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Linear Functions Preserving Multivariate and Directional Majorization

A. Armandnejad * and H. R. Afshin

Department of Mathematics, Vali-e-Asr University of Rafsanjan P. O. Box: 7713936417, Rafsanjan, Iran

> E-mail: armandnejad@mail.vru.ac.ir E-mail: afshin@mail.vru.ac.ir

ABSTRACT. Let V and W be two real vector spaces and let ~ be a relation on both V and W. A linear function $T: V \to W$ is said to be a linear preserver (respectively strong linear preserver) of ~ if $Tx \sim Ty$ whenever $x \sim y$ (respectively $Tx \sim Ty$ if and only if $x \sim y$). In this paper we characterize all linear functions $T: \mathbf{M}_{n,m} \to \mathbf{M}_{n,k}$ which preserve or strongly preserve multivariate and directional majorization.

Keywords: Doubly stochastic matrices, Directional majorization, Multivariate majorization, Linear preserver.

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1. INTRODUCTION

Let $\mathbf{M}_{n,m}$ be the vector space of all real $n \times m$ matrices. An $n \times n$ matrix $D = [d_{ij}]$ is called doubly stochastic provided that the entries of D are all nonnegative and $\sum_{k=1}^{n} d_{ik} = \sum_{k=1}^{n} d_{kj} = 1$ for every $i, j \in \{1, \dots, n\}$. Let X and Y belong to $\mathbf{M}_{n,m}$, we say X is multivariate majorized by Y (written $X \prec_m Y$) if X = DY for some $n \times n$ doubly stochastic matrix D. A generalized concept of multivariate majorization was introduced in [3]. For X and Y belong

^{*}Corresponding Author

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to $\mathbf{M}_{n,m}$, it is said that X is directional majorized by Y (written $X \prec_d Y$) if for every $a \in \mathbb{R}^m$ there exists a doubly stochastic matrix D_a such that $Xa = D_aYa$. When m = 1, the definition of multivariate majorization and directional majorization reduce to the classical concept of vector majorization. Vector majorization is a much studied concept in linear algebra and its applications, for more details about vector majorization see [1] and [5]. Some of our notations are explained next.

 \mathcal{P}_n ; The set of all $n \times n$ permutation matrices.

 ${\bf J}$; The $n\times n$ matrix with all entries equal to 1.

 $X = [x_1|\cdots|x_m]$; An $n \times m$ matrix with $x_j \in \mathbb{R}^n$ as the j^{th} column of X. trx; The summation of all components of a vector $x \in \mathbb{R}^n$.

About linear functions preserving multivariate and directional majorization on $\mathbf{M}_{n,m}$, Li and Poon obtained the following interesting result in [4].

Proposition 1.1. Let *T* be a linear operator on $M_{n,m}$. Then *T* preserves multivariate majorization if and only if *T* preserves directional majorization if and only if one of the following holds:

- (a) There exist $A_1, \dots, A_m \in \mathsf{M}_{n,m}$ such that $T(X) = \sum_{j=1}^m (trx_j)A_j$.
- (b) There exist $R, S \in M_m$ and $P \in \mathcal{P}_n$ such that T(X) = PXR + JXS.

The above proposition is in fact a generalization of the following proposition which has been proved by Ando in [2].

Proposition 1.2. A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ preserves vector majorization if and only if one of the following holds:

- (i) Tx = (trx)a for some $a \in \mathbb{R}^n$.
- (*ii*) $Tx = \alpha Px + \beta (trx)e = \alpha Px + \beta Jx$ for some $\alpha, \beta \in \mathbb{R}$ and $P \in \mathcal{P}_n$.

Our main result is a generalization of Proposition 1.1. In fact, we prove the following theorem.

Theorem 1.3. Let $T : M_{n,m} \to M_{n,k}$ be a linear function. Then *T* preserves multivariate majorization if and only if *T* preserves directional majorization if and only if one of the following holds:

(a) There exist $A_1, \dots, A_m \in M_{n,k}$ such that $TX = \sum_{i=1}^m (trx_i)A_i$.

(b) There exist $P \in \mathcal{P}_n$ and $R, S \in M_{m,k}$ such that TX = PXR + JXS.

2. Main result

We state the following statements to prove the main theorem. The following proposition is proved in [4].

Proposition 2.1. Let $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ be two linear preservers of vector majorization which satisfy :

(2.1)
$$T_1Qy + \gamma T_2Qy \prec T_1y + \gamma T_2y \quad \forall \gamma \in \mathsf{R}, \ \forall y \in \mathsf{R}^n, \ \forall Q \in \mathcal{P}_n.$$

Then T_1, T_2 are either of the form (i) or (ii) in Proposition 1.2 with the same P.

First, we prove a special case of Theorem 1.3, in which k = 1.

Lemma 2.2. A linear function $T : M_{n,m} \to \mathbb{R}^n$ preserves multivariate majorization if and only if one of the following holds:

- (i) There exist $a_1, \dots, a_m \in \mathsf{R}^n$ such that $TX = \sum_{j=1}^m (trx_j)a_j$.
- (*ii*) There exist $a, b \in \mathbb{R}^m$ and $P \in \mathcal{P}_n$ such that $TX = PXa + \mathsf{J}Xb$.

Proof. Define $T' : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ by T'X = [TX|0] where 0 denotes the $n \times (m-1)$ zero matrix. Clearly T' is a linear operator which preserves multivariate majorization. Then by Proposition [4], T' has one of the following forms; (a) $T'(X) = \sum_{i=1}^{m} (trx_i)B_i$ for some $B_1, \dots, B_m \in \mathbf{M}_{n,m}$. So $TX = \sum_{j=1}^{m} (trx_j)a_j$, where a_j is the first column of B_j for any j $(1 \le j \le n)$ and hence (i) holds. (b) T'(X) = PXR' + JXS' for some $P \in \mathcal{P}_n$ and some $R', S' \in \mathbf{M}_m$. So $TX = PXa + \mathbf{J}Xb$ where a and b are the first columns of R and S respectively, and hence (ii) holds.

Lemma 2.3. Let $T_1, T_2 : \mathsf{M}_{n,m} \to \mathsf{R}^n$ be two linear functions. If $T : \mathsf{M}_{n,m} \to \mathsf{M}_{n,2}$ debned by $TX = [T_1X|T_2X]$ preserves multivariate majorization, then T_1, T_2 are both either of the form (i) or (ii) in Lemma 2.2 with the same P.

Proof. If m = 1 then T_1, T_2 satisfy the conditions of Proposition 2.1 and hence T_1, T_2 are either of the form (i) or (ii) in Proposition 1.2 with the same P. If $m \ge 2$, define $T' : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ by T'(X) = [TX|0] where 0 denotes the $n \times (m-2)$ zero matrix. Clearly T' is an operator which preserves multivariate majorization. Therefore by Proposition 1.1, either $T'(X) = \sum_{i=1}^{m} (trx_i)B_i$, for some $B_1, \dots, B_m \in \mathbf{M}_{n,m}$ and hence $T(X) = \sum_{i=1}^{m} (trx_i)A_i$ where $B_i = [A_i|*]$ and * is an $n \times (m-2)$ block for every i $(1 \le i \le m)$, or T'(X) = PXR' + JXS' for some $P \in P_n, R', S' \in \mathbf{M}_m$ and hence T(X) = PXR + JXS where $R' = [R|*_1], S' = [S|*_2]$ and $*_1, *_2$ are two $n \times (m-2)$ blocks.

Proof of Theorem 1.3 If *T* satisfies (a) or (b), trivially *T* preserves multivariate and directional majorization. Conversely, let *T* be a linear preserver of multivariate majorization. Then there exist linear functions $T_i : \mathbf{M}_{n,m} \to \mathbf{R}^n$, $i = 1, \dots, k$ such that $TX = [T_1X|\cdots|T_kX]$. It is easy to see that T_i preserves multivariate majorization for every i $(1 \le i \le k)$. By Lemma 2.2 and Lemma 2.3, either every T_i satisfies condition (i) or (ii) of Lemma 2.2. If for every i $(1 \le i \le k)$ $T_i(X) = \sum_{j=1}^m (trx_j)a_j^i$ for some $a_1^i, \dots, a_m^i \in \mathbf{R}^n$, then $T(X) = [\sum_{j=1}^m (trx_j)a_j^1] \sum_{j=1}^m (trx_j)a_j^2] \cdots [\sum_{j=1}^m (trx_j)a_j^m] = \sum_{j=1}^m (trx_j)A_j$, for some $A_j \in \mathbf{M}_{n,k}$. Hence *T* satisfies condition (i). If for every i $(1 \le i \le k)$, $T_i(X) = PXa_i + JXb_i$ for some $a_i, b_i \in \mathbf{R}^m$ and $P \in \mathcal{P}_n$. Then $TX = [PXa_i + JXS_i] \cdots |PXa_k + JXS_k] = PX[a_1|\cdots|a_k] + JX[b_1|\cdots|b_k] = PXR + JXS$ for some $R, S \in \mathbf{M}_{m,k}$. Thus *T* satisfies condition (i). If *T* preserves directional

majorization it is easy to see that the following condition holds:

(2.2) $TX \prec_d TY$ whenever $X \prec_m Y$.

Now, if one replace multivariate majorization preserving by condition (2.2) in the previous lemmas, all proofs are valid. Then T satisfies conditions (a) or (b).

Now, we state the following lemma to characterize all strong linear preserver of multivariate and directional majorization from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$

Lemma 2.4. Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,k}$ be a linear function of the form T(X) = DXR + JXS, for some $R, S \in \mathbf{M}_{m,k}$ and invertible doubly stochastic $D \in \mathbf{M}_n$. Then T is injective if and only if R and (R + nS) are full-rank matrices.

Proof. Without loss of generality we may assume that D = I. Since dim(KerT) + rank(T) = nm, if k < m then $dim(KerT) \ge 1$. Therefore T is not injective. If $k \ge m$, the matrix representation of T with respect to the standard bases of $\mathbf{M}_{n,m}$ and $\mathbf{M}_{n,k}$ is similar to the following block matrix:

(2.3)
$$\begin{pmatrix} R+nS & & \\ & R & \\ & & \ddots & \\ & & \ddots & \\ & & & & R \end{pmatrix} \in \mathbf{M}_{nk,nm}$$

Therefore T is injective if and only if R and (R+nS) are full-rank matrices.

Theorem 2.5. Let $T : \mathsf{M}_{n,m} \to \mathsf{M}_{n,k}$ be a linear function. Then T strongly preserves multivariate majorization if and only if T strongly preserves directional majorization if and only if there exist $P \in \mathcal{P}_n$ and $R, S \in M_{m,k}$ such that R, (R + nS) are full-rank matrices and $TX = PXR + \mathsf{J}XS$.

Proof. It is clear that every strong linear preserver of multivariate majorization is injective. So by Theorem 1.3 and Lemma 2.4, $TX = PXR + \mathbf{J}XS$ for some $R, S \in M_{m,k}$ such that R, (R + nS) are full-rank matrices. The other side is trivial.

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