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C^*-Algebra Numerical Range of Quadratic Elements

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Abstract. It is shown that the result of Tso-Wu on the elliptical shape of the numerical range of quadratic operators holds also for the C*-algebra numerical range.

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1. Introduction

Let $\mathcal{A}$ be a C*-algebra with unit 1 and let $\mathcal{S}$ be the state space of $\mathcal{A}$, i.e. $\mathcal{S} = \{\varphi \in \mathcal{A}^* : \varphi \geq 0, \varphi(1) = 1\}$. For each $a \in \mathcal{A}$, the C*-algebra numerical range is defined by

$$V(a) := \{\varphi(a) : \varphi \in \mathcal{S}\}.$$

It is well known that $V(a)$ is non empty, compact and convex subset of the complex plane, $V(\alpha 1 + \beta a) = \alpha + \beta V(a)$ for $a \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, and if $z \in V(a)$, $|z| \leq \|a\|$ (For further details see [3]).

As an example, let $\mathcal{A}$ be the C*-algebra of all bounded linear operators on a complex Hilbert space H and $A \in \mathcal{A}$. It is well known that $V(A)$ is the closure of $W(A)$, where

$$W(A) := \{\langle Ax, x\rangle : x \in H, \|x\| = 1\},$$

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is the usual numerical range of the operator T.

In [7] the authors have proved that,

Theorem 1. *Let the operator A be quadratic i.e.;*

$$A^2 - 2\mu A - \lambda I = 0$$

with some μ, $\lambda \in \mathbb{C}$. Then $\overline{W(A)}$ is the elliptical disc with foci $z_{1,2} = \mu \pm \sqrt{\mu^2 + \lambda}$ and the major/minor axis of the length

$$s \pm |\mu^2 + \lambda| s^{-1}.$$

Here $s = \|A - \mu I\|$.

The purpose of this paper is to show that an analogous result holds for quadratic elements of any C*-algebra.

2. MAIN RESULT

Theorem 2. *If $\mathcal{A}$ is a C*-algebra with unity and $a \in \mathcal{A}$ is quadratic i.e.*

$$a^2 - 2\mu a - \lambda 1 = 0$$

with some μ, $\lambda \in \mathbb{C}$. Then $V(a)$ is the elliptical disc with foci $z_{1,2} = \mu \pm \sqrt{\mu^2 + \lambda}$ and the major/minor axis of the length

$$s \pm |\mu^2 + \lambda| s^{-1}.$$

Here $s = \|a - \mu 1\|$.

Proof. Let ρ be a state of $\mathcal{A}$. Then there exists a cyclic representation φ_ρ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\rho$ and a unit cyclic vector x_ρ for $\mathcal{H}_\rho$ such that

$$\rho(a) = \langle \varphi_\rho(a) x_\rho, x_\rho \rangle, \ a \in \mathcal{A}.$$

By Gelfand-Naimark Theorem the direct sum $\varphi : a \mapsto \sum_{\rho \in \mathcal{S}} \oplus \varphi_\rho(a)$ is a faithful representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H} = \sum_{\rho \in \mathcal{S}} \oplus \mathcal{H}_\rho$ (see [5]). Therefore for each $\rho \in \mathcal{S}, \rho(a) \in W(\varphi_\rho(a)) \subset W(\varphi(a))$ and hence $V(a)$ contained in $W(\varphi(a))$. On the other hand if x is a unit vector of $\mathcal{H}$, then the formula $\rho(b) = \langle \varphi(b)x, x \rangle, b \in \mathcal{A}$ defines a state on $\mathcal{A}$ and hence $\rho(a) = \langle \varphi(a)x, x \rangle \in V(a)$ and it follows that

$$W(T_a) = V(a) \tag{1}$$

where $T_a = \varphi(a)$. (see also Theorem 3 of [2]).

But $T_a^2 - 2\mu T_a - \lambda I = \varphi^2(a) - 2\mu\varphi(a) - \lambda\varphi(1) = \varphi(a^2 - 2\mu a - \lambda 1) = \varphi(0) = 0$. Then T_a is quadratic operator. So by Theorem 1, $W(T_a)$ is the elliptical disc with foci at $z_{1,2} = \mu \pm \sqrt{\mu^2 + \lambda}$ and the major/minor axis of the length

$$s \pm |\mu^2 + \lambda| s^{-1}.$$

where $s = \|T_a - \mu I\|$. Since φ is isometry, then $s = \|\varphi(a - \mu 1)\| = \|a - \mu 1\|$. Now the proof is completed by equation (1). $\square$

Corollary 3. *If a is a nontrivial self-inverse element in C^*-algebra $\mathcal{A}$ i.e. $a^2 = 1$, then $V(a)$ is a closed ellipse with foci at ± 1 and major/minor axis* $\|a\| \pm \frac{1}{\|a\|}$

Corollary 4. *If a is a nontrivial nilpotent element with nilpotency 2 i.e. $a^2 = 0$, then $V(a)$ is a closed disc with center at the origin and radius $\frac{\|a\|}{2}$.*

3. Hardy Space

Let $\mathbb{U}$ denote the open unit disc in the complex plane, and the *Hardy space* H^2 the functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ holomorphic in $\mathbb{U}$ such that $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$, with $\widehat{f}(n)$ denoting the n-th Taylor coefficient of f. The inner product inducing the norm of H^2 is given by $< f, g > := \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$. The inner product of two functions f and g in H^2 may also be computed by integration:

$$< f, g > = \frac{1}{2\pi i} \int_{\partial \mathbb{U}} f(z)\overline{g(z)} \frac{dz}{z}$$

where $\partial\mathbb{U}$ is positively oriented and f and g are defined a.e. on $\partial\mathbb{U}$ via radial limits.

For each holomorphic self map φ of $\mathbb{U}$ induces on H^2 a bounded *composition operator* C_φ defined by the equation $C_\varphi f = f \circ \varphi (f \in H^2)$. In fact (see [4])

$$\sqrt{\frac{1}{1 - |\varphi(0)|^2}} \leq \|\varphi\| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}$$

In the case $\varphi(0) \neq 0$ Joel H. Shapiro [9] has been shown that the second inequality changes to equality if and only if φ is an inner function.

A *conformal automorphism* is a univalent holomorphic mapping of $\mathbb{U}$ onto itself. Each such map is linear fractional, and can be represented as a product $w.\alpha_p$, where

$$\alpha_p(z) := \frac{p - z}{1 - \overline{p}z}, (z \in \mathbb{U}),$$

for some fixed $p \in \mathbb{U}$ and $w \in \partial\mathbb{U}$ (See [8]).
The map α_p interchanges the point p and the origin and it is a self-inverse automorphism of $\mathbb{U}$.
Therefore C_{α_p} is a self-inverse composition operator and by corollary 3 $\overline{W(C_{\alpha_p})}$ is an ellipse with foci at ± 1 and major axis $\|C_{\alpha_p}\| + \frac{1}{\|C_{\alpha_p}\|} = \frac{2}{\sqrt{1 - |p|^2}}$.
This is another proof of [1].

4. Dirichlet space

The Dirichlet space, which we denote by $\mathcal{D}$, is the set of all analytic functions f on the unit disc $\mathbb{U}$ for which

$$\int_{\mathbb{U}} |f^{'}(z)|^2 dA(z) < \infty,$$

where dA denote the normalized area measure. Equivalently an analytic function f is in $\mathcal{D}$ if $\sum_{n=1}^{\infty} n|\hat{f}(n)|^2 < \infty$, where $\hat{f}(n)$ denotes the n-th Taylor coefficients of f. The inner product inducing the norm of $\mathcal{D}$ is given by

$$< f, g >_{\mathcal{D}} := f(0)\overline{g(0)} + \int_{\mathbb{U}} f^{'}(z)\overline{g^{'}(z)} dA(z), \ f, g \in \mathcal{D}.$$

The inner product of two functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ in $\mathcal{D}$ may also be computed by

$$< f, g >_{\mathcal{D}} := f(0)\overline{g(0)} + \sum_{n=1}^{\infty} n\hat{f}(n)\overline{\hat{g}(n)}.$$

For each holomorphic self-map φ of $\mathbb{U}$ we define the composition operator C_φ by the equation $C_\varphi f = fo\varphi (f \in \mathcal{D})$. A univalent self-map φ of the unit disc is called a full map if it maps $\mathbb{U}$ onto its subset of full measure, i.e., $A(U \backslash \varphi(U)) = 0$. It is shown in [6] that for any univalent full map φ,

$$\|C_\varphi\| = \sqrt{\frac{L + 2 + \sqrt{L(4+L)}}{2}},$$

where $L = -\log(1 - |\varphi(0)|^2)$.
Thus we have the following:

The $\overline{W(C_{\alpha_p})}$ is ellipse with foci at ±1 and major/minor axis

$$\|C_{\alpha_p}\| \pm \frac{1}{\|C_{\alpha_p}\|} = \frac{L + 2 + \sqrt{L(4+L)} \pm 2}{\sqrt{2L + 4 + 2\sqrt{L(4+L)}}}.$$

It is easy to see that $\overline{W(C_{\alpha_0})} = [-1, 1]$.

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