

Quotient BCI-algebras induced by pseudo-valuations

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ABSTRACT. In this paper, we study pseudo-valuations on a BCI-algebra and obtain some related results. The relation between pseudo-valuations and ideals is investigated. We use a pseudo-metric induced by a pseudo-valuation to introduce a congruence relation on a BCI-algebra. We define the quotient algebra induced by this relation and prove that it is also a BCI-algebra and study its properties.

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1. INTRODUCTION

The notions of BCK and BCI-algebras were introduced by Imai and Iseki in [7, 8]. They are two important classes of logical algebras. BCI-algebras are generalization of BCK-algebras. Some properties of these structures were presented in [1, 4, 6, 10, 11, 12] and [13]. Recently, D. Busneag [2, 3] introduced the notion of a pseudo valuation and applied it to Hilbert-algebras and residuated lattices. Also, M. I. Doh and M. S. Kang [5] applied pseudo valuations

to BCK/BCI algebras and investigate some properties.

In the next section, some preliminary definitions and theorems are stated. In section 3, we study pseudo-valuation on BCI-algebras and investigate its properties which is not in [5]. We discuss the relation among pseudo-valuations and ideals of a BCI-algebra. We obtain some results of pseudo-metrics induced by pseudo-valuations on BCI-algebras and prove that a pseudo-metric induced by a pseudo-valuation v is a metric on a BCK-algebra if and only if v is a valuation but it may not be true in general for a BCI-algebra. In section 4, we use pseudo-metric induced by a pseudo-valuation to define the quotient algebra. We prove that this quotient algebra is also a BCI-algebra and obtain some related results.

2. PRELIMINARIES

Definition 2.1.[11] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCI-algebra*, if it satisfies the following conditions: for any $x, y, z \in X$:

$$(BCI\ 1) ((x * y) * (x * z)) * (z * y) = 0,$$

$$(BCI\ 2) x * 0 = x,$$

$$(BCI\ 3) x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y.$$

We call the binary operation $*$ on X the *multiplication* on X and the constant of X the *zero element* of X . We often write X instead of $X = (X, *, 0)$ for a BCI-algebra in brevity.

Theorem 2.2.[11] Let X be a BCI-algebra. Define a binary relation \leq on X by which $x \leq y$ if and only if $x * y = 0$ for any $x, y \in X$. Then (X, \leq) is a partially ordered set with 0 is a minimal element in the meaning that $x \leq 0$ implies $x = 0$.

A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$ is called a *BCK-algebra*. [10] The set of all positive elements of a BCI-algebra X is called the *BCK-part* of X and is denoted by $B(X)$.

Theorem 2.3.[10, 11] Let x, y, z be any elements in a BCI-algebra X . Then

- (1) $x \leq y$ implies $z * y \leq z * x$,
- (2) $x \leq y$ implies $x * z \leq y * z$,
- (3) $x * y \leq z$ if and only if $x * z \leq y$,
- (4) $x * (x * y) \leq y$,
- (5) $(x * y) * (z * y) \leq (x * z)$,
- (6) $(x * y) * (x * z) \leq (z * y)$,
- (7) $(x * y) * z = (x * z) * y$,
- (8) $x \leq x$,
- (9) $0 * (x * y) = (0 * x) * (0 * y)$.

A subset Y of a BCI-algebra X is called a *subalgebra* of X if constant 0 of X is in Y , and $(Y, *, 0)$ itself forms a BCI-algebra. $B(X)$ is a subalgebra of a BCI-algebra X .

Definition 2.4.[11] A subset I of a BCI-algebra X is called an *ideal* of X if

- (1) $0 \in I$,
- (2) $y \in I, x * y \in I$ imply $x \in I$ for any $x, y \in X$.

Any ideal I has the property: $y \in I$ and $x \leq y$ imply $x \in I$.

Definition 2.5.[11] An ideal I of a BCI-algebra X is called *closed* if I is closed under $*$ on X (i.e, I is a subalgebra of X).

Proposition 2.6.[11] An ideal I of a BCI-algebra X is closed if and only if $0 * x \in I$ for any $x \in I$.

Proposition 2.7.[11] Let X be a BCI-algebra. Then

- (i) If an ideal of X is a finite order, then it is closed, especially, if X is a finite order, then any ideal of X is closed.
- (ii) If X is a BCK-algebra, then any ideal of X is closed.

Definition 2.8.[11] Let X and Y be BCI-algebras. A map $f : X \rightarrow Y$ is called *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

f is called *epimorphism*, if it is a surjective homomorphism. f is called *monomorphism*, if it is a injective homomorphism. An *isomorphism* means that f is both of epimorphic and monomorphic. Moreover, we say X is *isomorphic* to Y , symbolically, $X \cong Y$, if there is an isomorphism from X to Y . For a homomorphism $f : X \rightarrow Y$, we have $f(0) = \acute{0}$ where 0 and $\acute{0}$ are zero elements of X and Y , respectively.

Definition 2.9.[11] An equivalence relation θ on a BCI-algebra X is called a *congruence relation* on X , if $(x, y) \in \theta$ implies $(x * z, y * z) \in \theta$ and $(z * x, z * y) \in \theta$ for all $x, y, z \in X$.

Theorem 2.10.[11] Let I be an ideal of a BCI-algebra X . Define a binary relation θ_I on X as follows: $(x, y) \in \theta_I$ if and only if $x * y, y * x \in I$, for all $x, y \in X$. Then θ_I is a congruence relation on X which is called the *ideal congruence* on X induced by the ideal I .

Theorem 2.11.[11] Let I be an ideal of a BCI-algebra X and θ_I be the ideal congruence relation. The set of all equivalence classes $[x]_I = \{y \in X : (x, y) \in \theta_I\}$ is denoted by X/I . On this set, we define $[x]_I * [y]_I = [x * y]_I$. Then

$(X/I, *, [0]_I)$ is a BCI-algebra.

3. PSEUDO-VALUATIONS ON BCI-ALGEBRAS

Definition 3.1.[4] A real function $v : X \rightarrow \mathfrak{R}$ is called a *pseudo-valuation* on a BCI-algebra X if it satisfies the following conditions:

(V1) $v(0) = 0$,

(V2) $v(x) \leq v(x * y) + v(y)$; for all $x, y \in X$.

The pseudo-valuation v is said to be a *valuation* if

(V3) $v(x) = 0$ implies $x = 0$.

Example 3.2.(i) Let X be an arbitrary BCI-algebra and $c \in \mathfrak{R}$ such that $c \geq 0$. Define $v : X \rightarrow \mathfrak{R}$ by $v(x) = c$ for all $x \in X - \{0\}$, and $v(0) = 0$. Then v is a pseudo-valuation on X . If $c = 0$, then v is called *zero pseudo-valuation*.
(ii) The set Z of integer, together with the binary operation $*$ defined by $x * y = x - y$ forms a BCI-algebra, where the operation $-$ is the subtraction as usual. Let $a \neq 0$ be an arbitrary element of Z . Then $v(x) = ax$ is a valuation on Z .

Theorem 3.3. Let v be a pseudo-valuation on a BCI-algebra X . Then

(1) $x \leq y$ implies $v(x) \leq v(y)$,

(2) $v(x * y) \leq v(x * z) + v(z * y)$,

(3) $0 \leq v(x * y) + v(y * x)$,

for all $x, y, z \in X$.

Proof. See Proposition 3.11 in [4]. □

Corollary 3.4. Let v be a pseudo-valuation on a BCI-algebra X . If $x \in B(X)$, then $v(x) \geq 0$.

Proof. Since $x \in B(X)$, then $0 \leq x$. By Theorem 3.3 part (1), we get that $0 = v(0) \leq v(x)$. □

In the following example, we will show that if v is a pseudo-valuation on a BCI-algebra X such that $v(x) \geq 0$ where $x \in X$, then it may not be true $x \in B(X)$ in general.

Example 3.5. Let X be a BCI-algebra with the universe $\{0, 1, a\}$ such that the operation $*$ is defined by the table below:

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

Define $v(0) = 0$, $v(1) = 3$ and $v(a) = 6$. Then v is a pseudo-valuation on X and $v(a) \geq 0$. But we have $a \notin B(X)$.

Theorem 3.6. Let I be an ideal of a BCI-algebra X and t be a positive element of \mathfrak{R} . Define $v_I : X \rightarrow \mathfrak{R}$,

$$v_I(x) = \begin{cases} 0 & x \in I \\ t & x \notin I \end{cases}$$

Then v_I is a pseudo-valuation on X which is called the *pseudo-valuation induced by ideal I* . Moreover v_I is a valuation if and only if $I = \{0\}$.

Proof. The proof is straightforward. □

Theorem 3.7. Let v be a pseudo-valuation on a BCI-algebra X . Then $I_v = \{x \in X : v(x) \leq 0\}$ is an ideal of X which is called the *ideal induced by pseudo-valuation v* .

Proof. Since $v(0) = 0$, we have $0 \in I_v$. Suppose that $y, x * y \in I_v$. Then $v(y), v(x * y) \leq 0$. We get that

$$v(x) \leq v(x * y) + v(y) \leq 0$$

Therefore $x \in I_v$ and I_v is an ideal of X . □

Corollary 3.8. Let v be a pseudo-valuation on a BCI-algebra X . If X is finite order or $X = B(X)$, then I_v is a closed ideal of X .

Proof. It follows from Theorem 3.7 and Proposition 2.7. □

Remark 3.9. The ideal induced by a pseudo-valuation v on a BCI-algebra X may not be closed. Consider Example 3.2 part (ii). If $v(x) = x$, for all $x \in Z$, then I_v is the set of negative integer which is not a closed ideal of Z .

Theorem 3.10. Let I be an ideal of a BCI-algebra X . Then $I_{v_I} = I$.

Proof. We have $I_{v_I} = \{x \in X : v_I(x) \leq 0\} = \{x \in X : x \in I\} = I$. □

Remark 3.11. The above Theorems do not furnish a one to one correspondence between ideals and pseudo-valuations, because two distinct pseudo-valuations of a given BCI-algebra may induce the same ideal. Consider the following example:

Example 3.12. Let X be a BCI-algebra with the universe $\{0, 1, 2, a, b\}$ such that the operation $*$ is defined by the table below:

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	b	a
a	a	a	a	0	0
b	b	b	a	2	0

Define $v_1(0) = v_1(1) = 0$, $v_1(2) = 4$, $v_1(a) = 3$, $v_1(b) = 5$ and $v_2(0) = v_2(1) = 0$, $v_2(2) = 4$, $v_2(a) = 2$, $v_2(b) = 3$. Then v_1 and v_2 are two pseudo-valuations on X such that $I_{v_1} = \{0, 1\} = I_{v_2}$.

Theorem 3.13. Let v be a pseudo-valuation on a BCI-algebra X . Define $d_v : X \times X \rightarrow \mathbb{R}$ by

$$d_v(x, y) = v(x * y) + v(y * x),$$

for $(x, y) \in X \times X$. Then d_v is a pseudo-metric on X which is called the *pseudo-metric induced by pseudo-valuation v* .

Proof. See Theorem 3.6 in [4]. □

Theorem 3.14. Let v be a pseudo-valuation on a BCI-algebra X such that I_v is a closed ideal of X . If d_v is a metric on X , then v is a valuation.

Proof. Suppose that v is not a valuation on X . Then there exists $x \in X$ such that $x \neq 0$ and $v(x) = 0$. Hence $0, x \in I_v$. Since I_v is a closed ideal of X , then $0 * x \in I_v$, that is $v(0 * x) \leq 0$. We have

$$0 = v(0) \leq v(0 * x) + v(x) = v(0 * x) \leq 0.$$

Hence $v(0 * x) = 0$. We get that $d_v(x, 0) = v(x * 0) + v(0 * x) = 0$. Since d_v is a metric on X , then $x = 0$ which is a contradiction. □

If I_v is not a closed ideal of X , then the above theorem may not be true. See the following example:

Example 3.15. Consider the set Z of integer, together with the binary operation $*$ defined by $x * y = x - y$. Let $a > 0$ be an arbitrary element of Z . Define $v_a(x) = a - x$, where $x \in Z - \{0\}$ and $v_a(0) = 0$. Then v_a is a pseudo-valuation

on a BCI-algebra Z , d_v is a metric space and $I_v = \{x \in X : a \leq x\} \cup \{0\}$ is not a closed ideal of Z . Since $v_a(a) = 0$, then v_a is not a valuation.

Theorem 3.16. Let v be a valuation on a BCI-algebra X such that $I_v = \{0\}$. Then d_v is a metric on X .

Proof. Since $I_v = \{0\}$, then $v(x) \geq 0$ for all $x \in X$. Hence d_v is a metric on X by Theorem 3.20 in [4]. \square

If $I_v \neq \{0\}$, then the above theorem may not be true. Consider v in Remark 3.9. Then $I_v \neq \{0\}$ and $d_v(0, 1) = 0$. Hence d_v is not a metric on X .

Corollary 3.17. Let v be a pseudo-valuation on a BCK-algebra X . Then v is a valuation if and only if d_v is a metric on X .

Proof. Since v is a valuation and X is a BCK-algebra, then $I_v = \{0\}$. By Theorem 3.16, d_v is a metric on X . Converse follows from Theorem 3.14 and Proposition 2.7. \square

Lemma 3.18. Let v be pseudo-valuation on a BCI-algebra X . Then

- (1) $d_v(x * z, y * z) \leq d_v(x, y)$,
 - (2) $d_v(z * x, z * y) \leq d_v(x, y)$,
 - (3) $d_v(x * y, z * w) \leq d_v(x * y, z * y) + d_v(z * y, z * w)$,
- for all $x, y, z, w \in X$.

Proof. See Proposition 3.17 in [4]. \square

4. QUOTIENT BCI-ALGEBRAS INDUCED BY PSEUDO VALUATIONS

Definition 4.1. Let v be a pseudo-valuation on a BCI-algebra X . Define the relation θ_v by:

$$(x, y) \in \theta_v \quad \text{if and only if} \quad d_v(x, y) = 0,$$

for all $x, y \in X$.

Proposition 4.2. Let v be a pseudo-valuation on a BCI-algebra X . Then θ_v is a congruence relation on X which is called the *congruence relation induced* by v .

Proof. Since θ_v induced by a pseudo-metric, it is an equivalence relation on X . Suppose that $(x, y), (z, w) \in \theta_v$. Then we have $d_v(x, y) = d_v(z, w) = 0$. By

Lemma 3.18 part (1), we have $d_v(x * z, y * z) \leq d_v(x, y) = 0$. By Theorem 3.3 part (3), we obtain that $0 \leq v((x * z) * (y * z)) + v((y * z) * (x * z)) = d_v(x * z, y * z)$. Hence $d_v(x * z, y * z) = 0$ and then $(x * z, y * z) \in \theta_v$. Similar proof gives $(y * z, y * w) \in \theta_v$. Since θ_v is transitive, then $(x * z, y * w) \in \theta_v$. Hence θ_v is a congruence relation on X . \square

Definition 4.3. Let v be a pseudo-valuation on a BCI-algebra X and θ_v be the congruence relation induced by v . The set of all equivalence classes $[x]_v = \{y \in A : (x, y) \in \theta_v\}$ is denoted by X/v . On this set, we define $[x]_v * [y]_v = [x * y]_v$. The resulting algebra is denoted by X/v and is called the *quotient algebra of X induced by pseudo-valuation v* .

Theorem 4.4. Let v be a pseudo-valuation on a BCI-algebra X . Then $(X/v, *, [0]_v)$ is a BCI-algebra and $d^*([x]_v, [y]_v) = d(x, y)$ is a metric on X/v . Moreover, the quotient topology on X/v coincide with the metric topology induced by d^* .

Proof. Since θ_v is a congruence relation, the operation $*$ is well defined. The proof of (BCI 1) and (BCI 2) is obvious. We only prove (BCI 3). Suppose that $[x]_v * [y]_v = [0]_v$ and $[y]_v * [x]_v = [0]_v$ for some $x, y \in X$. Then $[x * y]_v = [0]_v$ and $[y * x]_v = [0]_v$ by Definition 4.3. So $(x * y, 0), (y * x, 0) \in \theta_v$. By definition of θ_v , the following hold

$$v(x * y) + v(0 * (x * y)) = 0 \quad \text{and} \quad v(y * x) + v(0 * (y * x)) = 0.$$

By Theorem 2.3 part (9), we have $(0 * x) * (0 * y) = 0 * (x * y)$ and $(0 * y) * (0 * x) = 0 * (y * x)$. Since v is a pseudo-valuation and order preserving, we obtain that

$$\begin{aligned} v(0 * x) - v(0 * y) &\leq v((0 * x) * (0 * y)) = v(0 * (x * y)), \\ v(0 * y) - v(0 * x) &\leq v((0 * y) * (0 * x)) = v(0 * (y * x)). \end{aligned}$$

We get that

$$\begin{aligned} v(0 * x) - v(0 * y) + v(x * y) &\leq v(0 * (x * y)) + v(x * y) = 0, \\ v(0 * y) - v(0 * x) + v(y * x) &\leq v(0 * (y * x)) + v(y * x) = 0. \end{aligned}$$

Therefore $v(x * y) + v(y * x) \leq 0$. By Theorem 3.3 part (3), $v(x * y) + v(y * x) = 0$. It follows that $(x, y) \in \theta_v$, that is $[x]_v = [y]_v$. Hence $(X/v, *, [0]_v)$ is a BCI-algebra. \square

Proposition 4.5. Let v be a pseudo-valuation on a BCI-algebra X such that I_v is a closed ideal of X . Then $I_v \subseteq [0]_v$.

Proof. Let $x \in I_v$. Then $v(x) \leq 0$. Since I_v is a closed ideal of X , then $0 * x \in I_v$. By definition I_v , $v(0 * x) \leq 0$. We get that $v(0 * x) + v(x) \leq 0$. By Theorem 3.3 part (3), $v(0 * x) + v(x) = 0$. Hence $x \in [0]_v$. \square

If I_v is not a closed ideal of X , then the above theorem may not be true in general. For example, we have $I_v \not\subseteq [0]_v$ in Remark 3.9.

Proposition 4.6. Let v be a pseudo-valuation on a BCI-algebra X such that $v(x) \geq 0$ for all $x \in X$. Then $[0]_v \subseteq I_v$.

Proof. Let $x \in [0]_v$. Then $(0, x) \in \theta_v$. By definition θ_v , we have $v(0 * x) + v(x) = 0$. Since $v(x) \geq 0$ for all $x \in X$, we obtain $v(0 * x) = v(x) = 0$. Hence $x \in I_v$ by definition I_v . \square

If we do not have $v(x) \geq 0$ for all $x \in X$, then the above theorem may not be true. Consider Example 3.15, we have $I_v \not\subseteq [0]_v$.

Corollary 4.7. Let v be a pseudo-valuation on a BCI-algebra X such that $v(x) \geq 0$ for all $x \in X$ and I_v is a closed ideal of X . Then $I_v = [0]_v$.

Proof. It follows from Proposition 4.5 and Proposition 4.6. \square

Proposition 4.8. Let v be a pseudo-valuation on a BCI-algebra X and I_v be the ideal induced by v . Then $\theta_{I_v} \subseteq \theta_v$.

Proof. Let $(x, y) \in \theta_{I_v}$. Then $x * y, y * x \in I_v$. We have $v(x * y) \leq 0$ and $v(y * x) \leq 0$, by definition I_v . Thus $v(x * y) + v(y * x) \leq 0$. By Theorem 3.3 part (3), $v(x * y) + v(y * x) = 0$. It follows that $(x, y) \in \theta_v$. Hence $\theta_{I_v} \subseteq \theta_v$. \square

In the above theorem, the opposite inclusion may not hold. See Example 3.2 part (2).

Proposition 4.9. Let v be a pseudo-valuation on a BCI-algebra X such that $v(x) \geq 0$ for all $x \in X$ and I_v be the ideal induced by v . Then $\theta_v \subseteq \theta_{I_v}$.

Proof. Let $(x, y) \in \theta_v$. Then $v(x * y) + v(y * x) = 0$. Since $v(x) \geq 0$ for all $x \in X$, we obtain that $v(x * y) = 0$ and $v(y * x) = 0$. By definition I_v , we get that $x * y, y * x \in I_v$. It follows that $(x, y) \in \theta_{I_v}$. Hence $\theta_v \subseteq \theta_{I_v}$. \square

Proposition 4.10. Let I be an ideal of a BCI-algebra X . Then $\theta_I = \theta_{I_v}$.

Proof. Let $(x, y) \in \theta_I$. Then $x * y, y * x \in I$ by Theorem 2.10. We have $v_I(x * y) = v_I(y * x) = 0$, by Theorem 3.6. Hence $d_v(x, y) = 0$ and then $(x, y) \in \theta_{v_I}$.

Conversely, let $(x, y) \in \theta_{v_I}$. Then $v_I(x * y) + v_I(y * x) = 0$. Since $v_I(x) \geq 0$ for all $x \in X$, we obtain that $v_I(x * y) = v_I(y * x) = 0$, that is $x * y, y * x \in I$. Hence $(x, y) \in \theta_I$. \square

Theorem 4.11. Let v_1 and v_2 be two different pseudo-valuations on a BCI-algebra X such that $[0]_{v_1} = [0]_{v_2}$. Then θ_{v_1} and θ_{v_2} coincide, thus $X/v_1 = X/v_2$.

Proof. Let $(x, y) \in \theta_{v_1}$. Then $(x * y, 0) = (x * y, y * y) \in \theta_{v_1}$. It follows that $x * y \in [0]_{v_1}$. Similarly, we can show that $y * x \in [0]_{v_1}$. By assumption $[0]_{v_1} = [0]_{v_2}$, so we get that

$$[x]_{v_2} * [y]_{v_2} = [x * y]_{v_2} = [0]_{v_2} \quad \text{and} \quad [y]_{v_2} * [x]_{v_2} = [y * x]_{v_2} = [0]_{v_2}$$

Since X/v_2 is a BCI-algebra, then $[x]_{v_2} = [y]_{v_2}$. Hence $(x, y) \in \theta_{v_2}$ and then $X/v_1 = X/v_2$. It follows that $X/v_2 = X/v_1$. \square

Lemma 4.12. Let v be a pseudo-valuation on a BCI-algebra X and I be an ideal of X such that $[0]_v \subseteq I$. Denote $I/v = \{[x]_v : x \in I\}$. Then

- (1) $x \in I$ if and only if $[x]_v \in I/v$ for any $x \in X$,
- (2) I/v is an ideal of X/v .

Proof. (1) Suppose that $[x]_v \in I/v$. Then there exists $y \in I$ such that $[x]_v = [y]_v$. Hence $(x, y) \in \theta_v$. It follows that $(x * y, 0) \in \theta_v$. We get that $x * y \in [0]_v$. Since $[0]_v \subseteq I$, we have $x * y, y \in I$. Hence $x \in I$. The converse is trivial.

(2) Since $0 \in I$, then $[0]_v \in I/v$ by part (1). Let $[x]_v * [y]_v, [y]_v \in I/v$. By Definition 4.3, $[x]_v * [y]_v = [x * y]_v$. We have $x * y, y \in I$ by part (1). Since I is an ideal, $x \in I$. We get that $[x]_v \in I/v$. Therefore I/v is an ideal of X/v . \square

Lemma 4.13. Let v be a pseudo-valuation on a BCI-algebra X and J be an ideal of X/v . Then $I = \{x \in X : [x]_v \in J\}$ is an ideal of X such that $[0]_v \subseteq I$.

Proof. It is clear that $0 \in [0]_v \subseteq I$. Suppose that $x * y, y \in I$. Then $[y]_v, [x * y]_v = [x]_v * [y]_v \in J$. Since J is an ideal of X/v , then $[x]_v \in J$. By definition I , we obtain $x \in I$. Hence I is an ideal of X . \square

Theorem 4.14. Let v be a pseudo-valuation on a BCI-algebra X , $I(X, v)$ the collection of all ideals of X containing $[0]_v$, and $I(X/v)$ the collection of all ideals of X/v . Then $\varphi : I(X, v) \rightarrow I(X/v)$, $I \mapsto I/v$, is a bijection.

Proof. It follows from Lemma 4.12 and Lemma 4.13. \square

Lemma 4.15. Let X and Y be BCI-algebras, $f : X \rightarrow Y$ a homomorphism and v a pseudo-valuation on Y . Then $v \circ f : X \rightarrow \mathfrak{R}$ defined by $v \circ f(x) = v(f(x))$ for all $x \in X$ is a pseudo-valuation on X .

Proof. The proof is straightforward. \square

Theorem 4.16. Let X and Y be BCI-algebras, $f : X \rightarrow Y$ an epimorphism and v a pseudo-valuation on Y . Then $X/v \circ f \cong Y/v$.

Proof. By Lemma 4.15 and Theorem 4.4, we have $X/v \circ f$ and Y/v are BCI-algebras. Define $\psi : X/v \circ f \rightarrow Y/v$ by $\psi([x]_{v \circ f}) = [f(x)]_v$ for all $x \in X$.

(1) Suppose that $[x]_{v \circ f} = [y]_{v \circ f}$. Then $(v \circ f)(x * y) + (v \circ f)(y * x) = 0$. Since f is a homomorphism, then $v(f(x) * f(y)) + v(f(y) * f(x)) = 0$. We obtain that $[f(x)]_v = [f(y)]_v$. We get that $\psi([x]_{v \circ f}) = \psi([y]_{v \circ f})$, that is ψ is well define.

(2) We show that ψ is a homomorphism. Since f is a homomorphism,

$$(i) \psi([0]_{v \circ f}) = [f(0)]_v = [0]_v,$$

$$(ii) \psi([x]_{v \circ f} * [y]_{v \circ f}) = \psi([x * y]_{v \circ f}) = [f(x * y)]_v = [f(x) * f(y)]_v = [f(x)]_v * [f(y)]_v = \psi([x]_{v \circ f}) * \psi([y]_{v \circ f}).$$

(3) Let $[y]_v \in Y/v$ be arbitrary. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Hence $\psi([x]_{v \circ f}) = [f(x)]_v = [y]_v$ and ψ is surjective.

(4) We prove that ψ is one to one. Suppose that $\psi([x]_{v \circ f}) = \psi([y]_{v \circ f})$. Then $[f(x)]_v = [f(y)]_v$. We get that $v(f(x) * f(y)) + v(f(y) * f(x)) = 0$. Since f is a homomorphism, then $(v \circ f)(x * y) + (v \circ f)(y * x) = 0$. We obtain $[x]_{v \circ f} = [y]_{v \circ f}$. Hence $X/v \circ f \cong Y/v$. \square

Lemma 4.17. Let v be a pseudo-valuation on a BCI-algebra X and X/v be the corresponding quotient algebra. Then the map $\pi : X \rightarrow X/v$ defined by $\pi(x) = [x]_v$ for all $x \in X$ is an epimorphism.

Corollary 4.18. Let v be a pseudo-valuation on a BCI-algebra X and X/v the corresponding quotient algebra. For each pseudo-valuation \bar{v}_1 on a BCI-algebra X/v , there exists a pseudo-valuation v_1 on a BCI-algebra X , such that $v_1 = \bar{v}_1 \circ \pi$.

Proof. It follows from Lemma 4.15 and Lemma 4.17. \square

Theorem 4.19. Let v be a pseudo-valuation on a BCI-algebra X such that $v(x) \geq 0$ for all $x \in X$. Then $\bar{v} : X/v \rightarrow \mathfrak{R}$ define by $\bar{v}([x]_v) = v(x)$ is a pseudo-valuation on X/v .

Proof. It is enough to show that \bar{v} is well defined. Let $[x]_v = [y]_v$. Since $v(x) \geq 0$, then $v(x * y) = v(y * x) = 0$. We have $x * (x * y) \leq y$. By Theorem 3.3 part (1), $v(x * (x * y)) \leq v(y)$. It follows that $v(x * (x * y)) + v(x * y) \leq v(y) + v(x * y)$. Therefore $v(x) \leq v(x * (x * y)) + v(x * y) \leq v(y)$. Similarly, we can show that $v(y) \leq v(x)$. Therefore $v(y) = v(x)$ and then \bar{v} is well defined. \square

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