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On generalization of Čebyšev type inequalities

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Abstract. In this paper, we establish new Čebyšev type integral inequalities involving functions whose derivatives belong to L_p spaces via certain integral identities.

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1. Introduction

Let $f, g : [a, b] \to \mathbb{R}$ be two absolutely continuous functions whose derivatives $f', g' \in L_\infty[a, b]$. The Čebyšev functional is defined by:

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.1}$$

and the following inequality (see [3]) holds:

$$|T(f,g)| \le \frac{1}{12}(b-a)^2 \left\|f'\right\|_\infty \left\|g'\right\|_\infty . \tag{1.2}$$

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Many researchers have given considerable attention to (1.2) and a number of extensions, generalizations and variants have appeared in the literature, see ([1],[2],[4] [5] and [6]) and the references given therein.

Pachpatte in [5] established new inequalities of the Čebyšev type:

Theorem 1. Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$ with $f', g' \in L_2[a,b]$, then

$$|P(F,G,f,g)| \leq \frac{(b-a)^2}{12}\left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\|g'\|_2^2 - ([g;a,b])^2\right]^{\frac{1}{2}}$$

and

$$|P(A,B,f,g)| \leq \frac{(b-a)^2}{12}\left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\|g'\|_2^2 - ([g;a,b])^2\right]^{\frac{1}{2}}$$

where

$$P(\alpha,\beta,f,g) = \alpha\beta - \frac{1}{b-a}\left(\alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx\right) + \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \tag{1.3}$$

$$[f;a,b] = \frac{f(b)-f(a)}{b-a},\ F = \frac{f(a)+f(b)}{2},\ G = \frac{g(a)+g(b)}{2},\ A = f\left(\frac{a+b}{2}\right),\ B = g\left(\frac{a+b}{2}\right)$$

and

$$\|f\|_2 = \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}}.$$

In [6] Pachpatte presented an additional Čebyšev type inequality in the following theorem:

Theorem 2. Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a,b]$, $p > 1$, then we have

$$|P(C,D,f,g)| \leq \frac{1}{(b-a)^2}\left(\frac{\left(2^{q+1}+1\right)(b-a)^{q+1}}{3(q+1)6^q}\right)^{\frac{2}{q}}\|f'\|_p\|g'\|_p$$

where $P(\alpha,\beta,f,g)$ is as deÞned in (1.3),

$$C = \frac{1}{3}\left(\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right), \quad D = \frac{1}{3}\left(\frac{g(a)+g(b)}{2} + 2g\left(\frac{a+b}{2}\right)\right)$$

with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty.$$

The main purpose of the present note is to establish inequalities similar to the above inequalities.

2. Main Results

Theorem 3. Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a,b]$, $p > 1$, then we have

$$|P(F,G,f,g)| \le \left(\frac{2(b-a)}{(q+1)(q+2)}\right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p$$

where $P(\alpha, \beta, f, g)$ is as deÞned in (1.3),

$$F = \frac{f(a)+f(b)}{2}, \ G = \frac{g(a)+g(b)}{2}$$

with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

Proof. From the hypothesis of Theorem 3, we have the following identities (see, [1]),

$$F - \frac{1}{b-a}\int_a^b f(t)dt = L(f;a,b) \tag{2.1}$$

$$G - \frac{1}{b-a}\int_a^b g(t)dt = L(g;a,b) \tag{2.2}$$

where

$$L(f;a,b) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t) - f'(s))(t-s)\,dtds.$$

Multiplying the left sides and right sides of (2.1) and (2.2), we write

$$P(F,G,f,g) = L(f;a,b)\,L(g;a,b) \tag{2.3}$$

From (2.3), we get

$$|P(F,G,f,g)| \le |L(f;a,b)|\,|L(g;a,b)|. \tag{2.4}$$

Using the Hölder's integral inequality, we obtain

(2.5)

$$\begin{aligned} |L(f;a,b)| &\le \frac{1}{2(b-a)^2}\int_a^b\int_a^b |f'(t)-f'(s)|\,|t-s|\,dtds \\ &\le \frac{1}{2(b-a)^2}\left(\int_a^b\int_a^b |f'(t)-f'(s)|^p\,dtds\right)^{1/p}\left(\int_a^b\int_a^b |t-s|^q\,dtds\right)^{1/q}. \end{aligned}$$

By simple computation,

$$
\begin{aligned}
\int_a^b \int_a^b |t-s|^q \, dtds &= \int_a^b \left\{ \int_a^s (s-t)^q \, dt + \int_s^b (t-s)^q \, dt \right\} ds \\
&= \frac{1}{q+1} \int_a^b \left\{ -(s-t)^{q+1} \Big|_a^s + (t-s)^{q+1} \Big|_s^b \right\} ds \\
&= \frac{1}{q+1} \int_a^b \left\{ (s-a)^{q+1} + (b-s)^{q+1} \right\} ds \qquad (2.6) \\
&= \frac{1}{(q+1)(q+2)} \left((s-a)^{q+2} - (b-s)^{q+2} \right) \Big|_a^b \\
&= \frac{2(b-a)^{q+2}}{(q+1)(q+2)}
\end{aligned}
$$

and using $(a+b)^r \leq 2^{r-1}(a^r+b^r)$, $r \geq 1$, $a>0$, $b>0$, we have

(2.7)

$$
\begin{aligned}
\int_a^b \int_a^b |f'(t)-f'(s)|^p \, dtds &\leq \int_a^b \int_a^b \left\{ (|f'(t)|+|f'(s)|)^p \right\} dtds \\
&\leq 2^{p-1} \int_a^b \int_a^b \left(|f'(t)|^p + |f'(s)|^p \right) dtds \\
&= 2^{p-1} \left\{ \int_a^b \int_a^b |f'(t)|^p \, dtds + \int_a^b \int_a^b |f'(s)|^p \, dtds \right\} \\
&= 2^p (b-a) \|f'\|_p^p .
\end{aligned}
$$

Using (2.6) and (2.7) in (2.5), we obtain

$$
\begin{aligned}
|L(f;a,b)| &\leq \frac{1}{2(b-a)^2} \left(\frac{2(b-a)^{q+2}}{(q+1)(q+2)} \right)^{1/q} \left(2^p (b-a) \right)^{1/p} \|f'\|_p \qquad (2.8) \\
&= \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{1/q} \|f'\|_p .
\end{aligned}
$$

Similarly, we have

$$
|L(g;a,b)| \leq \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{1/q} \|g'\|_p . \qquad (2.9)
$$

Thus, using (2.8) and (2.9) in (2.4), we obtain

$$
|P(F,G,f,g)| \leq \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{2/q} \|f'\|_p \|g'\|_p .
$$

Theorem 4. Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$, then we have

$$|P(A, B, f, g)| \leq \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p \tag{2.10}$$

where $P(\alpha, \beta, f, g)$ is as deÞned in (1.3),

$$A = f(\frac{a+b}{2}), \; B = g(\frac{a+b}{2})$$

with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\|f\|_p = \left(\int\limits_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Proof. From the hypothesis of Theorem 4, we have the following identities (see, [2])

$$A - \frac{1}{b-a} \int_a^b f(t) dt = M\left(f; a, b\right) \tag{2.11}$$

$$B - \frac{1}{b-a} \int_a^b g(t) dt = M\left(g; a, b\right) \tag{2.12}$$

where

$$M\left(f; a, b\right) = \frac{1}{2\left(b-a\right)^2} \int_a^b \int_a^b \left(f'(t) - f'(s)\right) \left(m\left(t\right) - m\left(s\right)\right) dt ds,$$

and $m\left(t\right)$ involved in the notation $M\left(; a, b\right)$ is given by

$$m\left(t\right) = \begin{cases} t - a, & \text{if } t \in \left(a, \frac{a+b}{2}\right] \\ t - b, & \text{if } t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Multiplying the left sides and right sides of (2.11) and (2.12), we obtain

$$P\left(A, B, f, g\right) = M\left(f; a, b\right) M\left(g; a, b\right)$$

and thus

$$\left|P\left(A, B, f, g\right)\right| = \left|M\left(f; a, b\right)\right| \left|M\left(g; a, b\right)\right|. \tag{2.13}$$

From Hölder's integral inequality, we have

$$\left|M\left(f; a, b\right)\right| \leq \frac{1}{2\left(b-a\right)^2} \left(\int_a^b \int_a^b \left|f\left(t\right) - f\left(s\right)\right|^p dt ds\right)^{1/p} \left(\int_a^b \int_a^b \left|m\left(t\right) - m\left(s\right)\right|^q dt ds\right)^{1/q}. \tag{2.14}$$

On the other hand, we get,

$$\begin{aligned}
&\int_a^b \int_a^b |m(t) - m(s)|^q \, dtds \\
&= \int_a^b \left\{ \int_a^{(a+b)/2} |t - a - m(s)|^q \, dt + \int_{(a+b)/2}^b |t - b - m(s)|^q \, dt \right\} ds \\
&= \int_a^{(a+b)/2} \int_a^{(a+b)/2} |t - s|^q \, dtds + \int_a^{(a+b)/2} \int_{(a+b)/2}^b |t - s + a - b|^q \, dtds \\
&+ \int_{(a+b)/2}^b \int_a^{(a+b)/2} |t - s + b - a|^q \, dtds + \int_{(a+b)/2}^b \int_{(a+b)/2}^b |t - s|^q \, dtds \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.15}$$

Here, by simple computation, we deduce:

$$\begin{aligned}
I_1 &= \int_a^{(a+b)/2} \int_a^{(a+b)/2} |t - s|^q \, dtds \\
&= \int_a^{(a+b)/2} \left\{ \int_a^s (s - t)^q \, dt + \int_s^{(a+b)/2} (t - s)^q \, dt \right\} ds \\
&= \frac{1}{q+1} \int_a^{(a+b)/2} \left\{ -(s - t)^{q+1} \Big|_a^s + (t - s)^{q+1} \Big|_s^{(a+b)/2} \right\} ds \\
&= \frac{1}{q+1} \int_a^{(a+b)/2} \left\{ (s - a)^{q+1} - \left(\frac{a+b}{2} - s \right)^{q+1} \right\} ds \\
&= \frac{1}{(q+1)(q+2)} \left\{ (s - a)^{q+2} - \left(\frac{a+b}{2} - s \right)^{q+2} \right\} \Bigg|_a^{(a+b)/2} \\
&= \frac{2 \left(\frac{b-a}{2} \right)^{q+2}}{(q+1)(q+2)},
\end{aligned} \tag{2.16}$$

$$I_2 = \int_a^{(a+b)/2} \int_{(a+b)/2}^{b} |t - s + a - b|^q \, dtds$$

$$= \int_a^{(a+b)/2} \int_{(a+b)/2}^{b} (s - t + b - a)^q \, dtds$$

$$= \frac{1}{q+1} \int_a^{(a+b)/2} \left\{ -(s - t + b - a)^{q+1} \Big|_{(a+b)/2}^{b} \right\} ds$$

$$= \frac{1}{q+1} \int_a^{(a+b)/2} \left\{ -(s-a)^{q+1} + \left(s + \frac{b-3a}{2} \right)^{q+1} \right\} ds \tag{2.17}$$

$$= \frac{1}{(q+1)(q+2)} \left\{ -(s-a)^{q+2} + \left(s + \frac{b-3a}{2} \right)^{q+2} \right\} \Big|_{a}^{(a+b)/2}$$

$$= \frac{(b-a)^{q+2} - 2\left(\frac{b-a}{2}\right)^{q+2}}{(q+1)(q+2)},$$

(2.18)

$$I_3 = \int_{(a+b)/2}^{b} \int_a^{(a+b)/2} |t - s + b - a|^q \, dtds$$

$$= \int_{(a+b)/2}^{b} \int_a^{(a+b)/2} (t - s + b - a)^q \, dtdsds$$

$$= \frac{1}{q+1} \int_{(a+b)/2}^{b} \left\{ (t - s + b - a)^{q+1} \Big|_{a}^{(a+b)/2} \right\} ds$$

$$= \frac{1}{q+1} \int_{(a+b)/2}^{b} \left\{ \left(-s + \frac{3b-a}{2} \right)^{q+1} (b-s)^{q+1} \right\} ds$$

$$= \frac{1}{(q+1)(q+2)} \left\{ -\left(-s + \frac{3b-a}{2} \right)^{q+2} + (b-s)^{q+2} - \right\} \Big|_{(a+b)/2}^{b}$$

$$= \frac{(b-a)^{q+2} - 2\left(\frac{b-a}{2}\right)^{q+2}}{(q+1)(q+2)}$$

$$I_4 = \int_{(a+b)/2}^{b} \int_{(a+b)/2}^{b} |t - s|^q \, dtds$$

$$= \int_{(a+b)/2}^{b} \left\{ \int_{(a+b)/2}^{s} (s-t)^q \, dt + \int_s^b (t-s)^q \, dt \right\} ds$$

$$= \frac{1}{q+1} \int_{(a+b)/2}^{b} \left\{ -(s-t)^{q+1} \Big|_{(a+b)/2}^{s} + (t-s)^{q+1} \Big|_{s}^{b} \right\} ds$$

$$= \frac{1}{q+1} \int_{(a+b)/2}^{b} \left\{ \left(s - \frac{a+b}{2} \right)^{q+1} + (b-s)^{q+1} \right\} ds$$

$$= \frac{1}{(q+1)(q+2)} \left\{ \left(s - \frac{a+b}{2} \right)^{q+2} - (b-s)^{q+2} \right\} \Big|_{(a+b)/2}^{b} \tag{2.19}$$

$$= \frac{2\left(\frac{b-a}{2}\right)^{q+2}}{(q+1)(q+2)}$$

Using (2.16)-(2.19) in (2.15), we get

$$\left(\int_a^b \int_a^b |m(t) - m(s)|^q \, dtds \right)^{\frac{1}{q}} = \left(\frac{2(b-a)^{q+2}}{(q+1)(q+2)} \right)^{\frac{1}{q}} \tag{2.20}$$

Similar way in (2.7), we have

$$\left(\int_a^b \int_a^b |f'(t) - f'(s)|^p \, dtds \right)^{\frac{1}{p}} \le 2(b-a)^{\frac{1}{p}} \|f'\|_p . \tag{2.21}$$

Using (2.20) and (2.21) in (2.14), it follows that

$$|M(f;a,b)| \le \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{\frac{1}{q}} \|f'\|_p \tag{2.22}$$

and similarly,

$$|M(g;a,b)| \le \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{\frac{1}{q}} \|g'\|_p . \tag{2.23}$$

Using (2.22) and (2.23) in (2.13), we obtain (2.10).

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