

Clifford Wavelets and Clifford-valued MRAs

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ABSTRACT. In this paper using the Clifford algebra over \mathbb{R}^4 and its matrix representation, we construct Clifford scaling functions and Clifford wavelets. Then we compute related mask functions and filters, which arise in many applications such as quantum mechanics.

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1. INTRODUCTION

A complex-valued representation of a real 1-dimensional signal is an important tool in analysis of signal processing. The reason is that in its polar representation, the modulus of the complex signal is identified as a local quantitative measure of a signal, called local amplitude, and the argument of the complex signal is identified as a local measure for the qualitative information of a signal, called local phase. First step for generalizing such representation system was quaternion-valued representation, on which a signal can be expressed by four parameters as its local quantitative measures.

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On the other hand wavelets are a very useful and wide applied tools for practical applications in signal and image processing, multi-satellite measurements of electromagnetic wave fields, analysis of climate-related time-series and analysis space weather effects and so on. One usual way to construct wavelets pass through multiresolution analysis (MRA), which is a procedure for constructing wavelets from a scaling function. Now if the scaling function is a matrix of functions, we deal with matrix-valued MRAs. In this paper we show that any real or complex Clifford algebra can be identified with a suitable matrix algebra, then via this representation, Clifford-valued scaling functions, Clifford-valued MRAs and Clifford wavelets are given.

Notations. For an algebra \mathbb{K} , we denote its product with ”.”. \mathbb{R}, \mathbb{C} and \mathbb{H} are algebra of real numbers, complex numbers and quaternions, respectively. $\mathbb{K}[n]$ is the algebra of $n \times n$ matrices over field \mathbb{K} . $\otimes_{\mathbb{K}}$ denotes tensor product over field \mathbb{K} .

This paper is organized as follow: in second section we introduce the n -dimensional Clifford algebra (on brief) and some useful theorems on it, then we discuss the $Cl(\mathbb{R}^4)$ and $Cl(\mathbb{C}^4)$ (real and complex forms of Clifford algebra on \mathbb{R}^4 , resp.) and their matrix representations. Section 3 consists of multiresolution analysis (MRA) and Clifford wavelet structures. In section 4, we compute Clifford wavelets matrices on \mathbb{R}^4 .

2. CLIFFORD ALGEBRA

In this section we mention some definitions and basic facts about Clifford algebras.

Definition 2.1. let V be a finite dimensional vector space on the field \mathbb{F} . A quadratic form (q-form) on V is a function $h : V \times V \rightarrow \mathbb{F}$, such that
 $h(\alpha x_1 + x_2, y) = \alpha h(x_1, y) + h(x_2, y)$
 $h(x, \alpha y_1 + y_2) = \alpha h(x, y_1) + h(x, y_2)$.

Furthermore if $h(x, y) = h(y, x)$ then h is called symmetric. For any q-form h , there exists a matrix representation $A = (A_{ij})$ such that $A_{ij} = h(e_i, e_j)$ where $\{e_1, e_2, \dots, e_n\}$ is a basis for V . The q-form h is called nondegenerate, if $\det(h(e_i, e_j)) \neq 0$.

Let V be an n -dimensional vector space on the field F , and h be a nondegenerate symmetric q-form on V , then there exists an ordered basis $B = \{e_1, e_2, \dots, e_n\}$ for V such that $A = (A_{ij})$ is diagonal. In particular for $\mathbb{F} = \mathbb{R}$

$$h(e_i, e_j) = \begin{cases} \pm 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

If the matrix A have p -times 1 and q -times -1 on its diameter such that $p + q = n$, then h will be shown with $h(p, q)$. For h , a nondegenerate q-form on

real vector space V , the pair (V, h) is called a quadratic space (q-space). For describing the Clifford algebra on vector space V , consider the commutative tensor algebra $T(V) = \bigoplus_{r=0}^{\infty} \otimes^r V$ on real q-space (V, h) with unit 1. Let $I_h(V) = \langle V \otimes V + h(V, V) \rangle$ then I_h is a two-sided ideal in $T(V)$. The quotient space $\frac{T(V)}{I_h(V)}$ is called the *Clifford algebra* on V and is denoted by $Cl(V, h)$. The induced product, from tensor product on $T(V)$, is called *Clifford product* and will be shown with "·", $(Cl(V, h), "·")$ is again a commutative algebra with unit. If h is $h(p, q)$ then $Cl(V, h)$ will be shown by $Cl(p, q)$.

By considering the canonical projection map $\pi_h : T(V) \rightarrow Cl(V, h)$, one can find that the map $\theta_V : V \rightarrow Cl(V, h)$ is one-to-one. This fact says that $Cl(V, h)$ is generated by vector space $V \subset Cl(V, h)$ and identity 1, and its product satisfies the following relations:

- 1) $v \cdot v = -h(v, v)1$ for any $v \in Cl(V, h)$
- 2) $v \cdot w + w \cdot v = -2h(v, w)$.

In view of previous equations we can obtain the universal map for Clifford algebras as follow:

Proposition 2.1. *Let \mathcal{A} be a commutative \mathbb{K} -Algebra with unit 1, and $f : V \rightarrow \mathcal{A}$ be a linear map such that: $f(v) \cdot f(v) = -h(v, v)1$ for any $v \in V$, then f can be uniquely extended to the algebraic homomorphism $\tilde{f} : Cl(V, h) \rightarrow \mathcal{A}$. Furthermore, $Cl(V, h)$ is the unique associated \mathbb{K} -Algebra with this property.*

In other word if (V, h) is a q-space, then there exists a Clifford algebra associated to it and is unique up to an isomorphism. This is easy to show that if $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis for real vector space V , then the set $\{1, e_i, e_i e_j, e_i e_j e_k, \dots, e_1 e_2 e_3 \dots e_n : i+1 = j, j+1 = k\}$ is a basis for $Cl(V, h)$. Note that $Cl(V, h) = \frac{T(V)}{I_h} = \frac{\mathbb{R} \oplus V \oplus V \oplus V \oplus V \oplus \dots}{\langle V \otimes V + h(V)1 \rangle}$, and

$$T(V) = a_0 + \sum_{i=1}^n a_i e_i + \sum a_{ij} e_i \otimes e_j + \sum a_{ijk} e_i \otimes e_j \otimes e_k + \dots + a_{i_1 \dots i_n} e_1 \otimes e_2 \otimes \dots \otimes e_n.$$

Also $V \otimes V + h(V)1 = 0$ implies that $V \otimes V = -h(V)1$.

Example 2.2. *Let $V = \mathbb{R}^2$, and h be the quadratic form obtained by the matrix*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ i.e } V = \mathbb{R}^2 = \langle e_1, e_2 \rangle. \text{ Dim } V = 2, \text{ so } \dim Cl(V) = 4 \text{ and}$$

$$Cl(V) = Cl(\mathbb{R}^2) = \langle 1, e_1, e_2, e_1 e_2 \rangle \\ = \{a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_1 e_2 : e_1^2 = e_2^2 = -1, e_1 \cdot e_2 = -e_2 \cdot e_1\}$$

where $(e_1 \cdot e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = (-1)(-1)(-1) = -1$.

So if we define $\psi : Cl(\mathbb{R}^2) \rightarrow \mathbb{H}$ by

$$\psi(1) = 1, \psi(e_1) = i, \psi(e_2) = j, \psi(e_1 e_2) = \psi(e_3) = k$$

then, since ψ is an algebraic homomorphism, $Cl(\mathbb{R}^2) \cong \mathbb{H}$.

There are useful algebraic isomorphisms for $Cl(p, q)$ such as

$$(2.1) \quad \begin{aligned} Cl(n, 0) \otimes Cl(0, 2) &\cong Cl(0, n + 2) \\ Cl(0, n) \otimes Cl(2, 0) &\cong Cl(n + 2, 0) \\ Cl(p, q) \otimes Cl(1, 1) &\cong Cl(p + 1, q + 1), \end{aligned}$$

where $n, p, q \geq 0$ such that $n = p + q$.

Now we introduce a useful tool. Complexification is one of the important tools in linear algebra which make it more flexible. Let (V, h) be a real q-space. The complexification of V is the vector space $W = V \otimes_{\mathbb{C}} \mathbb{C}$ such that for $w \in W : w = v \otimes \lambda = v \otimes (a + ib) = v \otimes a + v \otimes ib = 1 \otimes av + i(1 \otimes bv)$. This means that any element of W can be written as $x + iy$ where $x, y \in V$. Now let g be a nondegenerate q-form on V . Then $g_W : W \times W \rightarrow \mathbb{C}$ is a nondegenerate q-form on $W = V \otimes \mathbb{C}$ defined by $g_W(x \otimes \lambda, y \otimes \gamma) = \lambda \gamma g(x, y)$. From this point of view the complexification of $Cl(V)$ is $Cl(V) \otimes \mathbb{C}$ and if $W = V \otimes_{\mathbb{C}} \mathbb{C}$ then $Cl(W) = Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$.

Lemma 2.3. *Let V be a real n -dimensional vector space, then*

$$Cl(V \oplus \mathbb{R}^2) \otimes \mathbb{C} \cong (Cl(V) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (Cl(\mathbb{R}^2) \otimes \mathbb{C}).$$

Proof. Let $\{\nu_1, \dots, \nu_n\}$ be an orthonormal basis for V and $\{e_1, e_2\}$ be the standard basis for \mathbb{R}^2 . Consider the real map $\theta : V \oplus \mathbb{R}^2 \rightarrow (Cl(V) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (Cl(\mathbb{R}^2) \otimes \mathbb{C})$ defined by

$$(\nu_j, 0) \mapsto i\nu_j \otimes e_1 e_2, \quad 1 \leq j \leq n, \quad (0, e_r) \mapsto 1 \otimes e_r \quad r = 1, 2,$$

so θ extends to algebra homomorphism $Cl(V \oplus \mathbb{R}^2) \otimes \mathbb{C} \cong (Cl(V) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (Cl(\mathbb{R}^2) \otimes \mathbb{C})$. On the other hand domain and range of θ have the same dimension and it is onto, so θ is isometry. □

the following lemma is the key tool for describing the complex Clifford algebras.

Lemma 2.4. *Let V be a real vector space such that $\dim V = 2n$, then $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the matrix algebra $\mathbb{C}[2^n]$. If $\dim V = 2n + 1$ then $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{C}[2^n] \oplus \mathbb{C}[2^n]$.*

Proof. We refer interested reader to [2], for an extended proof. □

2.1. Construction of Clifford Algebra on \mathbb{R}^4 . Now we are going to show that for $V = \mathbb{R}^4$, $Cl(V)$ is $\mathbb{H}[2] \cong \mathbb{C}[4]$. We know that, via the algebraic isomorphism

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + id & b + ic \\ -b + ic & a - id \end{pmatrix},$$

\mathbb{H} is isomorphic to $\mathbb{C}[2]$. Now if $V = \mathbb{R}^4 = \langle e_0, e_1, e_2, e_3 \rangle$ with Riemannian form

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ on it, then}$$

$$Cl(\mathbb{R}^4) = \{a_0 + \sum_{i=1}^4 a_i e_i + \sum_{i < j} a_{ij} e_i e_j + \sum_{i < j < k} a_{ijk} e_i e_j e_k + a_{1234} e_1 e_2 e_3 e_4 : e_i e_j = -e_j e_i, e_i^2 = -1, a_i \in \mathbb{R}\}$$

this means that $Cl(\mathbb{R}^4)$ is spanned by $2^4 = 16$ vectors:

$$1, E_1, E_2, E_3, E_4, E_1 E_2, E_1 E_3, E_1 E_4, E_2 E_3, E_2 E_4, E_3 E_4, \\ E_1 E_2 E_3, E_1 E_2 E_4, E_1 E_3 E_4, E_2 E_3 E_4, E_1 E_2 E_3 E_4,$$

as a basis. On the other hand

$$Cl(0, 2) = Cl(\mathbb{R}^2, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) = \langle e_0, e_1, e_2, e_3 = e_1 e_2 \rangle$$

where $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
such that $e_0^2 = e_1^2 = e_2^2 = 1, (e_1 e_2)^2 = -1$ and

$$Cl(2, 0) = Cl(\mathbb{R}^2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \cong \mathbb{H} = \langle e_0', e_1', e_2', e_3' = e_1' e_2' \rangle$$

where $e_0' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $e_1' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $e_2' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $e_3' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.
Now if in (2.1) we set $n = 2$ then

$$Cl(0, 2) \otimes Cl(2, 0) \cong Cl(4, 0).$$

Through the relation $A \otimes B = (A_{ij} B)$ between matrices we can find the matrix representation for $Cl(4, 0)$'s bases:

$$E_0 = e_0 \otimes e_0' = I, E_1 = e_0 \otimes e_3', E_2 = e_2 \otimes e_1', E_3 = e_1 \otimes e_1', E_4 = \\ e_0 \otimes e_2', E_1 E_2 = e_2 \otimes e_2', E_1 E_3 = e_1 \otimes e_2', E_1 E_4 = -(e_0 \otimes e_1'), E_2 E_3 = \\ e_3 \otimes e_0', E_2 E_4 = e_1 \otimes e_3', E_3 E_4 = e_2 \otimes e_3', E_1 E_2 E_3 = e_3 \otimes e_3', E_1 E_2 E_4 = \\ -(e_2 \otimes e_0'), E_2 E_3 E_4 = e_3 \otimes e_2', E_1 E_3 E_4 = -(e_1 \otimes e_0'), E_1 E_2 E_3 E_4 = -(e_3 \otimes \\ e_1').$$

This means that for any $\rho \in Cl(\mathbb{R}^4)$ we have

$$\rho = a_0 + a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4 + a_{12} E_1 E_2 + a_{13} E_1 E_3 + a_{14} E_1 E_4 + a_{24} E_2 E_4 \\ + a_{34} E_3 E_4 + a_{23} E_2 E_3 + a_{123} E_1 E_2 E_3 + a_{124} E_1 E_2 E_4 \\ + a_{234} E_2 E_3 E_4 + a_{134} E_1 E_3 E_4 + a_{1234} E_1 E_2 E_3 E_4$$

By the above matrix representation for E_i 's, associated matrix to ρ is:

$$\begin{pmatrix} a_0 + a_1 i + a_{34} i^- & a_2 + a_4 i + a_{12} i^- & -a_{24} i - a_{23} - a_{123} i^- & a_3 + a_{13} i a_{234} i^+ \\ a_{124} & a_{14} & a_{134} & a_{1234} \\ -a_2 + a_4 i + a_{12} i^+ & a_0 - a_1 i - a_{34} i^- & -a_3 + a_{13} i - a_{234} i^- & -a_{23} + a_{24} i + a_{123} i^- \\ a_{14} & a_{124} & a_{1234} & a_{134} \\ a_{23} + a_{24} i + a_{123} i^- & a_3 + a_{13} i + a_{234} i^- & a_0 + a_1 i - a_{34} i^+ & -a_2 + a_4 i - a_{12} i^- \\ a_{134} & a_{1234} & a_{124} & a_{14} \\ -a_3 + a_{13} i + a_{234} i^+ & a_{23} - a_{24} i - a_{123} i^- & a_2 + a_4 i - a_{12} i^+ & a_0 - a_1 i + a_{34} i^+ \\ a_{1234} & a_{134} a & a_{14} & a_{124} \end{pmatrix}.$$

Now if we set

$A_1 = a_0 + ia_1, B_1 = -a_{124} + ia_{34}, A_2 = a_2 + ia_4, B_2 = a_{14} + ia_{12}, A_3 = a_{23} + ia_{24}, B_3 = -a_{134} + ia_{123}, A_4 = a_3 + ia_{13}, B_4 = a_{1234} + ia_{234},$
 and then set $A = A_1 + B_1, B = A_1 - B_1, C = A_2 - \overline{B_2}, D = A_2 + \overline{B_2},$
 $E = A_3 + B_3, F = -A_3 + B_3, G = A_4 + \overline{B_4}, H = A_4 - \overline{B_4}, \rho$ can be shown as

$$(2.2) \quad \rho \cong \begin{pmatrix} A & -C & F & -G \\ \overline{C} & \overline{A} & \overline{G} & \overline{F} \\ E & -H & B & -D \\ \overline{H} & \overline{E} & \overline{D} & \overline{B} \end{pmatrix} := M_Q,$$

A simpler representation for ρ is $\rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \lambda \end{pmatrix}$, which is a 2×2 -matrix in \mathbb{H} , with $\alpha = A - j\overline{C}, \beta = F - j\overline{G}, \gamma = E - j\overline{H}, \lambda = B - j\overline{D}.$

Till now we've found the matrix representations for $Cl(\mathbb{R}^4)$ such that $\mathbb{H}[2] \cong \mathbb{C}[4].$ By considering the complexification of $Cl(\mathbb{R}^4)$ we will work with $\mathbb{C}[4],$ which is a more general and flexible case.

Let M_Q be the set of all 4×4 -matrices in $\mathbb{C}[4]$ which are like above then M_Q excepting the zero matrix is a subgroup of $GL(2, \mathbb{C})$ in the sense of matrix multiplication.

In next step we generalize these concepts to an MRA.

3. $Cl(\mathbb{R}^4)$ -VALUED MRA

3.1. General construction and mask functions. Let $L^2(\mathbb{R}, \mathbb{C}[r]) = \{\mathbf{F}(t) = (F_{m,n}(t)) : t \in \mathbb{R}, F_{m,n} \in L^2(\mathbb{R}), 1 \leq m, n \leq r\}$ be the space of matrix-valued functions defined on \mathbb{R} with values in $\mathbb{C}[r].$ The norm on $L^2(\mathbb{R}, \mathbb{C}[r])$ is the Ferobenious norm : $\|\mathbf{F}(t)\| = [\sum_{m,n} \int_{\mathbb{R}} |F_{m,n}(t)|^2 dt]^{\frac{1}{2}}$ and for $\mathbf{F}, \mathbf{G} \in L^2(\mathbb{R}, \mathbb{C}[r]),$ the "inner product" is defined by $\langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbb{R}, \mathbb{C}(r))} := \int_{\mathbb{R}} \mathbf{F}(t) \mathbf{G}^\dagger(t) dt$ where \mathbf{G}^\dagger is the complex conjugate transpose of $\mathbf{G}.$ As pointed out in [7] and [8] such operation, which is an integral of matrix product, is not really an inner product but it has the linear and commutative properties:

1. $\langle \mathbf{F}_1, a\mathbf{F}_2 + b\mathbf{F}_3 \rangle = a^\dagger \langle \mathbf{F}_1, \mathbf{F}_2 \rangle + b^\dagger \langle \mathbf{F}_1, \mathbf{F}_3 \rangle$
2. $\langle \mathbf{F}_1, \mathbf{F}_2 \rangle = \langle \mathbf{F}_2, \mathbf{F}_1 \rangle^\dagger.$

Here the orthogonality of \mathbf{F}_j and \mathbf{F}_k is identified with $\langle \mathbf{F}_j, \mathbf{F}_k \rangle = I_r \delta_{jk}$ where I_r is identity matrix and δ_{jk} the Kronecker delta. Now let $\mathbf{X}(t)$ be a $\mathcal{Cl}(\mathbb{R}^4)$ -valued function. Then $\mathbf{X}(t)$ via its components has a representation like M_Q , as shown in (2.2) and matrix representation of $\mathbf{X}(t)$ is shown with $M_Q(\mathbf{X})$. Define $L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4]) = \{M_Q(\mathbf{X}) : x_{ij} \in L^2(\mathbb{R}), 1 \leq i, j \leq 4\} \subseteq L^2(\mathbb{R}, \mathbb{C}[4])$, and

$$L^2(\mathbb{R}, \mathcal{Cl}(\mathbb{R}^4)) = \{\mathbf{X}(t) = x_0(t) + x_1(t)E_1 + \dots + x_{1234}(t)E_{1234} : x_i \in L^2(\mathbb{R})\},$$

then we can identify $L^2(\mathbb{R}, \mathcal{Cl}(\mathbb{R}^4))$ with $L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4])$ by $T : L^2(\mathbb{R}, \mathcal{Cl}(\mathbb{R}^4)) \rightarrow L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4])$ such that

$$\mathbf{X}(t) \mapsto \begin{pmatrix} x_A & -x_C & x_F & -x_G \\ \bar{x}_C & \bar{x}_A & \bar{x}_G & \bar{x}_F \\ x_E & -x_H & x_B & -x_D \\ \bar{x}_H & \bar{x}_E & \bar{x}_D & \bar{x}_B \end{pmatrix} = M_Q(\mathbf{X}),$$

where $x_A = x_0(t) + ix_1(t) + ix_{34}(t) - x_{124}(t)$ and all other entries are similar to M_Q 's entries.

Immediately we realize that $\langle \mathbf{X}, \mathbf{Y} \rangle_{L^2(\mathbb{R}, \mathcal{Cl}(\mathbb{R}^4))} \mapsto \langle M_Q(\mathbf{X}), M_Q(\mathbf{Y}) \rangle_{L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4])}$,

where $\langle \mathbf{X}, \mathbf{Y} \rangle_{L^2(\mathbb{R}, \mathcal{Cl}(\mathbb{R}^4))} = \int_{\mathbb{R}} \mathbf{X} \mathbf{Y}^\dagger dt$.

Now by considering $\mathcal{Cl}(\mathbb{R}^4) \cong \mathbb{C}[4]$, we will investigate some results in matrix-valued MRAs.

Definition 3.1. The matrix-valued function $\Phi(t) = (\varphi_{m,n}(t))_{r \times r} \in L^2(\mathbb{R}, \mathbb{C}[r])$ generates a matrix-valued multiresolution analysis for $L^2(\mathbb{R}, \mathbb{C}[r])$ if the subspaces $\mathbf{V}_j = \text{span}\{2^{\frac{j}{2}}\Phi(2^j t - k) : k \in \mathbb{Z}\}$ are nested: $\dots \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \mathbf{V}_2 \dots$, and the following conditions hold:

- 1) $\overline{\bigcup_{j \in \mathbb{Z}} \mathbf{V}_j} = L^2(\mathbb{R}, \mathbb{C}[r])$,
- 2) $\bigcap \mathbf{V}_j = 0_r$, in which 0_r is the $r \times r$ -zero matrix.
- 3) $\mathbf{X}(t) \in \mathbf{V}_0 \iff \mathbf{X}(2^j t) \in \mathbf{V}_j, \quad j \in \mathbb{Z}$,
- 4) $\mathbf{X}(t) \in \mathbf{V}_0 \iff \mathbf{X}(t - k) \in \mathbf{V}_0, \quad k \in \mathbb{Z}$,
- 5) $\{\Phi(t - k) : k \in \mathbb{Z}\}$ form an orthonormal basis for \mathbf{V}_0 .

Remark 3.1. : A sequence $\{\Phi_k\}_{k \in \mathbb{Z}}$ in $L^2(\mathbb{R}, \mathbb{C}(r))$ is called an orthonormal basis if it is an orthonormal set, $\langle \Phi_j, \Phi_k \rangle = I_r \delta_{jk}$, and for any $\mathbf{X}(t) \in L^2(\mathbb{R}, \mathbb{C}[r])$ there exists constant matrix-sequence $\{\mathbf{A}_k\}_{k \in \mathbb{Z}}$ such that $\mathbf{X}(t) = \sum_{k \in \mathbb{Z}} \mathbf{A}_k \Phi_k(t)$.

Condition (5) means that $X(t) = \sum_{k \in \mathbb{Z}} \mathbf{A}_k \Phi_k(t - k)$, which Ferobenious norm will guarantee the convergence of infinite sum, and $\mathbf{A}_k = \langle X, \Phi_k(t - k) \rangle$ by orthonormality. Also since $\Phi(t) \in \mathbf{V}_0 \subset \mathbf{V}_1$, then the two-scale matrix dilation equation is

$$(3.1) \quad \Phi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} \mathbf{G}_k \Phi(t - k)$$

which combined with orthonormality of Φ 's means

$$(3.2) \quad \sum_{k \in \mathbb{Z}} \mathbf{G}_k \mathbf{G}_{2l+k}^\dagger = \mathbf{I}_r \delta_{l0}, \quad l \in \mathbb{Z} .$$

Let $\widehat{\mathbf{G}}(f) = \sum_{k \in \mathbb{Z}} \mathbf{G}_k e^{-2\pi i k f}$ be the matrix mask function, then (3.2) implies that

$$(3.3) \quad \widehat{\mathbf{G}}(f) \widehat{\mathbf{G}}^\dagger(f) + \widehat{\mathbf{G}}(f + \frac{1}{2}) \widehat{\mathbf{G}}^\dagger(f + \frac{1}{2}) = 2\mathbf{I}_r,$$

Define matrix Fourier transform for $\Phi(t)$ by $\widehat{\Phi}(f) := \int_{\mathbb{R}} \Phi(t) e^{-2\pi i k f t} dt$. Then (3.1) gives $\widehat{\Phi}(f) = \frac{1}{\sqrt{2}} \widehat{\mathbf{G}}(\frac{f}{2}) \widehat{\phi}(\frac{f}{2})$, where by setting $f = 0$ we get $\widehat{\mathbf{G}}(0) = \sum \mathbf{G}_k = \sqrt{2} \mathbf{I}_r$, $\widehat{\mathbf{G}}(\frac{1}{2}) = 0$. Define the function matrix $\Psi(t) = (\psi_{m,n}(t))_{r \times r} \in L^2(\mathbb{R}, \mathbb{C}[r])$ and corresponding subspace $\mathbf{W}_j = span\{2^{\frac{j}{2}} \Psi(2^j t - k) : k \in \mathbb{Z}\}$. \mathbf{W}_j is orthogonal complement of \mathbf{V}_j in \mathbf{V}_{j+1} i.e. $\mathbf{V}_{j+1} = \mathbf{V}_j \oplus \mathbf{W}_j$, $\mathbf{V}_j \perp \mathbf{W}_j$ and $\bigoplus_{j \in \mathbb{Z}} \mathbf{W}_j = L^2(\mathbb{R}, \mathbb{C}[r])$. Since $\Psi(t) \in \mathbf{W}_0 \subseteq \mathbf{V}_1$, then $\Psi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} \mathbf{H}_k \Phi(2t - k)$. Combining this formula with (3.1) gives us

$$(3.4) \quad \sum_{k \in \mathbb{Z}} \mathbf{G}_k \mathbf{H}_{2l+k}^\dagger = 0_r, \quad l \in \mathbb{Z}.$$

Now if $\widehat{\mathbf{H}}(f) = \sum_{k \in \mathbb{Z}} \mathbf{H}_k e^{-2\pi i k f}$ then

$$(3.5) \quad \widehat{\mathbf{H}}(f) \widehat{\mathbf{G}}^\dagger(f) + \widehat{\mathbf{H}}(f + \frac{1}{2}) \widehat{\mathbf{G}}^\dagger(f + \frac{1}{2}) = 0_r,$$

and $\widehat{\Psi}(f) = \frac{1}{\sqrt{2}} \widehat{H}(\frac{f}{2}) \widehat{\phi}(\frac{f}{2})$. If $\{\Psi(t - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for \mathbf{W}_0 then

$$\langle \Psi, \Psi(t - k) \rangle = \int_{\mathbb{R}} \Psi(t) \Psi(t - k) dt = \mathbf{I}_r \delta_{k0} \quad k \in \mathbb{Z},$$

which implies the following relation for the matrix of wavelet mask function:

$$(3.6) \quad \sum_{k \in \mathbb{Z}} \mathbf{H}_k \mathbf{H}_{2l+k}^\dagger = \mathbf{I}_r \delta_{l0}, \quad l \in \mathbb{Z}.$$

This is equivalent to

$$(3.7) \quad \widehat{\mathbf{H}}(f) \widehat{\mathbf{H}}^\dagger(f) + \widehat{\mathbf{H}}(f + \frac{1}{2}) \widehat{\mathbf{H}}^\dagger(f + \frac{1}{2}) = 2\mathbf{I}_r.$$

Define $\widehat{\mathbf{M}}(f) = \begin{pmatrix} \widehat{\mathbf{G}}(f) & \widehat{\mathbf{G}}(f + \frac{1}{2}) \\ \widehat{\mathbf{H}}(f) & \widehat{\mathbf{H}}(f + \frac{1}{2}) \end{pmatrix}$ then equations (3.3),(3.5),(3.7) all together are equivalent to

$$(3.8) \quad \widehat{\mathbf{M}}(f) \widehat{\mathbf{M}}^\dagger(f) = 2\mathbf{I}_{2r},$$

which means $\widehat{\mathbf{M}}(f)$ is a paraunitary matrix.

3.2. Construction of filters. After constructing the mask function representation, now we are ready to describe and build filters. Suppose that $\widehat{\mathbf{G}}(f)$ is a finite polynomial matrix in $e^{-2\pi if}$, i.e. can be written in the form $\widehat{\mathbf{G}}(f) = \sum_{l=0}^{L'-1} \mathbf{G}_l e^{-2\pi ifl}$ with $\widehat{\mathbf{G}}(0) = \sqrt{2}\mathbf{I}_r$, and satisfies (3.1). Then from [8] if

$$(3.9) \quad \inf_{|f| \leq \frac{1}{4}} |\lambda_l[\widehat{\mathbf{G}}(f)]| > 0$$

for any eigenfunction $\lambda_l[\widehat{\mathbf{G}}(f)]$ of polynomial matrix $\widehat{\mathbf{G}}(f)$, the solution $\Phi(t)$ of the two-scale dilation equation is a matrix-valued scaling function for a matrix-valued MRA, and $\{\Psi_{j,k}(t) = 2^{\frac{j}{2}}\Psi(2^j t - k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for matrix-valued space $L^2(\mathbb{R}, \mathbb{C}[r])$. For designing the matrix filters with transforms $\widehat{\mathbf{G}}(f)$ and $\widehat{\mathbf{H}}(f)$ that satisfies (3.2) and for that $\widehat{\mathbf{M}}(f)$ is paraunitary, we consider

$$(3.10) \quad \widehat{\mathbf{G}}(f) = \frac{e^{2\pi if\gamma}}{\sqrt{2}}(\mathbf{I}_r + e^{\epsilon 2\pi if} \widehat{\mathbf{P}}(2f)), \quad \epsilon \in \{-1, 1\}$$

where γ is a finite integer and $\widehat{\mathbf{P}}(2f)$ is a (normalized) paraunitary matrix, i.e. $\widehat{\mathbf{P}}(f)\widehat{\mathbf{P}}^\dagger(f) = \mathbf{I}_r$ which satisfies $\widehat{\mathbf{P}}(f+1) = \widehat{\mathbf{P}}(f)$, and such that $\widehat{\mathbf{P}}(0) = \mathbf{I}_r$. The matrix $\widehat{\mathbf{G}}(f)$ satisfies conditions (3.1) and (3.2). Notice that the eigenvalues of the polynomial matrix $\widehat{\mathbf{G}}(f)$ are related to the eigenvalues of $\widehat{\mathbf{P}}(2f)$ via $\lambda_l[\widehat{\mathbf{G}}(f)] = \frac{e^{2\pi if\gamma}}{\sqrt{2}}\{1 + e^{\epsilon 2\pi if} \lambda_l[\widehat{\mathbf{P}}(2f)]\}$. Since $\widehat{\mathbf{M}}(f)$ is paraunitary, $\widehat{\mathbf{H}}(f)$ may be chosen as

$$(3.11) \quad \widehat{\mathbf{H}}(f) = e^{-2\pi if(L'-1+\delta)} \widehat{\mathbf{G}}^\dagger\left(f + \frac{1}{2}\right)$$

where L' is the design length of the filter \mathbf{G}_l , and $\delta \in \{0, 1\}$ is chosen so that $L' - 1 + \delta$ is odd, because by 3.5

$$\begin{aligned} & \widehat{\mathbf{H}}(f)\widehat{\mathbf{G}}^\dagger(f) + \widehat{\mathbf{H}}\left(f + \frac{1}{2}\right)\widehat{\mathbf{G}}^\dagger\left(f + \frac{1}{2}\right) \\ &= e^{-2\pi if(L'-1+\delta)}[\widehat{\mathbf{G}}^\dagger\left(f + \frac{1}{2}\right)\widehat{\mathbf{G}}^\dagger(f) + e^{-\pi i(L'-1+\delta)}\widehat{\mathbf{G}}^\dagger(f)\widehat{\mathbf{G}}^\dagger\left(f + \frac{1}{2}\right)] \\ &= e^{-2\pi if(L'-1+\delta)}[\widehat{\mathbf{G}}^\dagger\left(f + \frac{1}{2}\right)\widehat{\mathbf{G}}^\dagger(f) - \widehat{\mathbf{G}}^\dagger(f)\widehat{\mathbf{G}}^\dagger\left(f + \frac{1}{2}\right)] = 0_r, \end{aligned}$$

which provide $\widehat{\mathbf{G}}(f)$ is commutative in the sense that $\widehat{\mathbf{G}}(f)\widehat{\mathbf{G}}\left(f + \frac{1}{2}\right) = \widehat{\mathbf{G}}\left(f + \frac{1}{2}\right)\widehat{\mathbf{G}}(f)$, and indeed this condition holds when $\widehat{\mathbf{G}}(f)$ is defined as in (3.10).

The matrix $\widehat{\mathbf{H}}$ given by (3.11) is a polynomial which can be written in the form

$$\widehat{\mathbf{H}} = \sum_{m=\delta}^{L'-1+\delta} (-1)^{L'-1+\delta-m} \mathbf{G}_{L'-1+\delta-m}^\dagger e^{-2\pi ifm}.$$

If L' is even (and $\delta = 0$), then comparison with $\widehat{\mathbf{H}} = \sum_{l=0}^{L'-1} \mathbf{H}_l e^{-2\pi ifl}$ we obtain $\mathbf{H}_l = (-1)^{l+1} \mathbf{G}_{L'-l-1}^\dagger$ for $l = 0, 1, \dots, L' - 1$ and we set $L = L'$. If L' is odd ($\delta = 1$) we can increase the filter length to an even length $L' + 1$ by setting $\mathbf{G}_{L'} = 0_r$. Then we have $\mathbf{H}_l = (-1)^{l+1} \mathbf{G}_{(L'+1)-l-1}^\dagger$ for $l =$

$0, \dots, L'$, with $\mathbf{H}_0 = 0_r$. In this case we set $L = L' + 1$. For constructing the matrix $\widehat{\mathbf{P}}(f)$ we first consider the class of paraunitary matrices, defined by $\widehat{\mathbf{P}}(f) = \widehat{\mathbf{U}}(f)\widehat{\mathbf{D}}(f)\mathbf{U}^\dagger(f)$, where $\widehat{\mathbf{U}}(f)$ is an arbitrary (normalized) paraunitary polynomial matrix with $\widehat{\mathbf{U}}(0) = \mathbf{I}_r$, and $\widehat{\mathbf{D}}(f)$ is a diagonal matrix with diagonal elements $\widehat{\mathbf{D}}_{l,l} = e^{-2\pi i f k_l}$, $k_l \in \{0, 1\}$. Using the general lattice structure, the $r \times r$ -matrix $\widehat{\mathbf{U}}(f)$ may be constructed by $\widehat{\mathbf{U}}(f) = \widehat{\mathbf{U}}_q(f), \dots, \widehat{\mathbf{U}}_1(f)\mathbf{F}$, where q is a positive integer, \mathbf{F} is an $r \times r$ constant unitary matrix, i.e. $\mathbf{F}^\dagger\mathbf{F} = \mathbf{F}\mathbf{F}^\dagger$, and $\widehat{\mathbf{U}}_l(f) = \mathbf{I}_r + (e^{2\pi i f} - 1)\mathbf{z}_l\mathbf{z}_l^\dagger$ $l = 0, \dots, q$ with $\widehat{\mathbf{z}}_l^\dagger\mathbf{z}_l = 1$, unit-norm constant $r \times 1$ -vectors. The advantage of this construction is that the matrices $\widehat{\mathbf{D}}(f)$ and $\widehat{\mathbf{P}}(f)$ are similar and hence have the same eigenvalues, and those of $\widehat{\mathbf{D}}(f)$ are known. It is thus possible to compute the eigenvalues of $\widehat{\mathbf{G}}(f)$ to check that the sufficient condition (3.9) is satisfied.

4. MAIN RESULTS FOR $\mathbb{C}l(\mathbb{R}^4)$ -MRA

Case I:

Let $r = 4$, by the previous section $\widehat{\mathbf{D}}_{l,l} = e^{-2\pi i k f}$, $k \in \{0, 1\}$, $l = 1, 2, 3, 4$. So we have

$$\widehat{\mathbf{P}}(f) = \widehat{\mathbf{U}}(f)\widehat{\mathbf{D}}(f)\mathbf{U}^\dagger(f)$$

If $\widehat{\mathbf{U}}(f) = \mathbf{I}_4$, $\widehat{\mathbf{U}}$ is a paraunitary polynomial matrix which $\widehat{\mathbf{U}}(0) = \mathbf{I}_4$, so $\widehat{\mathbf{P}}(f) = e^{-2\pi i k f}\mathbf{I}_4$, this gives the diagonal matrix $\widehat{\mathbf{G}}(f) = \frac{e^{2\pi i f \gamma}}{\sqrt{2}}(1 + e^{(\epsilon - 2k)2\pi i f})\mathbf{I}_4$. $\widehat{\mathbf{G}}(f)$ has only one eigenvalue which is repeated and is $\lambda[\widehat{\mathbf{G}}(f)] = \frac{e^{2\pi i f \gamma}}{\sqrt{2}}(1 + e^{(\epsilon - 2k)2\pi i f})$. Now if we set $\epsilon = 1$ we obtain

$$\lambda[\widehat{\mathbf{G}}(f)] = \frac{e^{2\pi i f \gamma}}{\sqrt{2}}(1 + e^{2\pi i f}), (k = 0)$$

$$\lambda[\widehat{\mathbf{G}}(f)] = \frac{e^{2\pi i f \gamma}}{\sqrt{2}}(1 + e^{-2\pi i f}), (k = 1)$$

which in both case the condition $|\lambda[\widehat{\mathbf{G}}(f)]| = \sqrt{1 + \cos 2\pi f} > 0$, for $|f| \leq \frac{1}{4}$, is fullfaith. Hence the sufficient condition (3.9) is satisfied.

If we set $\gamma = 0, \epsilon = 1, k = 1$, then

$$\widehat{\mathbf{G}}(f) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + e^{-2\pi i f} & 0 & 0 & 0 \\ 0 & 1 + e^{-2\pi i f} & 0 & 0 \\ 0 & 0 & 1 + e^{-2\pi i f} & 0 \\ 0 & 0 & 0 & 1 + e^{-2\pi i f} \end{pmatrix}.$$

Let $f = 0$, then $\widehat{\mathbf{G}}(0) = \sqrt{2}\mathbf{I}_4$, $\widehat{\mathbf{G}}(\frac{1}{2}) = 0_4$ and in comparison with $\widehat{\mathbf{G}}(f) = \sum_{l=0}^{L'-1} \mathbf{G}_l e^{-2\pi i f l}$ we have

$$\widehat{\mathbf{G}}(f) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} e^{-2\pi i f}.$$

This means that $\mathbf{G}_0 = \mathbf{G}_1 = \frac{1}{\sqrt{2}}\mathbf{I}_4$ so, $\mathbf{H}_l = (-1)^{l+1}\mathbf{G}_{L-l-1}^\dagger$ for $l = 0, 1$.

Case II:

From now on we consider $\widehat{\mathbf{G}}(f) = \frac{e^{2\pi i f \gamma}}{\sqrt{2}}(\mathbf{I}_4 + e^{2\pi i f} \widehat{\mathbf{P}}(2f))$, we can make $\widehat{\mathbf{P}}(f)$ as

$$\widehat{\mathbf{P}}(f) = \widehat{\mathbf{U}}(f)\widehat{\mathbf{D}}(f)\mathbf{U}^\dagger(f)$$

(for $L_{\mathbf{M}_Q}^2(\mathbb{R}, \mathbb{C}[4])$) we set $\widehat{\mathbf{U}}(f) \in \mathbf{M}_Q \cap \mathbf{U}(4)$.

Set $q = 1$ and $\mathbf{F} = 4 \times 4$ -rotation matrix

$$\mathbf{F} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{bmatrix},$$

(note that $\mathbf{F} \in \mathbf{M}_Q$). Then $\widehat{\mathbf{U}}(f) = \widehat{\mathbf{U}}_1(f)\mathbf{F}$ such that $\widehat{\mathbf{U}}_1(f) = \mathbf{I}_4 + (e^{2\pi i f} - 1)\mathbf{z}_1\mathbf{z}_1^\dagger$.

Now let $\mathbf{z}_1 = \frac{e^{i\theta}}{\alpha}(a, b, c, d)^T$ so $\mathbf{z}_1^\dagger = \frac{e^{-i\theta}}{\alpha}(a, b, c, d)$ such that $\alpha = a^2 + b^2 + c^2 + d^2$.

For instant if $(a, b, c, d) = (0, 0, 0, \alpha)$, $\alpha \in \mathbb{R}$, then $\mathbf{z}_1\mathbf{z}_1^\dagger$ is a 4×4 -matrix with all entiers zero except $e_{4,4} = 1$, so $\mathbf{U}_1(f)$ is the same matrix with $e_{4,4} = e^{2\pi i f}$ and by choosing \mathbf{D} such that $\mathbf{D}_{1,1} = 1, \mathbf{D}_{2,2} = \mathbf{D}_{3,3} = \mathbf{D}_{4,4} = e^{-2\pi i f}$ finally we have:

$$(4.1) \quad \widehat{\mathbf{G}}(f) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos^2 \theta + e^{-2\pi i f} + e^{-4\pi i f} \sin^2 \theta & \sin \theta \cos \theta - e^{-4\pi i f} \sin \theta \cos \theta & 0 & 0 \\ \sin \theta \cos \theta - e^{-4\pi i f} \sin \theta \cos \theta & e^{-2\pi i f} + \sin^2 \theta + e^{-4\pi i f} \cos^2 \theta & 0 & 0 \\ 0 & 0 & 2e^{-2\pi i f} & 0 \\ 0 & 0 & 0 & 2e^{-2\pi i f} \end{pmatrix}.$$

This means that $\mathbf{G}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos^2 \theta & \sin \theta \cos \theta & 0 & 0 \\ \sin \theta \cos \theta & \sin^2 \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\mathbf{G}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$\mathbf{G}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin^2 \theta & \sin \theta \cos \theta & 0 & 0 \\ \sin \theta \cos \theta & \cos^2 \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and since $L' - 1 = 3$ then $L' = 4$ so $\delta = 0$.

Then we set $L = L' = 4$.

Now by $\mathbf{H}_l = (-1)^{l+1}\mathbf{G}_{L-l-1}^\dagger$, ($l = 0, 1, 2, 3$) we have

$$\mathbf{H}_0 = -\mathbf{G}_3^\dagger = 0_4, \mathbf{H}_1 = \mathbf{G}_2^\dagger, \mathbf{H}_2 = -\mathbf{G}_1^\dagger, \mathbf{H}_3 = \mathbf{G}_0^\dagger.$$

So from (3.1) and (3.2) we obtain the desired wavelets.

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