are shown to be monotonic; in epidemiology and combustion theory, they are called front waves. When $\lambda > 0$, we rather obtain pulses vanishing at infinity while v remains non-increasing. An attempt to adapt the upper and lower solution method to System (2) has led to the investigation of the following boundary value problem

$$(3)\qquad \begin{cases} -u'' + cu' + \lambda u &= f(x)g(u), \qquad -\infty < x < +\infty. \\ \lim\limits_{|x|\to+\infty} u(x) &= 0. \end{cases}$$

Are given c, $\lambda > 0$ positive real numbers and $f\colon \mathbb{R} \to \mathbb{R}^+$, $g\colon \mathbb{R} \to \mathbb{R}$ two continuous functions. Indeed, for a given function v, the first equation in (2) leads to $-u'' + cu' + \lambda u = f(x)u$ with $f(x) = v(x)g(v(x))$.

In this paper, we consider the polynomial growth case $0 < |g(u)| \le k|u|^p$ for some positive real numbers k and p. The case $g(u) = u$ stems from epidemiology: then the function f represents a density of susceptibles while u refers to the one of infectives. We will see that even in the linear case, tackling Problem (3) depends on whether it is considered on the half line or on the whole real line. The aim of this paper is two-fold. We first consider the linear case in Section 2 where a variational approach is used to prove existence of positive solutions on half-lines $(-a, +\infty)$ for any $a > 0$. The result of this section does not guarantee existence of solutions which are defined on the full real line. We also discuss how monotonicity of the function f affects existence or nonexistence of positive solutions. In Section 3, we deal with a general right-hand source term $h(x,u)$ with polynomial growth u^p in terms of the second argument. Existence of solutions, not necessarily positive and defined on the positive half-line is obtained via Schauder's fixed point theorem; this result complements a similar result obtained in [9]. The cases $p \neq 1$ or $p = 1$ are treated separately. An existence result of solution defined on the full real line is also given in Section 4. To illustrate the obtained results, some examples of applications are presented in

Section 5. We end the paper by giving some remarks in section 6 where we mention an open problem.

2. Problem on the negative half-line

Let $X\colon = C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, usually denoted $C^1_b(\mathbb{R})$, be the space of continuously differentiable functions with bounded derivatives. Equipped with the sup-norm, $\|u\|_X\colon = \sup_{x\in\mathsf{R}}\{|u(x)|, |u'(x)|\}$, it is a Banach space. In this section, we investigate the question of existence and nonexistence of positive solution $u \in X$ to the following linear problem

$$(4)\qquad \begin{cases} E_u \quad -u'' + cu' + \lambda u & = & f(x)u, \qquad -\infty < x < +\infty. \\ \displaystyle\lim_{|x|\to+\infty} u(x) & = & 0. \end{cases}$$

Here $f\colon \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function.

Despite its linearity, Problem (4) presents some difficulties due to the space to which the solution is required to belong and to the fact that u is not allowed to change sign. In this section, we are interested in questions of existence, nonexistence, regularity, and positivity of solutions. We first give some simple properties involving nonexistence results and regularity; this is the content of the first subsection. Then, we prove, in the second one, an existence result under some assumptions on the function f; in particular, f will not be assumed monotonic and boundary condition at positive infinity will not be derived. However, we will rather show that $\lim_{x\to+\infty} u(x)e^{-\frac{cx}{2}} = 0$.

2.1. **Case of nonexistence and qualitative properties.** We briefly mention three natural results; the first two results are concerned with simple nonexistence criteria. The third one specifies the regularity of the solution u with respect to that of the function f. Hereafter, $C_0(\mathbb{R}, \mathbb{R})$ refers to the Banach space of continuous functions vanishing at infinity with usual norm $\|u\|_0 = \sup_{x\in\mathsf{R}} |u(x)|$. The norm in the Banach space $H^1(\mathbb{R})$ is denoted by $\|.\|$, while the norm in the Lebesgue space $L^p(\mathbb{R})$ for $1 \le p < \infty$ will be referred to as $|.|_p$:

$$|u|_p = \left(\int_{-\infty}^{+\infty} |u(x)|^p\, dx\right)^{\frac{1}{p}}.$$

The function $\phi^+ = \max(\phi, 0)$ (respectively $\phi^- = -\min(\phi, 0)$) stands for the positive part (respectively the negative part) of the function ϕ. Recall that $L^p(\mathbb{R})$ is the Banach space of p^{th} power integrable functions on $\mathbb{R}$; and $H^1(\mathbb{R})$ is the Sobolev space of all functions in $L^2(\mathbb{R})$ with distribution derivatives up to the first order also in $L^2(\mathbb{R})$. In addition, the notation $:=$ means throughout this article to be defined equal to.

Proposition 2.1. Let $f \in C^1(\mathbb{R})$ be a nondecreasing function. Then Problem (4) has only the trivial solution in X.

Proof. First, note that $u \in X$ implies $u'(\pm\infty) = 0$; for this, it suffices to check that they exist as finite limits; we do it at positive infinity. Let

$$\underline{\ell}: = \liminf_{x\to+\infty} u'(x) \leq \bar{\ell}: = \limsup_{x\to+\infty} u'(x).$$

Then, by a classical fluctuation lemma (see Lemma 4.2 in [10]), there exist two sequences $(x_n)_{n\in\mathsf{N}}$ and $(y_n)_{n\in\mathsf{N}}$ tending to $+\infty$ such that $\underline{\ell} = \lim_{n\to\infty} u'(x_n)$ and $\bar{\ell} = \lim_{n\to\infty} u'(y_n)$ whereas $\lim_{n\to\infty} u''(x_n) = \lim_{n\to\infty} u''(y_n) = 0$.
Inserting into Equation E_u, we find $c\underline{l} = c\bar{l} = 0$; hence $\underline{\ell} = \bar{\ell} = 0$ for $c > 0$. The fact that u'' vanishes at positive infinity follows directly from Equation E_u itself.

Now, multiply Equation E_u by u' and then integrate by parts over $\mathbb{R}$; we find the identity

$$(5) \qquad \begin{aligned} 0 < c\int_{-\infty}^{+\infty} |u'|^2(x)dx &= \int_{-\infty}^{+\infty} f(x)(\tfrac{u^2(x)}{2})'\,dx \\ &= -\tfrac{1}{2}\int_{-\infty}^{+\infty} f'(x)|u|^2(x)\,dx. \end{aligned}$$

A contradiction with $f' \geq 0$ is then reached.

Proposition 2.2. Let $u \in X$ be a positive solution of Problem (4) corresponding to an eigenvalueλ and assume$f \in L^\infty(\mathbb{R})$. Then, it holds that

$$(6) \qquad 0 \leq \inf_{x\in\mathsf{R}} f(x) < \lambda < \sup_{x\in\mathsf{R}} f(x).$$

Proof. A straightforward integration of Equation E_u on the full real line yields:

$$\int_{-\infty}^{+\infty} (\lambda - f(x))u(x)\,dx = 0.$$

Since u is a nontrivial solution which does not change sign, the result follows.

Proposition 2.3. Let $u \in X$ be a solution and set$m: = \inf(1, \lambda)$. We have:
(a) If f is $L^1(\mathbb{R})$, then $u \in H_0^1(\mathbb{R})$ and

$$(7) \qquad \sqrt{m}\|u\| \leq \|u\|_0\sqrt{|f|_1}.$$

(b) Moreover, when $f \in L^2(\mathbb{R})\cap L^1(\mathbb{R})$, the following estimate holds true

$$(8) \qquad m\|u\| \leq \|u\|_0|f|_2.$$

Proof. (a) Given f in $L^1(\mathbb{R})$, multiply equation E_u by u and integrate over $\mathbb{R}$; we get the identity:

$$(9) \qquad \int_{-\infty}^{+\infty} |u'|^2(x)\,dx + \lambda\int_{-\infty}^{+\infty} |u|^2(x)\,dx = \int_{-\infty}^{+\infty} f(x)|u|^2(x)\,dx.$$

Hence, $m\|u\|^2 \leq \|u\|_0^2 |f|_1$. Then $u \in H^1(\mathbb{R}) = H_0^1(\mathbb{R})$ and the estimate in part (a) follows.
(b) If further $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then the Cauchy-Schwartz Inequality applied in (9)

$$m\|u\|^2 \leq \|u\|_0 \int_{\infty}^{+\infty} f(x)u(x)\,dx \leq \|u\|_0 |f|_2 |u|_2,$$

providing estimate (8).

2.2. **Existence of solutions.** The main result in this section is

Theorem 2.4. Assume the following conditions are ful“lled

$$\frac{c^2}{4} - f \in L^1(\mathbb{R}), \qquad (a)$$

$$\int_{-\infty}^{+\infty} \left(\frac{c^2}{4} - f(x)\right) dx < 0. \qquad (b) \tag{10}$$

There exists $a > 0$ such that $f(x) \leq \frac{c^2}{4}; \quad \forall\, x,\ |x| \geq a. \quad (c)$

Then, Equation E_u admits a nontrivial positive solution u such that $ue^{-\frac{cx}{2}} \in H^1(\mathbb{R})$.

In order to use a variational method, we first begin by writing Equation E_u under its normal form, by setting $v(x) = e^{-\frac{cx}{2}} u(x)$. We thus integrate out the damping term so that Problem (4) reduces to the one of seeking a positive solution v of the problem:

$$\begin{cases} v \in H^1(\mathbb{R}), \quad v \geq 0 \\ E_v \quad -v'' + (\lambda + \frac{c^2}{4})v = f(x)v, \quad -\infty < x < +\infty. \end{cases} \tag{11}$$

Remark 2.1. **(a)** When f is uniformly continuous, we know that Assumption (10)(a) implies $\lim_{x\to\pm\infty} f(x) = \frac{c^2}{4}$.
(b) It would be interesting to consider, instead of Assumption (10)(b), the rather weaker one: $\exists\, x_0 \in \mathbb{R},\ \frac{c^2}{4} - f(x_0) < 0$.
(c) Assumption (10)(c) is veri“ed whenever $\lim_{x\to\pm\infty} f(x) := f_\pm < \frac{c^2}{4}$.

Remark 2.2. With Hypotheses (10)(b), (c), Problem (11) has no positive solution whenever $f \in C^1(\mathbb{R})$ is a non-increasing function; this is easily seen as in Proposition 2.1, showing that the case in which f is monotonic is not covered by this theorem. Notice further that Problems (4) and (11) are not equivalent.

Proof of Theorem 2.4. Let us introduce the functional

$$\Im_n(\phi) := \int_{-\infty}^{+\infty} |\phi'(x)|^2\,dx + \int_{-\infty}^{+\infty} \left(\frac{c^2}{4} - f(x)\right) |\phi(x)|^2\,dx, \tag{12}$$

as well as the set of constrains

$$S := \{\phi \in H^1(\mathbb{R}),\ \int_{-\infty}^{+\infty} |\phi(x)|^2\,dx = 1\}. \tag{13}$$

Problem (11) then amounts to seeking critical points for the functional $\Im$ on the set S; let $\mu\colon=\inf_{\phi\in S}\Im(\phi)$. In the sequel, we seek for a negative Lagrange multiplier μ which is achieved for some $v\in H^1(\mathbb{R})$, that is satisfying Euler's Equation (see [11]): $\Im(\mathrm{v})=\mu\mathrm{F}'(\mathrm{v})$ with $\mathrm{F}(\mathrm{v})=\int_{\mathsf{R}}|\mathrm{v}(\mathrm{x})|^2\,\mathrm{dx}-1$. Then $\lambda=-\mu$ will be an eigenvalue to Problem (11). However, the remainder of the proof relies heavily on the following two lemmas:

Lemma 2.5. Under Assumptions (10)(a) and (b), we have that $\mu<0$.

Lemma 2.6. Assume $\mu<0$ and (10)(c) holds true. Then μ is achieved.

Proof of Lemma 2.5. Consider the test function $\phi_\beta(x)=e^{-\beta|x|}$ with some $\beta>0$. Then, $\Im(\phi_\beta)=\beta+\int_{-\infty}^{+\infty}\left(\frac{c^2}{4}-\mathrm{f(x)}\right)\mathrm{e}^{-2\beta|\mathrm{x}|}\,\mathrm{dx}$. Assumption (10)(b) implies that $\left|\left(\frac{c^2}{4}-f(x)\right)\right|e^{-2\beta|x|}\leq\left|\frac{c^2}{4}-f(x)\right|\in L^1(\mathbb{R})$. Moreover, we can pass to the limit as $\beta\to0^+$ and find $\lim_{\beta\to0^+}\int_{-\infty}^{+\infty}\left(\frac{c^2}{4}-f(x)\right)e^{-2\beta|x|}dx=\int_{-\infty}^{+\infty}\left(\frac{c^2}{4}-f(x)\right)dx<0$.

Therefore, $\lim_{\beta\to0^+}\Im(\phi_\beta)<0$; and by continuity, we infer the existence of some $\beta_0>0$ such that $\Im(\phi_{\beta_0})<0$. Furthermore, since $\Im(\mathrm{k}\phi)=\mathrm{k}^2\Im(\phi),\ \forall\,\mathrm{k}\in\mathbb{R}$, the function $\tilde{\phi}_{\beta_0}=\frac{\phi_{\beta_0}}{|\phi_{\beta_0}|_2}$ lies in S and still satisfies $\Im(\phi_{\beta_0})<0$, proving that $\mu<0$.

Proof of Lemma 2.6. Consider a minimizing sequence $v_n\in S$, that is such that $\lim_{n\to+\infty}\Im(v_n)=\mu$. Up to a new subsequence, we may assume, without loss of generality, $v_n\geq0,\ \forall\ n\in\mathbb{N}$. Since $\mu<0$, we may also suppose $\Im(v_n)\leq0$ for sufficiently large n. Therefore, with (12), we find the estimate

$$\int_{-\infty}^{+\infty}|v_n'(x)|^2\,dx-\left|\left(\frac{c^2}{4}-f\right)^-\right|_\infty\leq\Im(v_n)\leq0.$$

Then,

$$\int_{-\infty}^{+\infty}|v_n'(x)|^2\,dx\leq\left|\left(\frac{c^2}{4}-f\right)^-\right|_\infty.$$

Since v_n belongs to the set S defined by (13), the sequence $(v_n)_{n\in\mathsf{N}}$ is bounded in $H^1(\mathbb{R})$. Extracting, if necessary, a new subsequence, we infer the following convergence (see [3]):

$$(14)\qquad\begin{cases}v_n\rightharpoonup v,\ \text{weakly in }H^1(\mathbb{R}). & (a)\\ v_n\to v,\ \text{strongly in }C^0_{\text{loc}}(\mathbb{R}). & (b)\\ v_n\to v,\ \text{strongly in }L^2_{\text{loc}}(\mathbb{R}). & (c)\\ v_n\to v,\ \text{a.e.} & (d)\end{cases}$$

The remaining of the proof will be carried over in two steps:

Step 1. To check $v \not\equiv 0$, rewrite, in (12), the functional $\Im(v_n)$ as

$$\Im(v_n) = \int_{-\infty}^{+\infty} |v_n'(x)|^2\,dx + \int_{-\infty}^{+\infty} \left(\tfrac{c^2}{4} - f(x)\right)^+ |v_n(x)|^2\,dx \tag{15}$$
$$- \int_{-\infty}^{+\infty} \left(\tfrac{c^2}{4} - f(x)\right)^- |v_n(x)|^2\,dx.$$

Assumption (10)(c) tells us that $\left(\frac{c^2}{4} - f(x)\right)^- = 0$, for $|x| \geq a$ and hence we obtain the lower bound

$$\Im(v_n) \geq -\int_{-a}^{+a} \left(\frac{c^2}{4} - f(x)\right)^- |v_n(x)|^2\,dx. \tag{16}$$

Therefore, $v \not\equiv 0$ as claimed; otherwise, passing to the lim inf in (16), as n goes to infinity, and making use of (14)(c), we find that $\mu \geq 0$, which is a contradiction.

Step 2. Owing to (14)(a), we have the upper bound

$$\Im(v) \leq \liminf_{n\to+\infty} \Im(v_n) = \mu; \tag{17}$$

which, thanks to (15), may be expanded as follows

$$\mu = \liminf_{n\to+\infty} \Im(v_n) \geq \int_{-\infty}^{+\infty} |v'(x)|^2\,dx$$
$$+ \int_{-\infty}^{+\infty} \left(\tfrac{c^2}{4} - f(x)\right)^+ |v(x)|^2\,dx \tag{18}$$
$$- \int_{-a}^{+a} \left(\tfrac{c^2}{4} - f(x)\right)^- |v(x)|^2\,dx.$$

Moreover, the equality $\mu = \Im(v)$ holds true. To see this, we first obtain, with the aid of (14)(a), the estimate

$$\int_{-\infty}^{+\infty} |v(x)|^2\,dx \leq \liminf_{n\to+\infty} \int_{-\infty}^{+\infty} |v_n(x)|^2\,dx = 1. \tag{19}$$

We claim that $\int_{-\infty}^{+\infty} |v(x)|^2\,dx = 1$. On the contrary, assume that

$$\int_{-\infty}^{+\infty} |v(x)|^2\,dx < 1$$

and set $\tilde{v} := \frac{v}{|v|_2}$ which lies in S. Therefore,

$$\mu \leq \Im(\tilde{v}) = \frac{\Im(v)}{|v|_2^2} \leq \frac{\mu}{|v|_2^2}.$$

Hence $\mu\left(1 - \frac{1}{|v|_2^2}\right) \leq 0$ which implies $|v|_2 \geq 1$, contradicting our assumption. This shows that $v \in S$, and then $\mu \leq \Im(v)$; with (17), we finally deduce the needed equality $\mu = \Im(v)$, ending thereby the proof of Lemma 2.6.

To end the proof of Theorem 2.4, it remains to check that v is positive. To this end, rewrite Equation ε_v as

$$-v'' + \left(\left(\frac{c^2}{4} - f\right)^+ - \mu\right) v = (\frac{c^2}{4} - f)^- v.$$

The right-hand side is nonnegative. The positivity of v then follows from the Maximum Principle because $\mu < 0$. Moreover, $v \not\equiv 0$ for $\int_{-\infty}^{+\infty} v^2(x)\,dx = 1$. Finally, assume $v(x_0) = 0$ for some $x_0 \in \mathbb{R}$; then $v'(x_0) = 0$ since $v \geq 0$ (as will be seen in the remark below, notice that v belongs to $C^1(\mathbb{R})$). As for $v''(x_0) = 0$, it follows from equation E_v itself, contradicting uniqueness to the initial value problem. Therefore, $v(x) > 0$ for any $x \in \mathbb{R}$.

Remark 2.3. (a) Since $v \in H^1(\mathbb{R})$, we have that $v \in C^0(\mathbb{R})$. In addition $v'' \in C^0(\mathbb{R})$ for f is continuous; we infer that $v' \in C^0(\mathbb{R})$ [3] whence $v \in C^2(\mathbb{R})$.
(b) When $f \in L^\infty(\mathbb{R})$, we can argue as in part (a) to deduce that $v \in H^2(\mathbb{R})$.
(c) Finally, we deduce from parts (a) and (b) that $u \in C^2(\mathbb{R}) \cap H^2(]-\infty, a[)$ $\cap\, W^{2,\infty}(]-\infty, a[)$, for any $a \in \mathbb{R}$.

Remark 2.4. (a) Theorem 2.4 tells us that $u = o(e^{\frac{cx}{2}})$ as $x \to -\infty$ but gives no information about the behavior of u at positive in“nity; in particular, we do not know whether or not $u(+\infty) = 0$; the latter remains an open question. In this respect, we can say that Problem (4) has only been solved on the negative half line.
(b) Assume $\lim\limits_{x\to+\infty} f(x) := f_+$ exists. Then, from linearity of Equation E_u, a straightforward argument yields the following behavior of positive solutions, near positive in“nity:

$$u(x) \sim K \exp\Big(x\Big(\frac{c}{2} - \sqrt{\lambda + \frac{c^2}{4} - f^+}\Big)\Big),$$

for some positive constant K. Note that, from Assumption (10)(c), the function $\frac{c^2}{4} - f^+$ is nonnegative and so is the function $\lambda + \frac{c^2}{4} - f^+$. Therefore

$$\lim_{x\to+\infty} u(x) = 0 \iff \lambda > f^+.$$

Unfortunately, we ignore the position of μ, and thus the one of λ, with respect to f^+, except in the case in which $f^+ = 0$. However, the latter is ruled out whenever for instance $\lim\limits_{x\to+\infty} f(x) = \frac{c^2}{4}$ (see Remark 2.1(a)).

3. Problem on the positive half-line

3.1. **Setting of the problem.** Consider the boundary value problem with general right-hand term:

$$(20)\qquad \begin{cases} -u'' + cu' + \lambda u &= h(x, u(x)), \quad x \in I \\ u(0) = u(+\infty) &= 0. \end{cases}$$

Here $h\colon I \times \mathbb{R} \to \mathbb{R}$ is a continuous function not necessarily positive. The interval I denotes $]0,+\infty[$, the set of positive real numbers. Setting $k\colon = \sqrt{\lambda + \frac{c^2}{4}}$, Problem (20) is rewritten for the function $v(x) = e^{-\frac{c}{2}x}u(x)$:

$$\text{(21)} \qquad \begin{cases} -v'' + k^2 v &= e^{-\frac{c}{2}x} h(x, e^{\frac{c}{2}x}v(x)), \quad x \in I \\ v(0) = v(+\infty) &= 0. \end{cases}$$

Equivalently, the new unknown v satisfies the integral equation:

$$v(x) = \int_0^{+\infty} G(x,s) e^{-\frac{c}{2}s} h(s, e^{\frac{c}{2}s}v(s))\, ds$$

with Green's function

$$\text{(22)} \qquad G(x,s) = \frac{1}{2k} \begin{cases} e^{-ks}(e^{kx} - e^{-kx}) & x \le s \\ e^{-kx}(e^{ks} - e^{-ks}) & x \ge s. \end{cases}$$

The unknown u is then solution of the integral equation:

$$u(x) = \int_0^{+\infty} e^{\frac{c}{2}(x-s)} G(x,s) h(s, u(s))\, ds.$$

The following lemma provides estimates of the Green function G and will play an important role in the sequel; we omit the proof:

Lemma 3.1. For any $x,\ s \in I$, we have that

$$\text{(23)} \qquad 0 < G(x,s) \le \tfrac{1}{2k} \text{ and } 0 < G(x,s)e^{-\mu x} \le G(s,s)e^{-ks}, \ \ \forall\, \mu \ge k.$$

Under suitable assumptions on the nonlinear function h, we shall prove the existence of a solution to Problem (20). The proof relies on Schauder's fixed point theorem and Zima's compactness criterion (see [17]); but first of all, let us recall some

3.2. **Preliminaries.** Let $p\colon I \longrightarrow]0,+\infty[$ be a continuous function. Denote by X the Banach space consisting of all weighted functions u continuous on I and satisfying

$$\sup_{x \in I}\{p(x)|u(x)|\} < \infty,$$

equipped with the norm $\|u\| = \sup_{x\in I}\{p(x)|u(x)|\}$. We have

Definition 3.1. A set of functions $u \in \Omega \subset X$ are said to be almost equi-continuous on I if they are equi-continuous on each interval $[0,T]$, $0 < T < +\infty$.

Lemma 3.2. ([18]) If the functions $u \in \Omega$ are almost equi-continuous on I and uniformly bounded in the sense of the norm

$$\|u\|_q = \sup_{x \in I}\{q(x)|u(x)|\}$$

where the function q is positive, continuous on I and satisfies

$$\lim_{x\to+\infty}\frac{p(x)}{q(x)}=0,$$

then Ω is relatively compact in X.

Finally, recall Schauder's Fixed Point Theorem for the reader's convenience.

Theorem A. Let E be a Banach space and $C\subset E$ a bounded, closed and convex subset of E. Let $F\colon\ C\longrightarrow C$ be a completely continuous mapping. Then F has a fixed point in C.

3.3. **Main result.** Having disposed of these auxiliary results, we are in position to prove

Theorem 3.3. Assume that:

$$(24)\qquad \begin{cases} h\colon\ I\times\mathbb{R}\longrightarrow\mathbb{R} \ \text{ is a continuous function,}\\ \exists\, p>0:\ p\neq 1,\ |h(x,u)|\leq a(x)+b(x)|u|^p,\quad \forall\,(x,u)\in I\times\mathbb{R},\\ \text{where } a,b: I\longrightarrow\mathbb{R}^+ \text{ are continuous positive functions.}\end{cases}$$

$$(25)\qquad \begin{cases} \text{There exists } \theta>k+\frac{c}{2} \text{ such that}\\ M_1\colon\ =\int_0^{+\infty}e^{-(k+\frac{c}{2})s}a(s)\,ds<\infty,\\ M_2\colon\ =\int_0^{+\infty}e^{(p\theta-k-\frac{c}{2})s}b(s)\,ds<\infty.\end{cases}$$

$$(26)\qquad 2k\left(\frac{2k}{pM_2}\right)^{\frac{1}{p-1}}-M_2\left(\frac{2k}{pM_2}\right)^{\frac{p}{p-1}}-M_1\geq 0,\ \ \text{when } p>1.$$

Then Problem (20) has at least one solution $u\in C(I;\mathbb{R})$.

Remark 3.1. If the coefficients a, b belong to $C(I,\mathbb{R}^+)\cap L^\infty(I,\mathbb{R}^+)$, then the first condition in Assumption (25) is obviously satisfied. As for the second one, it holds true whenever $p<1$ since we may always choose a real number θ such that $k+\frac{c}{2}<\theta<\frac{1}{p}\left(k+\frac{c}{2}\right)$. In case $p>1$, we have that $p\theta-k-\frac{c}{2}>0$ and then the function b should decay exponentially to zero as $s\to+\infty$ very faster in order that the second condition be fulfilled.

Lemma 3.4. Assumption (26) is equivalent to the following condition:

$$(27)\qquad \exists\ R_0>0,\quad \frac{1}{2k}(M_1+M_2R_0^p)\leq R_0.$$

Proof. Define the function $\omega(x)\colon\ =x-\frac{M_2}{2k}x^p=x\left(1-\frac{M_2}{2k}x^{p-1}\right)$. Then $\omega(0)=0$ and $\lim\limits_{x\to+\infty}\omega(x)=-\infty$ for any $p>1$. Moreover, the function ω changes monotonicity and admits a maximum at some value x_0 such that

$x_0^{p-1} = \frac{1}{p}\frac{2k}{M_2}$ and $\omega(x_0) = \frac{p-1}{p}x_0$. Hence, Assumption (27) is verified if and only if $\frac{p-1}{p}x_0 \geq \frac{M_1}{2k}$, that is:

$$\frac{M_1}{2k}\left(\frac{pM_2}{2k}\right)^{\frac{1}{p-1}} \leq \frac{p-1}{p} = 1 - \frac{1}{p} \tag{28}$$

which may be rewritten as

$$M_1 \leq 2k\left(\frac{2k}{pM_2}\right)^{\frac{1}{p-1}} - M_2\left(\frac{2k}{pM_2}\right)^{\frac{p}{p-1}}$$

for $\frac{2k}{p}\left(\frac{2k}{pM_2}\right)^{\frac{1}{p-1}} = M_2\left(\frac{2k}{pM_2}\right)^{\frac{p}{p-1}}$, whence (26) and the lemma is proved.

Remark 3.2. In case $p < 1$, the function ω defined above is such that $\omega(0) = 0$ and $\lim\limits_{x\to+\infty} \omega(x) = +\infty$. Then, for all $A > 0$, there exists an $R > 0$ such that $\omega(R) > A$. It follows that the condition

$$2k\left(\frac{2k}{pM_2}\right)^{\frac{1}{p-1}} - M_2\left(\frac{2k}{pM_2}\right)^{\frac{p}{p-1}} \leq M_1,$$

which is also equivalent to (27), is always satisfied.

Proof of Theorem 3.3:
Let $\theta \in \mathbb{R}$ be as in Assumption (25) and consider the weighted space

$$X = \{u \in C(I;\mathbb{R}) : \sup_{x\in I}\{e^{-\theta x}|u(x)|\} < \infty\}$$

endowed with the weighted sup-norm:

$$\|u\|_\theta = \sup_{x\in I}\{e^{-\theta x}|u(x)|\}.$$

Next define on X the operator F by:

$$Fu(x) = \int_0^{+\infty} e^{\frac{c}{2}(x-s)}G(x,s)h(s,u(s))\,ds.$$

We show that F satisfies the following properties:

- **(a):** F maps any closed ball $\overline{B}_R = \{u \in X : \|u\| \leq R\}$ into the space X.
- **(b):** F is completely continuous in X.
- **(c):** There exists a closed ball $\overline{B}_{R_0}$ such that $F(\overline{B}_{R_0}) \subset \overline{B}_{R_0}$.

• **Claim 1**: For any $u \in \overline{B}_R$, $\sup_{x\in I} e^{-\theta x}|Fu(x)| < \infty$, that is $F(\overline{B}_R) \subset X$. Indeed, choosing $\mu = \theta - \frac{c}{2}$ in (23), noting that $\mu \geq k$ by (25), and using (24), we have the estimates

$$\begin{aligned}
e^{-\theta x}|Fu(x)| &\leq e^{-\theta x}\int_0^{+\infty} e^{\frac{c}{2}(x-s)}G(x,s)|h(s,u(s))|\,ds \\
&= e^{-(\theta-\frac{c}{2})x}\int_0^{+\infty} e^{\frac{-c}{2}s}G(x,s)|h(s,u(s))|\,ds \\
&\leq \int_0^{+\infty} e^{-(k+\frac{c}{2})s}G(s,s)\,[a(s)+b(s)|u(s)|^p]\,ds \\
&\leq \tfrac{1}{2k}\int_0^{+\infty} e^{-(k+\frac{c}{2})s}a(s)\,ds \;+\; \tfrac{1}{2k}\,\|u\|_\theta^p\int_0^{+\infty} e^{(p\theta-k-\frac{c}{2})s}b(s)\,ds \\
&\leq \tfrac{1}{2k}(M_1+M_2\,\|u\|_\theta^p)<\infty \\
&\leq \tfrac{1}{2k}(M_1+M_2R^p)<\infty.
\end{aligned}$$

• **Claim 2**: F is completely continuous:
(a) Let $\theta_1 \in \mathbb{R}$ be such that $\theta_1 < \theta$ and consider the functions $u \in \overline{B}_R$ for some positive real number R. Then the family $\{Fu\}$ is uniformly bounded with respect to the norm $\|.\|_{\theta_1}$. Indeed, as shown in Claim 1, we have that $\|Fu\|_{\theta_1} \leq \frac{1}{2k}(M_1+M_2R^p)$, for any $u \in \overline{B}_R$.
(b) The functions $\{Fu\}$ for $u \in \overline{B}_R$ are almost equi-continuous on I. The proof follows that in Theorem 5.1, [9] and is omitted.
(c) Applying Lemma 3.2 with weight $q(x) = e^{-\theta_1 x}$, we deduce that the mapping F is completely continuous on $\overline{B}_R$.

• **Claim 3**: Let R_0 be as in (27) and check that $F(\overline{B}_{R_0}) \subset \overline{B}_{R_0}$. For any $u \in \overline{B}_{R_0}$, we have

$$0 \leq e^{-\theta x}|Fu(x)| \leq \frac{1}{2k}(M_1+M_2\|u\|_\theta^p) \leq \frac{1}{2k}(M_1+M_2R_0^p) \leq R_0.$$

By Theorem A, the operator F has a fixed point in $\overline{B}_{R_0}$ and so Problem (20) admits a solution u in $\overline{B}_{R_0}$.

Remark 3.3. To ensure existence of nontrivial solutions, one must add, further to the assumptions in Theorem 3.3, the condition:

$$\exists\, x_0 \in I,\ h(x_0,0) \neq 0.$$

Remark 3.4. In [9], the Krasnosels•kii “xed point theorem in cones has been used and a result similar to Theorem 3.3 has been proved with the following additional assumption:

$$\begin{cases} \text{There exist } \alpha>0,\ \gamma,\delta>0 \text{ and } x_0\in I \text{ such that:} \\ \min\limits_{x\in[\gamma,\delta],\ u\in[m\alpha,\alpha e\quad]} h(x,u) \geq \alpha e^{\theta x_0}\left[\int_\gamma^\delta e^{\frac{c}{2}(x_0-s)}G(x_0,s)\ ds\right]^{-1} \end{cases}$$

where $m: \ = \min\left\{e^{-k\delta}, e^{k\gamma}-e^{-k\gamma}\right\}$ and h is a positive continuous function.

In general, the latter assumption is not easy to be verified; however, existence of solutions which are both nontrivial and positive has been obtained. This condition has been weakened in [8] in case of positive right-hand term. Finally, note that when $p = 1$, an existence result of positive solutions is also given in [19].

3.4. **The case $p = 1$.** The case $p = 1$ may be treated as in Theorem 3.3 and the following result of a solution with arbitrary sign can be proved; we omit the proof.

Theorem 3.5. Suppose the following assumptions are fulfilled:

$$(\Re 1) \quad \begin{cases} h\colon\ I \times \mathbb{R} \longrightarrow \mathbb{R} \ \text{ is a continuous function,} \\ |h(x,u)| \leq a(x) + b(x)|u|, \quad \forall\,(x,u) \in I \times \mathbb{R}, \\ \text{where } a, b\colon\ I \longrightarrow \mathbb{R}^+ \ \text{ are continuous positive functions} \end{cases}$$

$$(\Re 2) \quad \begin{cases} \text{There exists } \theta > k + \frac{c}{2} \ \text{ such that} \\ M_1\colon = \int_0^{+\infty} e^{-(k+\frac{c}{2})s} a(s)\, ds < \infty, \\ M_2\colon = \int_0^{+\infty} e^{(\theta-k-c/2)s} b(s)\, ds < 2k. \end{cases}$$

Then Problem (20) has at least one solution $u \in C(I;\mathbb{R})$.

Remark 3.5. Notice that the condition $M_2 < 2k$ guarantees the existence of a ball B such that the mapping F sends B into itself.

Corollary 3.6. The problem

$$(29) \qquad \begin{cases} -u'' + cu' + \lambda u &= f(x)u, \quad 0 < x < +\infty. \\ u(0) = u(+\infty) &= 0 \end{cases}$$

where $f\colon\ \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists $\theta > c + \frac{k}{2}$ with $\int_0^{+\infty} e^{(\theta-k-\frac{c}{2})s}|f(s)|\, ds < 2k$ has at least one solution $u \in C(I,\mathbb{R})$.

4. Problem on the full real line

The sub-linear problem can be also treated on $(-\infty,+\infty)$ and u may be regarded as a solution of the integral equation:

$$u(x) = \int_{-\infty}^{+\infty} K(x,s) h(s,u(s)) ds$$

with new Green's function:

$$K(x,s) = \frac{1}{r_1 - r_2} \begin{cases} e^{r_1(x-s)}, & x \leq s \\ e^{r_2(x-s)}, & x \geq s, \end{cases}$$

and characteristic roots:

$$r_1 = \frac{c+\sqrt{c^2+4\lambda}}{2},\ r_2 = \frac{c-\sqrt{c^2+4\lambda}}{2}.$$

Consider the Banach space:

$$E = C_0(\mathbb{R},\mathbb{R}) = \{u \in C(\mathbb{R},\mathbb{R}):\ \ u(-\infty) = u(+\infty) = 0\}$$

equipped with the sup-norm:

$$\|u\| = \sup_{x\in\mathsf{R}} |u(x)|\,.$$

We state without proof the following result for a right-hand term $h(x,u)$ for the following problem:

$$(30)\qquad \begin{cases} -u'' + cu' + \lambda u &= h(x,u), \quad -\infty < x < +\infty. \\ u(-\infty) = u(+\infty) &= 0, \end{cases}$$

The proof runs parallel to that of Theorem 3.3.

Theorem 4.1. Problem (30) has a solution in E provided there exist positive functions $(a,b) \in E^2$ such that:

$$(a) \quad |h(x,u)| \le a(x) + b(x)|u|, \ \forall\, x, u \in \mathbb{R}.$$
$$(b) \quad a^* = \int_{-\infty}^{+\infty} a(x)dx < \infty, \quad b^* = \int_{-\infty}^{+\infty} b(x)dx < \sqrt{c^2 + 4\lambda}.$$

Corollary 4.2. The problem

$$(31)\qquad \begin{cases} -u'' + cu' + \lambda u &= f(x)u, \quad -\infty < x < +\infty. \\ u(-\infty) = u(+\infty) &= 0, \end{cases}$$

where the function $f \in C(\mathbb{R},\mathbb{R}) \cap L^1(\mathbb{R},\mathbb{R})$ satis“es $\int_{-\infty}^{+\infty} |f(s)|\, ds < \sqrt{c^2 + 4\lambda}$, admits at least one solution $u \in C_0(\mathbb{R},\mathbb{R})$.

5. Applications

To illustrate the results of Theorems 3.3 and 3.5, take in Problem (20) the values $c = 2$ and $\lambda = 3$ and discuss three situations.

Example 5.1. Consider the sub-linear case where h is de“ned by

$$(32)\qquad h(x,u) = \begin{cases} e^{2x}, & x \in \mathbb{R}^+, \ u \in [0,1] \\ e^{2x} + e^{\frac{x}{2}} \sqrt{\ln(u)}, & x \in \mathbb{R}^+, \ u \ge 1. \end{cases}$$

Here $p = \frac{1}{2}$ and all assumptions in Theorem 3.3 are met since we may take any $3 < \theta < 5$ in order that (25) be satis“ed; indeed, in this case $M_1 = 1$ and $M_2 = \frac{2}{5-\theta}$. Then, Problem (20) has a solution. This solution is nontrivial for $h(x,0) \neq 0$ for any $x \in I$ (see Remark 3.3). In addition, it is positive by the weak maximum principle.

Example 5.2. Consider the boundary value problem:

$$(33)\qquad \begin{cases} -u'' + 2u' + 3u &= e^{\alpha x} + e^{-\beta x} g(u), \quad 0 < x < \infty \\ u(0) = u(+\infty) &= 0 \end{cases}$$

with some $0 < \alpha < 3$, $\beta > 3$ and a function $g \in C(\mathbb{R},\mathbb{R})$ with quadratic growth $|g(u)| \le |u|^2$. Then, $M_1 = \frac{1}{3-\alpha}$ and $M_2 = \frac{1}{3+\beta-2\theta}$ for some $3 < \theta < \frac{3+\beta}{2}$. Assumption (28) is ful“lled whenever $4(3+\beta-2\theta)(3-\alpha) \ge 1$ and $3 < \theta \le \frac{3+\beta}{2} - \frac{1}{8(3-\alpha)}$ which is possible for any $0 < \alpha < \frac{71}{24}$. Therefore, Problem (33) has at least a nontrivial solution by Theorem 3.3.

Example 5.3. Consider the problem:

$$(34)\qquad \begin{cases} -u'' + 2u' + 3u &= e^{\alpha x} + e^{-\beta x} g(u), \quad 0 < x < \infty \\ u(0) = u(+\infty) &= 0 \end{cases}$$

with some $0 < \alpha < 3$, $\beta > 3$ and a function $g(u) = \frac{2}{\pi}\frac{\ln(1+u^2)\arctan(u)}{1+|u|}$ which satis“es $|g(u)| \leq |u|$. Then, $M_1 = \frac{1}{3-\alpha}$ and $M_2 = \frac{1}{3+\beta-\theta} \leq 4$ for any $3 < \theta < \beta + \frac{11}{4}$. By Theorem 3.5, Problem (34) has at least a nontrivial solution.

6. Concluding remarks

(a) We believe that this work has contributed to the study of Dirichlet boundary value problems posed on unbounded intervals of the real line and associated with the class of second-order differential equations $-u''+cu'+\lambda u = h(x,u)$. In this work, we have discussed polynomial-like growth of the nonlinear term h in terms of the unknown u : $|h(x,u)| \leq a(x) + b(x)|u|^p$ $(p \in \mathbb{R}^+)$. Example 5.1 shows that in the strict sub-linear case $p < 1$, the coefficient b need not tend to zero at positive infinity and then the nonlinear term h is not committed to die out as x gets very large. Also, a special attention has been paid to the linear case which arises more particularly in epidemiology.
(b) Corollary 4.2 proves existence of solutions to Problem (4) defined on the whole real line under condition that $f \in C_0(\mathbb{R},\mathbb{R})$ and $\int_{-\infty}^{+\infty} f(x)dx < \sqrt{c^2+4\lambda}$ but does neither encompass the nonincreasing case, nor does it yield positive solutions to this problem.
(c) However, when f is nonincreasing, the linear problem $h(x) = f(x)u$ set on the full real line is, as far as we know, an open question. In 2001, the first author proposed this question in Problem Section of Electronic Journal of Differential Equations, Problem (2001-2) and is still unsolved
(see http://math.uc.edu/ode/odesols/p2001.htm).

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