

**The System of Vector Variational-like Inequalities with  
Weakly Relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  Pseudomonotone Mappings in  
Banach Spaces**

M. Abbasi\* and Kh. Pourbarat

Department of Mathematics, Faculty of Science, University of Kashan, Iran

E-mail: malek.abbasi@yahoo.com

E-mail: pourbara@kashanu.ac.ir

ABSTRACT. In this paper, we introduce two concepts of weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone and demipseudomonotone mappings in Banach spaces. Then we obtain some results of the solutions existence for a system of vector variational-like inequalities with weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone and demipseudomonotone mappings in reflexive Banach spaces. Finally we show that our results improve and extend some corresponding results of Ref [6].

**Keywords:** Variational-like Inequality, Relaxed  $\eta - \alpha$  Pseudomonotone, Relaxed  $\eta - \alpha$  Demipseudomonotone,  $\eta$ -hemicontinuous Mapping.

**2000 Mathematics subject classification:** 32K05.

1. INTRODUCTION AND AN OUTLINE OF THE PREVIOUS WORKS

The first who introduced and studied vector variational inequalities was Giannessi [4] in the setting of finite dimensional Euclidean spaces. Ever since

---

\*Corresponding Author

then, there have been various extensions and generalizations in different directions for Giannessi's work which monotonicity is a central and common concept in all of them. In the recent years many generalizations of monotonicity have been introduced to study various classes of variational inequalities. Briefly, a historical order of these researches are: Chen [2] who introduced the concept of semimonotonicity and applied it in the semimonotone scalar variational inequalities in Banach space, Fang and Huang [3] who introduced a new concept of relaxed  $\eta - \alpha$  monotonicity and obtained some existence theorems of solutions for variational-like inequalities with relaxed  $\eta - \alpha$  monotone mappings in reflexive Banach spaces, Bai, Zhou and Ni [1] who introduced a new concept of relaxed  $\eta - \alpha$  pseudomonotone and obtained some existence of the solutions for variational-like inequalities with relaxed  $\eta - \alpha$  pseudomonotone mappings in reflexive Banach spaces and very recently the works of Wu and Huang [6] (also the authors of this paper [5]) who introduced the new concepts of relaxed  $\eta - \alpha$  pseudomonotone and demipseudomonotone mappings and obtained some existence results for solutions of vector variational-like inequalities with relaxed  $\eta - \alpha$  pseudomonotone and demipseudomonotone mappings by means of KKM technique and Glicksberg fixed point theorem in reflexive Banach spaces.

Inspired and motivated by the works mentioned above, in this paper we introduce two concepts of relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone and demipseudomonotone mappings in Banach spaces. We then obtain some existence results of the solutions for a system of variational-like inequalities with relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone and demipseudomonotone mappings in reflexive Banach spaces. We think this is significant, since only a simple technique is applied to extend and improve some corresponding results of the previous works.

Consider a Banach space  $X$  and a pointed convex closed cone  $P$  with  $\text{int}P \neq \emptyset$ , where  $\text{int}P$  is the interior of  $P$ . We now define

$$\begin{aligned} x \geq y &\iff x - y \in P \\ x \not\geq y &\iff x - y \notin P \\ x > y &\iff x - y \in \text{int}P \\ x \not> y &\iff x - y \notin \text{int}P. \end{aligned}$$

Throughout this section, unless otherwise specified, suppose that  $K$  is a nonempty closed convex subset of  $X$ . Let  $D$  be a Banach space induced by the convex closed cone  $P$  such that  $(D, \leq)$  is an ordered Banach space. Denote the space of all bounded linear operator from  $X$  to  $D$  by  $L(X, D)$

**Definition 1.1.** ([6], [5]) A mapping  $T : K \rightarrow L(X, D)$  is said to be relaxed  $\eta - \alpha$  pseudomonotone if there exist the mappings  $\eta : K \times K \rightarrow X$  and

$\alpha : X \rightarrow D$  with  $\alpha(tz) = t^p\alpha(z)$  for all  $t > 0$  and  $z \in X$  such that

$$(1) \quad \langle Ty, \eta(x, y) \rangle \not\leq 0 \implies \langle Tx, \eta(x, y) \rangle \geq \alpha(x - y)$$

where  $p > 1$  is a constant.

**Definition 1.2.** A mapping  $T : K \rightarrow L(X, D)$  is said to be relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone if there exists a family  $\{\eta_\gamma, \alpha_\gamma\}_{\gamma \in \Gamma}$  of mappings  $\eta_\gamma : K \times K \rightarrow X, \alpha_\gamma : X \rightarrow D$  with  $\alpha_\gamma(tz) = t^p\alpha_\gamma(z)$  for all  $t > 0, \gamma \in \Gamma$  and  $z \in X$  such that

- (i)  $T$  is relaxed  $\eta_\gamma - \alpha_\gamma$  pseudomonotone mapping, for all  $\gamma \in \Gamma$ ;
- (ii)  $d(Co(H_\Gamma(x, y)), \partial P) = 0$  implies  $H_\Gamma(x, y) \cap P \subseteq \partial P$  for all  $x, y \in K$  where  $H_\Gamma(x, y) = \{-\langle Ty, \eta_\gamma(x, y) \rangle; \gamma \in \Gamma\}$ ,  $\partial P$  denotes the boundary of  $P$  and  $d(Co(H_\Gamma(x, y)), \partial P)$  is the distance between the sets  $Co(H_\Gamma(x, y))$  and  $\partial P$ .

where  $p > 1$  is a constant which doesn't depend on the  $\gamma \in \Gamma$ .

**Definition 1.3.** A mapping  $T : K \rightarrow L(X, D)$  is said to be weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone if there exists a family  $\{\eta_\gamma, \alpha_\gamma\}_{\gamma \in \Gamma}$  of mappings  $\eta_\gamma : K \times K \rightarrow X, \alpha_\gamma : X \rightarrow D$  with  $\alpha_\gamma(tz) = t^p\alpha_\gamma(z)$  for all  $t > 0, \gamma \in \Gamma$  and  $z \in X$  such that

- (i)  $T$  is relaxed  $\eta_\gamma - \alpha_\gamma$  pseudomonotone mapping, for all  $\gamma \in \Gamma$ ;
- (ii)  $d(Co(H_\Delta(x, y)), \partial P) = 0$  implies  $H_\Delta(x, y) \cap P \subseteq \partial P$  for all  $x, y \in K, \Delta \in \Sigma$  where  $H_\Delta(x, y) = \{-\langle Ty, \eta_\gamma(x, y) \rangle; \gamma \in \Delta\}$  and  $\Sigma$  denotes a set including all finite subsets of  $\Gamma$ .

Here,  $p > 1$  is a constant which doesn't depend on the  $\gamma \in \Gamma$ .

**Remark 1.1.** Obviously if in Definition 1.3  $\Gamma$  is a finite set, then it coincides with Definition 1.2.

**Example 1.1.** Let  $K = (-\infty, +\infty)$  and  $\Gamma = \{1, 2\}$ . Define  $\eta_1(x, y) = x - y, \eta_2(x, y) = e^x - e^y, \alpha_1(z) = \alpha_2(z) = -\frac{z^2}{2}$  and

$$T(x) = \begin{cases} \frac{3x}{2} & x \geq 0; \\ -\frac{x}{2} & x < 0. \end{cases}$$

Easily we can verify that this mapping is weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma=1,2}$  pseudomonotone.

**Lemma 1.1.** Let the mapping  $T : K \rightarrow L(X, D)$  be relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone. Then there exists a family  $\{\eta_\gamma, \alpha_\gamma\}_{\gamma \in \Gamma}$  of mappings  $\eta_\gamma : K \times K \rightarrow X, \alpha_\gamma : X \rightarrow D$  with  $\alpha_\gamma(tz) = t^p\alpha_\gamma(z)$  for all  $t > 0, \gamma \in \Gamma$  and  $z \in X$  such that

- (i)  $T$  is relaxed  $\eta_\gamma - \alpha_\gamma$  pseudomonotone mapping, for all  $\gamma \in \Gamma$ ;

(ii) if  $\langle Ty, \eta_{\gamma_0}(x, y) \rangle \not\leq 0$ , for some  $\gamma_0 \in \Gamma$  and  $x, y \in K$ , then

$$\langle Ty, \eta_\gamma(x, y) \rangle \not\leq 0 \quad \forall \gamma \in \Gamma,$$

where  $p > 1$  is a constant which doesn't depend on the  $\gamma \in \Gamma$ .

*Proof.* Obviously, the first condition is the same with definition. So we must only verify the second. Suppose on the contrary that the second condition doesn't hold. Hence there exists  $\gamma \in \Gamma$  such that  $s = -\langle Ty, \eta_\gamma(x, y) \rangle \in \text{int}P$ . Let  $t = -\langle Ty, \eta_{\gamma_0}(x, y) \rangle$  and we see that  $t \notin \text{int}P$ . So either  $t \in \partial P$  or  $t \in \text{int}P^c$ . If  $t \in \partial P$  then  $d(\text{Co}(H_\Gamma(x, y)), \partial P) = 0$  and thus by definition  $H_\Gamma(x, y) \cap \text{int}P \subseteq \partial P$ . This implies  $s \in \partial P$  which is impossible. So we must have  $t \in \text{int}P^c$ . We now define the function  $f : [0, 1] \rightarrow \mathbb{R}$  given by

$$f(\lambda) = \begin{cases} d(v(\lambda), \partial P) & v(\lambda) \in \text{int}P \\ -d(v(\lambda), \partial P) & v(\lambda) \in \text{int}P^c \\ 0 & v(\lambda) \in \partial P \end{cases}$$

where  $v(\lambda) = \lambda t + (1 - \lambda)s$ . Easily we can verify that  $f$  is a continuous function. Furthermore  $\partial P$  is closed and thus  $f(0) = d(s, \partial P) > 0$ ,  $f(1) = -d(t, \partial P) < 0$ . Therefore  $f(1) < 0 < f(0)$ . By mean value theorem we may deduce that there exists  $\lambda \in [0, 1]$  such that  $f(\lambda) = 0$  and thus  $v(\lambda) \in \partial P$ . This implies  $d(\text{Co}(H_\Gamma(x, y)), \partial P) = 0$  and by a similar argument as above, this leads to a contradiction. This completes the proof.  $\square$

**Remark 1.2.** Definition 1.2 generalizes Definition 1.1. Indeed, by letting  $\Gamma = \{1\}$  we easily can see that this definition reduce to Definition 1.1.

**Definition 1.4.** ([6], [5]) Let  $T : K \rightarrow L(X, D)$  and  $\eta : K \times K \rightarrow X$  be two mappings.  $T$  is said to be  $\eta$ -hemicontinuous if, for any  $x, y \in K$  the mapping

$$t \mapsto \langle T(x + t(y - x)), \eta(y, x) \rangle$$

is continuous at  $0^+$ .

**Definition 1.5.** Let  $T : K \rightarrow L(X, D)$  be a mapping and  $\{\eta_\gamma\}_{\gamma \in \Gamma}$  be a family of mappings  $\eta_\gamma : K \times K \rightarrow X$ ,  $\gamma \in \Gamma$ .  $T$  is said to be  $\{\eta_\gamma\}_{\gamma \in \Gamma}$ -hemicontinuous if, for any  $\gamma \in \Gamma$ ,  $T$  is  $\eta_\gamma$ -hemicontinuous.

**Remark 1.3.** It is easy to verify that Definition 1.5 generalizes Definition 1.4.

**Definition 1.6.** ([6], [5]) A mapping  $T : K \rightarrow D$  is said to be completely continuous if, for any net  $\{x_\lambda\} \in K$ ,  $x_\lambda \rightharpoonup x_0$  (weakly convergence), then  $Tx_\lambda \rightarrow Tx_0$  in norm.

**Lemma 1.2.** ([6]) Let  $K$  be a nonempty bounded closed convex set and  $T : K \rightarrow L(X, D)$  be  $\eta$ -hemicontinuous and relaxed  $\eta - \alpha$  pseudomonotone. Suppose that

- (i)  $\eta(x, x) = 0, \forall x \in K$ ;
- (ii) for any given points  $y, z \in K$ , the mapping  $x \mapsto \langle Tz, \eta(x, y) \rangle$  is convex and the mapping  $x \mapsto \langle Tz, \eta(y, x) \rangle$  is completely continuous;
- (iii)  $\alpha : X \rightarrow D$  is completely continuous.

Then the following problem is solvable: find  $x \in K$  such that

$$(2) \quad \langle Tx, \eta(y, x) \rangle \not\leq 0 \quad \forall y \in K.$$

**Lemma 1.3.** ([6]) Let  $K$  be a nonempty unbounded closed convex set and  $T : K \rightarrow L(X, D)$  be  $\eta$ -hemicontinuous and relaxed  $\eta - \alpha$  pseudomonotone. Suppose that

- (i) there exist a constant  $r > 0$  and  $y_0 \in K$  with  $\|y_0\| = r$  such that

$$\langle Tz, \eta(x, y_0) \rangle > 0, \quad \forall z \in K \text{ with } \|z\| = r;$$

- (ii)  $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in K$ ;
- (iii) for any given points  $y, z \in K$ , the mapping  $x \mapsto \langle Tz, \eta(x, y) \rangle$  is convex and completely continuous;
- (iv)  $\alpha : X \rightarrow D$  is completely continuous.

Then the problem (2) is solvable.

**Lemma 1.4.** Let  $T : K \rightarrow L(X, D)$  be a relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone mapping, which  $\Gamma$  is a countable set of indexes. Let  $\eta = \sum_{\gamma \in \Gamma} \eta_\gamma$  and  $\alpha = \sum_{\gamma \in \Gamma} \alpha_\gamma$ , with  $\sum_{\gamma \in \Gamma} \|\eta_\gamma\| < \infty, \sum_{\gamma \in \Gamma} \|\alpha_\gamma\| < \infty$ . Then  $T$  is a relaxed  $\eta - \alpha$  pseudomonotone mapping.

*Proof.* First, the conditions  $\sum_{\gamma \in \Gamma} \|\eta_\gamma\| < \infty$  and  $\sum_{\gamma \in \Gamma} \|\alpha_\gamma\| < \infty$  guarantee that  $\eta$  and  $\alpha$  exist. We have

$$\begin{aligned} \alpha(tz) &= \sum_{\gamma \in \Gamma} \alpha_\gamma(tz) \\ &= \sum_{\gamma \in \Gamma} t^p \alpha_\gamma(z) \\ &= t^p \left( \sum_{\gamma} \alpha_\gamma \right) (z) \\ &= t^p \alpha(z). \end{aligned}$$

We now suppose that  $\langle Ty, \eta(x, y) \rangle \not\leq 0$ , for all  $x, y \in K$ . Hence

$$\begin{aligned} &\langle Ty, \sum_{\gamma} \eta_\gamma(x, y) \rangle \not\leq 0 \quad \forall x, y \in K \\ \Rightarrow &\sum_{\gamma} \langle Ty, \eta_\gamma(x, y) \rangle \not\leq 0 \quad \forall x, y \in K \\ \Rightarrow &\exists \gamma_0 \in \Gamma \text{ such that } \langle Ty, \eta_{\gamma_0}(x, y) \rangle \not\leq 0 \quad \forall x, y \in K. \end{aligned}$$

By Lemma 1.1 we know that

$$\langle Ty, \eta_\gamma(x, y) \rangle \not\leq 0, \quad \forall \gamma \in \Gamma, x, y \in K.$$

Now by condition (i) in Definition 1.2 we know that

$$\langle Tx, \eta_\gamma(x, y) \rangle \geq \alpha_\gamma(x - y), \quad \forall \gamma \in \Gamma, x, y \in K.$$

Hence we have

$$\sum_{\gamma} \langle Tx, \eta_\gamma(x, y) \rangle \geq \sum_{\gamma} \alpha_\gamma(x - y).$$

This completes the proof.  $\square$

## 2. THE SYSTEM OF VECTOR VARIATIONAL-LIKE INEQUALITIES WITH WEAKLY RELAXED $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$ PSEUDOMONOTONE MAPPINGS

In this section we suppose that  $K$  is a nonempty closed convex subset of a real reflexive Banach space  $X$  and  $(D, \leq)$  is an ordered Banach space induced by the pointed closed convex cone  $P$  with  $\text{int}P \neq \emptyset$ . We denote the space of all bounded linear operators from  $X$  to  $D$  with  $L(X, D)$ . Now the following system is discussed: find  $x \in K$  such that

$$(3) \quad \langle Tx, \eta_\gamma(y, x) \rangle \not\leq 0, \quad \forall y \in K, \gamma \in \Gamma$$

**Theorem 2.1.** *Let  $K$  be a nonempty bounded closed convex subset of  $X$ . Suppose that  $T : K \rightarrow L(X, D)$  be  $\{\eta_\gamma\}_{\gamma \in \Gamma}$ -hemicontinuous and weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone mapping. Let  $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$  and  $S_n = \{1, 2, \dots, n\}$ . Let  $\eta_n = \sum_{m \in S_n} \eta_{\gamma_m}$  and  $\alpha_n = \sum_{m \in S_n} \alpha_{\gamma_m}$ . Suppose that the following conditions hold*

- (i)  $\eta_n(x, x) = 0, \forall x \in K, n \in \mathbb{N}$ ;
- (ii) for any given points  $y, z \in K$  and each  $\gamma \in \Gamma$ , the mapping  $x \mapsto \langle Tz, \eta_\gamma(x, y) \rangle$  is convex and the mapping  $x \mapsto \langle Tz, \eta_\gamma(y, x) \rangle$  is completely continuous;
- (iii)  $\alpha_\gamma : X \rightarrow D$  is completely continuous, for all  $\gamma \in \Gamma$ .

Then the system (3) has a solution.

*Proof.* Based on Lemma 1.4,  $T$  is relaxed  $\eta_n - \alpha_n$  pseudomonotone, for all  $n \in \mathbb{N}$ . Easily we can verify that, for each  $n$  the mapping  $x \mapsto \langle Tz, \eta_n(x, y) \rangle$  is convex and the mapping  $x \mapsto \langle Tz, \eta_n(y, x) \rangle$  is completely continuous and  $T$  is  $\eta_n$ -hemicontinuous. We can also easily investigate that for each  $n$ ,  $\alpha_n$  is completely continuous. Set

$$R_n = \{x : \langle Tx, \eta_{\gamma_m}(y, x) \rangle \not\leq 0, \forall y \in K, m \in S_n\}$$

For each  $n$ ,  $R_n \neq \emptyset$ . In fact, based on Lemma 1.2, for each  $n$ , there exists a solution, say  $x_0$ , such that

$$\langle Tx_0, \eta_n(y, x_0) \rangle \not\leq 0, \quad \forall y \in K.$$

Thus

$$\begin{aligned} & \langle Tx_0, \sum_{m \in S_n} \eta_{\gamma_m}(y, x_0) \rangle \not\leq 0, \forall y \in K \\ \Rightarrow & \exists u \in S_n \text{ such that } \langle Tx_0, \eta_{\gamma_u}(y, x_0) \rangle \not\leq 0, \forall y \in K. \end{aligned}$$

By hypothesis,  $T$  is weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_\gamma$  pseudomonotone mapping and each  $S_n$  is finite, thus by Lemma 1.1 we must have

$$\langle Tx_0, \eta_{\gamma_m}(y, x_0) \rangle \not\leq 0, \forall y \in K, m \in S_n.$$

This implies each  $R_n$  is not empty. Furthermore  $R_{n+1} \subseteq R_n$  for all  $n$ . Easily we can verify that each  $R_n$  is weakly closed. From this and the conditions of theorem it follows that each  $R_n$  is weakly compact and hence the family  $\{R_n\}$  has finite intersection property. Thus  $\bigcap_{n \in \mathbb{N}} R_n \neq \emptyset$ . Now if  $x_0 \in \bigcap_{n \in \mathbb{N}} R_n$ , then  $x_0$  satisfies the system (3). This completes the proof.  $\square$

**Theorem 2.2.** *Let  $K$  be a nonempty unbounded closed convex subset of  $X$ . Suppose that  $T : K \rightarrow L(X, D)$  be  $\{\eta_\gamma\}_\gamma$ -hemicontinuous and weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_\gamma$  pseudomonotone. Let  $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$  and  $S_n = \{1, 2, \dots, n\}$ . Let  $\eta_n = \sum_{m \in S_n} \eta_{\gamma_m}$  and  $\alpha_n = \sum_{m \in S_n} \alpha_{\gamma_m}$ . Furthermore assume that the following hold*

(i) *there exist a constant  $r > 0$  and  $y_0 \in K$  with  $\|y_0\| \leq r$  such that*

$$\langle Tz, \eta_\gamma(z, y_0) \rangle > 0, \forall z \in K, \gamma \in \Gamma \text{ with } \|z\| = r;$$

- (ii)  $\eta_n(x, y) + \eta_n(y, x) = 0, \forall x, y \in K, n \in \mathbb{N}$ ;  
 (iii) *for any given points  $y, z \in K$ , and each  $\gamma \in \Gamma$  the mapping  $x \mapsto \langle Tz, \eta_\gamma(x, y) \rangle$  is convex and completely continuous;*  
 (iv)  $\alpha_\gamma : X \rightarrow D$  is completely continuous, for all  $\gamma \in \Gamma$ .

*Then the system (3) has a solution.*

*Proof.* From these conditions we know that

- (i) for each  $n$ ,  $T$  is  $\eta_n$ -hemicontinuous and relaxed  $\eta_n - \alpha_n$  pseudomonotone;  
 (ii)  $\langle Tz, \eta_n(z, y_0) \rangle > 0, \forall z \in K, n \in \mathbb{N}$  with  $\|z\| = r$ ;  
 (iii) for any given points  $y, z \in K, n \in \mathbb{N}$  the mapping  $x \mapsto \langle Tz, \eta_n(x, y) \rangle$  is convex and completely continuous;  
 (iv)  $\alpha_n$  is completely continuous, for all  $n$ .

Applying Lemma 1.3 and a similar discussion as the previous theorem we may deduce that the desired system has a solution. This completes the proof.  $\square$

### 3. THE SYSTEM OF VECTOR VARIATIONAL-LIKE INEQUALITIES WITH WEAKLY RELAXED $\{\eta_\gamma - \alpha_\gamma\}_\gamma$ DEMIPSEUDO MONOTONE MAPPINGS

Throughout this section, Let  $X$  be a real reflexive Banach space,  $K \subset X$  be a nonempty closed convex set and  $(D, \leq)$  be an ordered Banach space induced

by the pointed closed convex cone  $P$  with  $\text{int}P \neq \emptyset$ . Denote by  $L(X, D)$  the space of all bounded linear operators from  $X$  to  $D$ . We discuss the following system: find  $x \in K$  such that

$$(4) \quad \langle A(x, x), \eta_\gamma(y, x) \rangle \not\leq 0, \quad \forall y \in K, \gamma \in \Gamma$$

**Definition 3.1.** A mapping  $A : K \times K \rightarrow L(X, D)$  is said to be weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  demipseudomonotone if the following conditions hold

- (i) for any  $u \in K$ , the mapping  $s \mapsto A(u, s)$  is weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  pseudomonotone;
- (ii) for any  $v \in K$  and  $w \in X$  the mapping  $z \mapsto \langle A(z, v), w \rangle$  is completely continuous.

**Lemma 3.1.** ([6]) Let  $K \subset X$  be a nonempty bounded closed convex subset of  $X$  and the mapping  $A : K \times K \rightarrow L(X, D)$  be nonlinear. Suppose that

- (i)  $A$  is relaxed  $\eta - \alpha$  demipseudomonotone;
- (ii) for each  $x \in K$  the mapping  $s \mapsto A(x, s)$  is finite-dimension continuous; i.e., for any finite-dimension subspace  $F \subset X$ , the mapping  $A(x, \cdot) : A \cap F \rightarrow L(X, D)$  is continuous;
- (iii)  $\eta(x, y) + \eta(y, x) = 0$ ;
- (iv) for any given points  $w, y, z \in K$ , the mapping  $x \mapsto \langle A(w, z), \eta(x, y) \rangle$  is convex and the mapping  $x \mapsto \eta(x, y)$  is completely continuous;
- (v)  $\alpha : X \rightarrow D$  is completely continuous.

Then there exists  $x_0 \in K$  such that

$$\langle A(x_0, x_0), \eta(v, x_0) \rangle \not\leq 0, \quad \forall v \in K.$$

**Lemma 3.2.** ([6]) Let  $K \subset X$  be a nonempty unbounded closed convex subset of  $X$  and the mapping  $A : K \times K \rightarrow L(X, D)$  be nonlinear. Suppose that

- (i)  $A$  is relaxed  $\eta - \alpha$  demipseudomonotone;
  - (ii) for each  $x \in K$  the mapping  $s \mapsto A(x, s)$  is finite-dimension continuous;
  - (iii)  $\eta(x, y) + \eta(y, x) = 0$ ;
  - (iv) for any given points  $w, y, z \in K$ , the mapping  $x \mapsto \langle A(w, z), \eta(x, y) \rangle$  is convex and the mapping  $x \mapsto \eta(x, y)$  is completely continuous;
  - (v)  $\alpha : X \rightarrow D$  is convex and completely continuous;
  - (vi) there exist a constant  $r > 0$  and  $y_0 \in K$  with  $\|y_0\| \leq r$  such that
- $$(5) \quad \langle A(z, z), \eta(z, y_0) \rangle > 0, \quad \forall z \in K \text{ with } \|z\| = r.$$

Then there exists  $x_0 \in K$  such that

$$\langle A(x_0, x_0), \eta(z, x_0) \rangle \not\leq 0, \quad \forall z \in K.$$

**Lemma 3.3.** Let  $K \subset X$  be a nonempty closed convex set and the mapping  $A : K \times K \rightarrow L(X, D)$  be weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  demipseudomonotone. Let  $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$  and  $S_n = \{1, 2, \dots, n\}$ . Let  $\eta_n = \sum_{m \in S_n} \eta_{\gamma_m}$  and  $\alpha_n = \sum_{m \in S_n} \alpha_{\gamma_m}$ . Then for each  $n$ ,  $A$  is relaxed  $\eta_n - \alpha_n$  demipseudomonotone.



*Proof.* By Lemma 1.4 we know that, for any  $u \in K$  the mapping  $s \mapsto A(u, s)$  is relaxed  $\eta_n - \alpha_n$  pseudomonotone. This completes the proof.  $\square$

**Theorem 3.1.** *Let  $K \subset X$  be a nonempty bounded closed convex set and the mapping  $A : K \times K \rightarrow L(X, D)$  be nonlinear. Let  $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$  and  $S_n = \{1, 2, \dots, n\}$ . Let  $\eta_n = \sum_{m \in S_n} \eta_{\gamma_m}$  and  $\alpha_n = \sum_{m \in S_n} \alpha_{\gamma_m}$ . Suppose that the following hold*

- (i)  $A$  is weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  demipseudomonotone;
- (ii) for each  $x \in K$  the mapping  $s \mapsto A(x, s)$  is finite-dimension continuous;
- (iii)  $\eta_n(x, y) + \eta_n(y, x) = 0$ , for all  $x, y \in K, n \in \mathbb{N}$ ;
- (iv) for any given points  $w, y, z \in K$  and  $\gamma \in \Gamma$  the mapping

$$x \mapsto \langle A(w, z), \eta_\gamma(x, y) \rangle$$

is convex and the mapping  $x \mapsto \eta_\gamma(x, y)$  is completely continuous;

- (v)  $\alpha_\gamma : X \rightarrow D$  is convex and completely continuous, for each  $\gamma \in \Gamma$ ;

Then the system (4) is solvable.

*Proof.* Based on Lemma 3.3, for each  $n$ ,  $A$  is relaxed  $\eta_n - \alpha_n$  demipseudomonotone. Also for each  $n$  the mapping  $x \mapsto \langle A(w, z), \eta_n(x, y) \rangle$  is completely continuous. Easily we can verify that for each  $n$ ,  $\alpha_n$  is also completely continuous. Thus all conditions of Lemma 3.1 are satisfied and hence for each  $n$  there exists  $x_0 \in K$  such that

$$\langle A(x_0, x_0), \eta_n(v, x_0) \rangle \not\leq 0, \forall v \in K.$$

In other words, for each  $n$ ,  $\langle A(x_0, x_0), \sum_{m \in S_n} \eta_{\gamma_m}(v, x_0) \rangle \not\leq 0$ , for all  $v \in K$ . It follows that there exists  $u \in S_n$  such that

$$\langle A(x_0, x_0), \eta_{\gamma_u}(v, x_0) \rangle \not\leq 0, \forall v \in K.$$

By hypothesis,  $A(x_0, \cdot)$  is weakly relaxed  $\{\eta_{\gamma_0} - \alpha_{\gamma_0}\}_{\gamma \in \Gamma}$  pseudomonotone. Hence we can deduce that

$$\langle A(x_0, x_0), \eta_{\gamma_m}(v, x_0) \rangle \not\leq 0, \forall v \in K, m \in S_n.$$

Set

$$R_n = \{x : \langle A(x, x), \eta_{\gamma_m}(v, x) \rangle \not\leq 0, \forall v \in K, m \in S_n.\}$$

The above argument implies each  $R_n$  is not empty and  $R_{n+1} \subseteq R_n$ . Furthermore from hypothesis we know that each  $R_n$  is weakly compact and thus  $\bigcap_{n \in \mathbb{N}} R_n \neq \emptyset$ . This completes the proof.  $\square$

**Theorem 3.2.** *Let  $K \subset X$  be a nonempty unbounded and closed convex set and the mapping  $A : K \times K \rightarrow L(X, D)$  be nonlinear. Let  $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$  and  $S_n = \{1, 2, \dots, n\}$ . Let  $\eta_n = \sum_{m \in S_n} \eta_{\gamma_m}$  and  $\alpha_n = \sum_{m \in S_n} \alpha_{\gamma_m}$ . Suppose that the following hold*

- (i)  $A$  is weakly relaxed  $\{\eta_\gamma - \alpha_\gamma\}_{\gamma \in \Gamma}$  demipseudomonotone;
- (ii) for each  $x \in K$  the mapping  $s \mapsto A(x, s)$  is finite-dimension continuous;

- (iii)  $\eta_n(x, y) + \eta_n(y, x) = 0$ , for all  $x, y \in K, n \in \mathbb{N}$ ;
- (iv) for any given points  $w, y, z \in K$  and  $\gamma \in \Gamma$  the mapping

$$x \mapsto \langle A(w, z), \eta_\gamma(x, y) \rangle,$$

is convex and the mapping  $x \mapsto \eta_\gamma(x, y)$  is completely continuous;

- (v)  $\alpha_\gamma : X \rightarrow D$  is convex and completely continuous, for each  $\gamma \in \Gamma$ ;
- (vi) there exist a constant  $r > 0$  and  $y_0 \in K$  with  $\|y_0\| \leq r$  such that

$$\langle A(z, z), \eta_\gamma(z, y_0) \rangle > 0, \quad \forall z \in K, \quad \gamma \in \Gamma \text{ with } \|z\| = r.$$

Then the system (4) is solvable.

*Proof.* Easily we can verify that the conditions in Lemma 3.3 are all satisfied. Hence, based on Lemma 3.2, for each  $n$ , there exists  $x_0 \in X$  such that

$$\langle A(x_0, x_0), \eta_n(y, x_0) \rangle \not\leq 0, \quad \forall y \in K, n \in \mathbb{N}.$$

This implies that there exists  $u \in S_n$  such that

$$\langle A(x_0, x_0), \eta_{\gamma_u}(y, x_0) \rangle \not\leq 0, \quad \forall y \in K.$$

By hypothesis, we can deduce that

$$\langle A(x_0, x_0), \eta_{\gamma_m}(y, x_0) \rangle \not\leq 0, \quad \forall y \in K, m \in S_n.$$

Again, for each  $n$ , we can define a set  $R_n = \{x : \langle A(x, x), \eta_{\gamma_m}(y, x) \rangle \not\leq 0, \forall y \in K, m \in S_n\}$  and then conclude that each  $R_n \neq \emptyset$  and  $R_{n+1} \subseteq R_n$ . Now since each  $R_n$  is weakly compact, thus  $\bigcap_{n \in \mathbb{N}} R_n \neq \emptyset$ . This completes the proof.  $\square$

#### 4. APPLICATIONS

As we have already claimed, the results appeared in this paper improve the results of many corresponding authors. In fact, we can show that each of Theorems 2.2, 3.1 and 3.2 are extensions of Lemma 1.3, 1.4 and 3.1, respectively. These Lemmas are indeed, the main results of the [6].

In what follows, we briefly show that for instance Theorem 2.2 improves Lemma 1.3. Toward this end, suppose that the all conditions in Lemma 1.3 hold. Let  $\Gamma = \{1, 2\}$ ,  $\eta_1(x, y) = \eta(x, y)$ ,  $\eta_2(x, y) = -\eta(y, x)$  and  $\alpha_1 = \alpha_2 = \alpha$ . We now verify the conditions of Theorem 2.2. In fact by the first condition of this lemma there exists  $y_0 \in K$  and  $r > 0$  such that

$$\langle Tz, \eta(z, y_0) \rangle > 0,$$

for all  $z \in K$  with  $\|z\| = r$ . Thus

$$\langle Tz, \eta_1(z, y_0) \rangle > 0,$$

for all  $z \in K$  with  $\|z\| = r$ . Based on the second condition of this lemma  $\eta_1(z, y_0) = -\eta_1(y_0, z)$  and hence  $\eta_2(z, y_0) = \eta_1(z, y_0)$ . It follows that

$$\langle Tz, \eta_2(z, y_0) \rangle > 0.$$

Therefore the first condition of Theorem 2.2 is satisfied. Let

$$\tilde{\eta} = \eta_1 + \eta_2.$$

Easily we can deduce that

$$\tilde{\eta}(x, y) = \eta(x, y) - \eta(y, x).$$

Whence

$$\tilde{\eta}(x, y) + \tilde{\eta}(y, x) = 0,$$

for all  $x, y \in K$ . Therefore the second condition of the theorem is satisfied. On the other hand, the mapping  $x \mapsto \langle Tz, \eta(x, y) \rangle$  is convex. In other words the mapping  $x \mapsto \langle Tz, \eta_1(x, y) \rangle$  is convex. Applying the second condition of the lemma we can deduce that the mapping  $x \mapsto \langle Tz, \eta_2(x, y) \rangle$  is also convex. Furthermore, with the help of a similar argument we can deduce that these two mappings are completely continuous and hence the second condition of the theorem holds. Since  $\alpha_1 = \alpha_2 = \alpha$  we see that the last condition of the theorem equals with the last condition of the lemma. Furthermore,  $T$  is both relaxed  $\{\eta_\gamma - \alpha_\gamma\}_\gamma$  pseudomonotone and  $\{\eta_\gamma\}_\gamma$ -hemicontinuous, where  $\eta_\gamma$ 's,  $\alpha_\gamma$ 's and the set  $\Gamma$  have been given above. Therefore based on this theorem, there exists a solution, say  $x_0$ , such that

$$\langle Tx_0, \eta_1(y, x_0) \rangle \not\leq 0,$$

for all  $y \in K$  and hence  $\langle Tx_0, \eta(y, x_0) \rangle \not\leq 0$ , for all  $y \in K$ , as desired.

**Acknowledgement.** I appreciate the referees for their valuable suggestions, improving the structure and increasing the beauty of the paper.

#### REFERENCES

- [1] M. R. Bai, S. Z. Zhou and G. Y. Ni, Variational-like inequalities with relaxed  $\eta - \alpha$  pseudomonotone mappings in Banach spaces, *Appl. Math. Lett.*, **19** (2006), 547-554.
- [2] Y. Q. Chen, On the semimonotone operator theory and applications, *J. Math. Anal. Appl.*, **231** (1999), 177-192.
- [3] Y. P. Fang and N. J. Huang, The vector  $F$ -complementary problems with demipseudomonotone mappings Banach spaces, *J. Optim. Theory Appl.*, **118** (2003), 327-338.
- [4] F. Giannessi, *Theorems of alternative, quadratic programs and complementarity problems*, in: R. W. Cottle, F. Giannessi AND J. L. Lions (Ed.), *Variational Inequalities and Complementarity problems*, pp. 151-186, Wiley, New York, 1980.
- [5] Kh. Pourbarat and M. Abbasi, On the Vector Variational-like Inequalities with Relaxed  $\eta - \alpha$  Pseudomonotone Mappings, *Iranian Journal of Mathematical Sciences and Informatics*, **1** (2009), 37-42.
- [6] K. Q. Wu and N. J. Huang, Vector variational-like inequalities with relaxed  $\eta - \alpha$  Pseudomonotone mappings in Banach Spaces, *J. Math. Inequal.*, **1** (2007), 281-290.