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Energy of Graphs, Matroids and Fibonacci Numbers

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Abstract. The energy $E(G)$ of a graph G is the sum of the absolute values of the eigenvalues of G. In this article we consider the problem whether generalized Fibonacci constants φ_n $(n \geq 2)$ can be the energy of graphs. We show that φ_n cannot be the energy of graphs. Also we prove that all natural powers of φ_{2n} cannot be the energy of a matroid.

Keywords: Graph energy, Fibonacci numbers, Matroid.

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1. Introduction

Let $G = (V, E)$ be a simple and finite graph of order n where V and E be vertex and edge sets of G, respectively. If A is the adjacency matrix of G, then the eigenvalues of A, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are said to be the eigenvalues of the graph G. These are the roots of the characteristic polynomial $\phi(G, \lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$. An interval I is called a zero-free interval for a characteristic polynomial $\phi(G, \lambda)$ if $\phi(G, \lambda)$ has no root in I.

The energy of the graph G is defined as $E = E(G) = \sum_{i=1}^{n} |\lambda_i|$. This definition was put forward by I. Gutman [6] and was motivated by earlier

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results in theoretical chemistry [7]. It is easy to see that if a undirected graph G has positive eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, then $E = 2\Sigma_{i=1}^{m}\lambda_i$.

A matroid M consists of a non-empty finite set E and a non-empty collection I of subsets of E, called independent sets, satisfying the following properties:

(i) any subset of an independent set is independent,
(ii) if I and J are independent sets with $|J| > |I|$ then there is an element e, contained in J but not in I such that $I \cup \{e\}$ is independent.

Let $M = (E, I)$ be a matroid defined in terms of its independent sets. Then a subset of E is dependent if it is not independent and a minimal dependent set is called a cycle. If $M(G)$ is the cycle matroid of a graph G then the cycles of $M(G)$ are precisely the cycles of G. A graphic matroid is a matroid $M(G)$ on the set of edges of a graph G by taking the cycles of G as the cycles of the matroid. For a subset A of E, the rank of A denoted by $r(A)$, is the size of the largest independent set contained in A. Note that the rank of M is equal to $r(E)$ since a subset A of E is independent if and only if $r(A) = |A|$. Recall that a complex number ζ is called an algebraic number (respectively, algebraic integer) if it is a zero of some monic polynomial with rational (respectively, integer) coefficients (see [13]). Corresponding to any algebraic number ζ, there is a unique monic polynomial p with rational coefficients, called the minimal polynomial of ζ (over the rationals), with the property that p divides every polynomial with rational coefficients having ζ as a zero. (The minimal polynomial of ζ has integer coefficients if and only if ζ is an algebraic integer.) Since the characteristic polynomial is a monic polynomial in λ with integer coefficients, its zeros are, by definition, algebraic integers. This naturally raises the question: Which algebraic integers can occur as the energy of a graph?

In 2004 Bapat and Pati [2] obtained the following result:

Theorem 1.1. *The energy of a graph cannot be an odd integer.*

In 2008 Pirzada and Gutman communicated an interesting result:

Theorem 1.2. ([10]) *The energy of a graph cannot be the square root of an odd integer.*

Also [1] and [11] contribute to the question of which numbers can be graph energies.

In this paper we prove some further results of this kind.

In Section 2, we prove that τ, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, cannot be the energy of a graph. Also we generalize this result and prove that all n-anacci numbers cannot be energy of graphs. In Section 3, we study natural powers of $2n$-anacci constants as the energy of a matroid. We show that all natural powers of $2n$-anacci constants cannot be the energy of a matroid.

2. Energy of graph and the golden ratio

In this section, we investigate the quantity τ, where τ is the golden ratio as a graph energy. We show that τ cannot be a graph energy. Also we prove that all n-anacci constants cannot be a graph energy. We need the following theorem:

Theorem 2.1. ([3]) *If graph G with order n has no isolated vertices, then $E(G) \geq 2\sqrt{n-1}$, with equality for stars.*

The following theorem is an immediate consequence of Theorem 2.1.

Theorem 2.2. *The golden ratio cannot be the energy of a graph.*

Proof. Since for every $n \geq 2$, $2\sqrt{n-1} > \frac{1+\sqrt{5}}{2}$, the result is true for every graphs of order $n \geq 2$. Since τ is not the energy of K_1, therefore we have the result by Theorem 2.1. □

Fibonacci numbers are terms of the sequence defined in a quite simple recursive fashion.

An n-step ($n \geq 2$) Fibonacci sequence $F_k^{(n)}$, $k = 1, 2, 3, \ldots$ is defined by letting $F_1^{(n)} = F_2^{(n)} = \ldots = F_n^{(n)} = 1$ and other terms according to the linear recurrence equation $F_k^{(n)} = \sum_{i=1}^{k-1} F_{k-i}^{(n)}$, $(k > 2)$. The limit $\varphi_n = \lim_{k\to\infty} \dfrac{F_k^{(n)}}{F_{k-1}^{(n)}}$ is called the n-anacci constant.

It is easy to see that φ_n is the real positive zero of $f_n(x) = x^n - x^{n-1} - \ldots - x - 1$, and this polynomial is the minimal polynomial of φ_n over $\mathbb{Z}[x]$. It is obvious that φ_n is a zero of $g_n(x) = x^n(2-x) - 1$. Note that $\varphi_2 = \tau$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and $\lim_{n\to\infty} \varphi_n = 2$ (see [9]).

Theorem 2.3. *For every integer $n \geq 2$, the n-anacci numbers φ_n cannot be the energy of a graph.*

Proof. By above statements, for every $n \geq 2$, $\{\varphi_n\}$ is an increasing sequence and $\varphi_n < 2$. Therefore we have the result similar to the proof of Theorem 2.2. □

3. $2n$-anacci and energy of matroid

In this section we shall study natural powers of $2n$-anacci numbers as the energy of a matroid. Characteristic polynomials of matroids were first studied by Rota [12]. Heron [8] defined chromatic polynomials of matroids and showed that they are equivalent to characteristic polynomials.

We need the following theorem which is about the zeros of characteristic polynomials of matroids

Theorem 3.1. ([5]) *Let M be a loopless matroid with rank r and characteristic polynomial $P(M,t)$. Then*

(i) $P(M,t) = t^r - |M|t^{r-1} + k_{r-2}t^{r-2} - \cdots + (-1)^r k_0$ *where* $k_0, \cdots, k_{r-2}$ *are positive integers*

(ii) $(-1)^r P(M,t) > (1-t)^r$ *for* $t \in (-\infty, 1)$

(iii) $P(M,1) = 0$, *and the multiplicity of* 1 *as a zero of* $P(M,t)$ *is equal to the number of components of* M.

(iv) *if* $r(M)$ *and* $c(M)$ *be rank and the number of components of* M *respectively, then for* $t \in (1, \frac{32}{27}]$, *we have* $(-1)^{r(M)+c(M)}P(M,t) \geq (t-1)^{r(M)}$.

By Theorem 3.1, we deduce that the maximal zero-free intervals for characteristic polynomials of loopless matroids are precisely $(-\infty, 1)$ and $(1, \frac{32}{27}]$.

Using the terminology and notation from the book [4], we define two operations with graphs. By $V(G)$ and $E(G)$ are denoted the vertex and edge sets, respectively, of the graph G. Let G_1 and G_2 be two graphs with disjoint vertex sets of orders n_1 and n_2, respectively. The direct product of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices $(x_1, x_2) \in V(G_1 \times G_2)$ and $(y_1, y_2) \in V(G_1 \times G_2)$ are adjacent if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$. The sum of G_1 and G_2, (or Cartesian product) denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices $(x_1, x_2) \in V(G_1 + G_2)$ and $(y_1, y_2) \in V(G_1 + G_2)$ are adjacent if and only if either $(x_1, y_1) \in E(G_1)$ and $x_2 = y_2$ or $(x_2, y_2) \in E(G_2)$ and $x_1 = y_1$. The above specified two graph products have the following spectral properties (see [4], p.70). Let $\lambda_i^{(1)}$, $i = 1, \cdots, n_1$, and $\lambda_j^{(2)}$, $j = 1, \cdots, n_2$, be, respectively, the eigenvalues of the graphs G_1 and G_2.

Lemma 3.1. *The eigenvalues of* $G_1 \times G_2$ *are* $\lambda_i^{(1)}\lambda_j^{(2)}$, $i = 1, \cdots, n_1$; $j = 1, \cdots, n_2$.

Lemma 3.2. *The eigenvalues of* $G_1 + G_2$ *are* $\lambda_i^{(1)} + \lambda_j^{(2)}$, $i = 1, \cdots, n_1$; $j = 1, \cdots, n_2$.

Now, we state and prove the following theorem:

Theorem 3.2. *If* α *is not a root of any characteristic polynomial of graph, then* α *cannot be the energy of a graph.*

Proof. Suppose that there exist a graph G such that $E(G) = \alpha$. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be positive eigenvalues of G. Then in view of the fact that the sum of all eigenvalues of any graph is equal to zero, $E(G) = 2\sum_{i=1}^{m} \lambda_i$. Denote $\lambda_1 + \lambda_2 + \ldots + \lambda_m$ by λ. By Lemma 3.1 a λ is an eigenvalue of some graph H isomorphic to the sum of m disjoint copies of the graph G. By Lemma 3.2, 2λ is an eigenvalues of the product of P_2 and H. Therefore α is an eigenvalue of $H \times P_2$, a contradiction. Hence we have the result. □

We need the following theorem to show our main results in this section.

Theorem 3.3. ([9]) *The polynomial* $f_n(x) = x^n - x^{n-1} - \cdots - x - 1$ *is an irreducible polynomial over* $\mathbb{Q}$.

Here, we may prove that all natural powers of $2n$-anacci constants cannot be the energy of a matroid.

Theorem 3.4. *All natural powers of φ_{2n} cannot be energy of matroids.*

Proof. By Theorem 3.2, it suffices to prove that $\varphi_{2n}^m (m \in \mathbb{N})$ cannot be a characteristic zero. Suppose that $\varphi_{2n}^m (m \in \mathbb{N})$ is a characteristic zero, that is there exists a characteristic polynomial

$$P(G, \lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda$$

such that $P(G, \varphi_{2n}^m) = 0$. Therefore,

$$\varphi_{2n}^{mk} + a_{k-1}\varphi_{2n}^{m(k-1)} + \cdots + a_1\varphi_{2n}^m = 0.$$

Hence φ_{2n} is a zero of the polynomial,

$$Q(\lambda) = \lambda^{mk} + a_{k-1}\lambda^{mk-m} + \cdots + a_1\lambda^m$$

in $\mathbb{Z}[x]$. But $f_{2n}(\lambda) = \lambda^{2n} - \lambda^{2n-1} - \cdots - \lambda - 1$ is the minimal polynomial of φ_{2n} over $\mathbb{Z}[x]$. Therefore $f_{2n}(\lambda)$ divides $Q(\lambda)$. Since $f_{2n}(0) = -1 < 0$ and $f_{2n}(-1) = 1 > 0$, $f_{2n}(\lambda)$ and so $Q(\lambda)$ must have a zero say α, in $(-1, 0)$. Therefore, α^m is a root of $P(G, \lambda)$. Since $\alpha^m \in (-1, 0) \cup (0, 1)$, we have a contradiction. □

Conjecture 3.1. *Let $n \in \mathbb{N}$. Then all natural powers of $2n+1$-annaci numbers cannot be the energy of a graph and a matroid.*

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References

[1] C. Adiga, Z. Khoshbakht and I. Gutman, More graphs whose energy exceeds the number of vertices, *Iranian Journal of Mathematical Sciences and Informatics*, **2**(2) (2007), 57−62.

[2] R. B. Bapat, S. Pati, Energy of a graph is never an odd integer, *Bull. Kerala Math. Assoc.*, **1** (2004), 129−132.

[3] A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *New results in Algebraic Combinatorics and Applications*, Springer-Verlag, Berlin, (2001), 196−211.

[4] D. Cvetkovic, M. Doob, H. Sachs, *Spectra of Graphs Theory and Application*, Academic Press, New York, 1980; 2nd revised ed.: Barth, Heidelberg, 1995.

[5] H. Edwards, R. Hierons and B. Jackson, The Zero-Free Intervals for Characteristic Polynomials of Matroids, *Combinatorics, Probability and Computing*, **7** (1998), 153−165.

[6] I. Gutman, The energy of a graph *Ber. Math. Statist. Sekt. Forshungsz. Graz* **103** (1978), 1−22.

[7] I. Gutman and O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer Verlag, Berlin, 1986.

[8] A. P. Heron, Matroid polynomials, In Combinatorics (D. J. A. Welsh and D. R. Woodall, Eds.), *Inst. Math. Appl. Southend-on-Sea.* (1972), 164−203.

[9] P. A. Martin, The Galois group of $x^n - x^{n-1} - \ldots - x - 1$, *Journal of Pure and Applied Algebra* **190** (2004), 213−223.

[10] S. Pirzada, I. Gutman, Energy of graph is never the square root of an odd integer, *Appl. Anal. Discr. Math.* **2** (2008), 118−121.

[11] O. Rojo, L. Medina, Constructing graphs with energy $\sqrt{r}\,E(G)$ where G is a bipartite graph, *MATCH Commun. Math. Comput. Chem.* **62** (2009), 465−472.

[12] G. C. Rota, On the foundations of combinatorial theory I. Theory of Mobius functions. *Z. Wahrscheinlichkeitstheorie* **2** (1964), 340−368.

[13] I. Stewart, D. Tall, *Algebraic Number Theory*, 2nd ed., Chapman and Hall, London-New York, 1987.