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Classification based on 3-similarity

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Abstract. Similarity concept, finding the resemblance or classifying some groups of objects and study their common properties has been the interest of many researchers. Basically, in the studies the similarity between two objects or phenomena, 2-similarity in our words, has been discussed. In this paper, we consider the case when the resemblance or similarity among three objects or phenomena of a set, 3-similarity in our terminology, is desired. After defining 3-equivalence relation and 3-similarity, some common and different points between them are investigated. We will see that in some special cases we can reach from 3-similarity to 2-similarity.

Keywords: Similarity relations, 3-equivalence relations, 3-similarity, n-similarity, Dissimilarity.

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1. Introduction

In recent years, lots of work have been developed on the concept of similarity and dissimilarity. Charkraborty and Das [2], Valverde [20], Trillas and Valverde [19], and Ovchinnikov [12, 13] have widely studied the similarities in various contexts. Loia et al. [11] used similarity relations in an Internet e-mail application. In psychology, similarity concept is used to determine why and how entities are grouped to categories, and why some categories are comparable to each other while others are not [6]. Also, the artificial intelligence community have started using computational similarity models as a new method for information retrieval [15]. Alguliev and Aliguliyev in [1] evaluated the performance of different similarity measures in the context of document summarization. Also, lots of works have been done on the application of similarity measures to fuzzy sets [22] which is an important tool in fuzzy mathematics [21], decision making, market prediction, and pattern recognition [23, 3, 14]. In all these, the measure of similarity between two objects or phenomena, i.e. 2-similarity, has been discussed and computed. In this paper, the similarity of three or more objects, out of a group of objects, 3-similarity, in our terminology, is investigated. It is well known that the set of all 2-equivalence relations on a set like U have one to one correspondence to the set of all partitions on U. On the other hand, every 2-similarity gives a class of 2-equivalence relations on U. We will define the 3-equivalence relation and will investigate the relation between 3-similarity and 3-equivalence relations.

After the introduction section, in Section 2, we will review the theoretical aspects of the 2-similarity relations, then we will define the 3-similarity relation. Thereafter, we continue with bringing the definitions, propositions, theorems, and lemmas regarding 3-similarity relations. We will show that most of the definitions and propositions related to 2-similarity relations could also be stated for 3-similarity relations; analogies between them will be studied as well. Also, we will show that if we have a 2-similarity, under certain conditions, 3-similarity could obtained. We have shown how T-norms can be used to generalize the concept of 2-similarity to 3-similarity. In this paper some open questions have been posed. The first being, the possibility of constructing a set X based on U, such that all 3-equivalence relations on X correspond to a subset of all partitions on X. The second open question is the possibility of redefining 3-equivalence relations in such a way that the set of all the 3-equivalence relations correspond to the set of all partitions on $U \times U$. The third one is, defining the conditions on which a 3-similarity can create a 2-similarity. Finally we will show that the idea can be generalized towards the n-similarity relations. Throughout the paper, various examples have been brought to support our ideas. Section 3 concludes the paper.

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2. Similarity Relations

Similarity relation [11] is a mathematical notion that provides a way to manage alternative instances of an entity that can be considered "equal" to other entities with a given degree [9, 22].

Definition 2.1. A 2-similarity on a domain U is a function $S : U \times U \longrightarrow [0, 1]$ such that the following properties hold [11]:

(i) $S(x, x) = 1$ for any $x \in U$(reflexivity).
(ii) $S(x, y) = S(y, x)$ for any $x, y \in U$(symmetry).
(iii) $S(x, z) \geq S(x, y) \wedge S(y, z)$ for any $x, y, z \in U$ (transitivity), where $\wedge$ is a minimum operator.
We say that S is strict if the following implication is also verified:
(iv) $S(x, z) = 1 \Rightarrow x = z$.

For further definitions and propositions regarding 2-similarity, refer to [7].

In 3-similarity, a three member similarity, we expect the permutation of the members dose not have any effect on their similarity. Moreover, if all 3 members of the group are exactly identical, their similarity also be maximum. Regarding these points we define a 3-similarity as follows:

Definition 2.2. A 3-similarity on a domain U is a function $S : U \times U \times U \longrightarrow [0, 1]$ such that the following properties hold:

(i) $S(x, x, x) = 1$ for any $x \in U$(reflexivity).
(ii) $S(x_1, x_2, x_3) = S(x_{i_1}, x_{i_2}, x_{i_3})$ for any $x_1, x_2, x_3 \in U$ (symmetry), where (i_1, i_2, i_3) is an arbitrary permutation of (1,2,3).
(iii) $S(x_1, x_2, x_3) \geq S(t, x_2, x_3) \wedge S(x_1, t, x_3) \wedge S(x_1, x_2, t)$, for any $t, x_1, x_2, x_3 \in U$ (transitivity), where $\wedge$ is a minimum operator.
We say that S is strict if the following implication is also verified:
(iv) $S(x_1, x_2, x_3) = 1 \Rightarrow x_1 = x_2 = x_3$.

The following example shows a strict 3-similarity.

Example 2.3. Suppose $U = \{1, 2, 3\}$, then we define

$$S(x, y, z) = \begin{cases} \frac{1}{2} & \text{if } x \neq y \text{ or } x \neq z \\ 1 & \text{if x=y=z} \end{cases}$$

It is obvious that S is a 3-similarity and it satisfies the property (iv) above.

The following example on integer numbers, $\mathbb{Z}$, clarifies the concept of 3-similarity.

Example 2.4. Let $\mathbb{Z}$ be the set of integer numbers and let n_1, n_2, $n_3 \in \mathbb{Z}$. Consider $n_j \equiv i_j \pmod 3$, where $1 \leq j \leq 3$ and $0 \leq i_j \leq 2$.
Set $A = \{i_1, i_2, i_3\}$. It is obvious that $1 \leq |A| \leq 3$, where $|A|$ denotes the

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cardinality of A.
Define

$$S(n_1, n_2, n_3) = \frac{3 - |A|}{2}.$$

We show that S is a 3-similarity on $\mathbb{Z}$. To this end we define the sets $\mathbb{Z}_0$, $\mathbb{Z}_1$, and $\mathbb{Z}_2$ as

$$\mathbb{Z}_i = \{3k + i : k \in \mathbb{Z}\},\ i = 0, 1, 2$$

It is obvious that $\mathbb{Z}$ is partitioned to classes $\mathbb{Z}_0$, $\mathbb{Z}_1$, and $\mathbb{Z}_2$. Now consider three integers numbers, n_1, n_2, and n_3. 3 cases are possible as follows:
Case 1) The integer numbers belong to the same class, in this case, by dividing any of these members into 3, we get the same remainder. This means A is an one-element set and we have $S(n_1, n_2, n_3) = \frac{3-1}{2} = 1$. As an example, if n_1, n_2, and n_3 all are multiples of 3, then the remainder of division of these numbers into 3 is equal to zero. So $A = \{0\}$ and $|A| = 1$.
Case 2) If n_1, n_2, and n_3 come from two different classes, i.e. two of them from one class, and the other one from another. Then $|A| = 2$, and $S(n_1, n_2, n_3) = \frac{3-2}{2} = 0.5$.
Case 3) If n_1, n_2, and n_3 each come from a different class, then $|A| = 3$, and $S(n_1, n_2, n_3) = \frac{3-3}{2} = 0$. So, we showed that by Definition 2.2 the degree of similarity of three integer numbers would be either 0, 0.5, or 1. With these explanations the reflexivity and symmetry properties of S are clear. For the transitivity property of S, we show that for all $n_1, n_2, n_3, t \in \mathbb{Z}$, we have

$$S(n_1, n_2, n_3) \geq \min\{S(t, n_2, n_3), S(n_1, t, n_3), S(n_1, n_2, t)\}. \qquad (1)$$

Let $n_j \equiv i_j (\text{mod } 3)$ for $j = 1, 2, 3$ where $0 \leq i_j \leq 2$. Also, let $t \equiv m (\text{mod } 3)$ for $0 \leq m \leq 2$.
If $i_1 = i_2 = i_3$, then $S(n_1, n_2, n_3) = 1$ and (1) holds.
If $i_1 = i_2$ and $i_1 \neq i_3$, then $S(n_1, n_2, n_3) = 0.5$. Now we may encounter three different cases:
Case 1: $m = i_1$, then $m \neq i_3$ and $S(t, n_2, n_3) = 0.5$. Therefore, (1) holds.
Case 2: $m = i_3$, then $S(n_1, n_2, t) = 0.5$, and (1) holds.
Case 3: $m \neq i_1$, and $m \neq i_3$, then $S(n_1, t, n_3) = 0$, and (1) holds.
If i_1, i_2, and i_3 are three different elements of $\{0, 1, 2\}$, then $S(n_1, n_2, n_3)$ is equal to 0 and m is equal to one of them. Therefore, one of the similarities of the right hand side of (1) is zero, hence (1) holds.

Similarities and their dual, dissimilarities, and their relationships have been discussed in [7] and in [8]. Also, various applications of 3-similarity relations have been brought in [8].

Definition 2.5. A subset R of $U \times U \times U$, is called a 3-equivalence relation on U if,

(i) $(x, x, x) \in R$, for all $x \in U$.

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(ii) If $(x_1, x_2, x_3) \in R$, then $(x_{i_1}, x_{i_2}, x_{i_3}) \in R$ for all permutations (i_1, i_2, i_3) of $(1, 2, 3)$.

(iii) $(t, y, z), (x, t, z)$ *and* $(x, y, t) \in R$ implies that $(x, y, z) \in R$ for all $x, y, z, t \in U$, where by a 3-relation we mean any non-empty subset of $U \times U \times U$.

It is well known that every 2-equivalence relation corresponds to a partition on U. Our definition of 3-equivalence relation does not make any partition on U. If each partition on U corresponds to a 3-equivalence relation, then, there must be a one-to-one correspondence between 2-equivalence and 3-equivalence relations. However, in the following example we show that, such correspondence cannot exist.

Example 2.6. Suppose $U = \{1, 2, 3\}$, then there are five 2-equivalence relations on U as follows:
$R_0 = \{(1, 1), (2, 2), (3, 3)\}$,
$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$,
$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$,
$R_3 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$,
$R_4 = U \times U$.

Whereas, the number of 3-equivalence relations exceeds 5. Here we list only a few of them, R'_0 to R'_7 as follows:
$R'_0 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$,
$R'_1 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (1, 2, 1), (2, 1, 1)\}$,
$R'_2 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 3), (1, 3, 1), (3, 1, 1)\}$,
$R'_3 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 2), (2, 2, 1), (2, 1, 2)\}$,
$R'_4 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 3, 3), (3, 3, 1), (3, 1, 3)\}$,
$R'_5 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (2, 3, 3), (3, 3, 2), (3, 2, 3)\}$,
$R'_6 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (2, 2, 3), (2, 3, 2), (3, 2, 2)\}$.
$R'_7 = U \times U \times U$.

On the other hand, there is not any correspondence between the set of all 3-equivalence relations on U, and the set of all partitions on $U \times U$. To see this, let again $U = \{1, 2, 3\}$. Then $U \times U$ has 9 members, and the number of partitions on a 9-member set exceeds 10000, whereas, the number of 3-equivalence relations on a 3-member set like U is less than 2^7. Consequently, we cannot create a one-to-one correspondence between the set of all 3-equivalence relations and all the partitions on $U \times U$.

Here, we pose 2 open questions:

Open Question 1: Is it possible to construct a set X based on U, such that all 3-equivalence relations on X corresponds to a subset of all partitions on X?

Open Question 2: Can we redefine 3-equivalence relations in such a way that

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the set of all the 3-equivalence relations correspond to the set of all partitions on $U \times U$?

Definition 2.7. Let U be a set and $S : U \times U \times U \longrightarrow [0,1]$ be a 3-similarity on U. Then for any $\lambda \in [0,1]$, the 3-relation $\cong_{S,\lambda}$ in U is defined as $(x,y,z) \in \cong_{S,\lambda}$, if $S(x,y,z) \geq \lambda$. The set $\cong_{S,\lambda}$ is called cut of level λ of S or $\lambda - cut$ of S.

Lemma 2.8. *Let $S : U \times U \times U \longrightarrow [0,1]$ be a map and let for any $\lambda \in [0,1]$, $\cong_{S,\lambda} := \{(x,y,z) \in U \times U \times U : S(x,y,z) \geq \lambda\}$, the λ-cut of S be a 3-relation. Then S is a 3-similarity on U if and only if for any $\lambda \in [0,1]$, $\cong_{S,\lambda}$ is a 3-equivalence relation.*

Proof. Let S be a 3-similarity and $\lambda \in [0,1]$. We show $\cong_{S,\lambda}$ is a 3-equivalence relation.

(a) $S(x,x,x) = 1 \geq \lambda$, implies that $(x,x,x) \in \cong_{S,\lambda}$ for all $x \in U$ and for all $\lambda \leq 1$.

(b) If $(x_1,x_2,x_3) \in \cong_{S,\lambda}$, then $S(x_{i_1},x_{i_2},x_{i_3}) = S(x_1,x_2,x_3) \geq \lambda$. Hence, $(x_{i_1},x_{i_2},x_{i_3}) \in \cong_{S,\lambda}$ for all permutations (i_1,i_2,i_3) of *(1,2,3)*.

(c) For $t,x,y,z \in U$, let $(t,y,z),(x,t,z)$ *and* $(x,y,t) \in \cong_{S,\lambda}$. Then $S(t,y,z) \geq \lambda$, $S(x,t,z) \geq \lambda$, and $S(x,y,t) \geq \lambda$. Now by definition of 3-similarity, $S(x,y,z) \geq \lambda$ and $(x,y,z) \in \cong_{S,\lambda}$. Therefore, $\cong_{S,\lambda}$ is a 3-equivalence relation.

Conversely, for any $\lambda \in [0,1]$, let $\cong_{S,\lambda}$ be a 3-equivalence relation, we show S is a 3-similarity.

(a) Reflexivity: Since for all $\lambda \in [0,1]$, $(x,x,x) \in \cong_{S,\lambda}$, $S(x,x,x) \geq \lambda$. Hence, $S(x,x,x) = 1$.

(b) Symmetry: Let (i_1,i_2,i_3) be an arbitrary permutation of *(1,2,3)*, and let $\alpha = S(x_1,x_2,x_3)$, and $\beta = S(x_{i_1},x_{i_2},x_{i_3})$ where x_1, x_2 and $x_3 \in U$. Now by the definition of λ-cuts, $(x_1,x_2,x_3) \in \cong_{S,\lambda}$ for each $0 \leq \lambda \leq \alpha$ and $(x_{i_1},x_{i_2},x_{i_3}) \in \cong_{S,\lambda}$ for each $0 \leq \lambda \leq \beta$. On the other hand, by the definition of 3-equivalence relations, $(x_1,x_2,x_3) \in \cong_{S,\lambda}$ if and only if $(x_{i_1},x_{i_2},x_{i_3}) \in \cong_{S,\lambda}$. Therefore, $\alpha = \beta$.

(c) Transitivity: Assume that x,y,z,t are arbitrary elements of U.

We have to show

$$S(x,y,z) \geq S(t,y,z) \wedge S(x,t,z) \wedge S(x,y,t). \qquad (2)$$

Suppose, $\alpha = S(x,y,z)$, $\alpha_1 = S(t,y,z)$, $\alpha_2 = S(x,t,z)$, and $\alpha_3 = S(x,y,t)$. If on the contrary (2) does not hold, then we have $\alpha < \min\{\alpha_1,\alpha_2,\alpha_3\}$. Then there is a β such that $\alpha < \beta < \min\{\alpha_1,\alpha_2,\alpha_3\}$. Since $S(t,y,z) = \alpha_1$, $S(t,y,z) \geq \lambda$ for every $\lambda \in [0,\alpha_1]$, and $\beta < \alpha_1$, therefore, $(t,y,z) \in \cong_{S,\beta}$. By the same argument, $(x,t,y) \in \cong_{S,\beta}$, and $(x,y,t) \in \cong_{S,\beta}$. Now, by the definition of 3-equivalence relation, $(x,y,z) \in \cong_{S,\beta}$ and hence $\alpha = S(x,y,z) \geq \beta$. It contradicts with $\alpha < \beta$. Therefore, (2) holds and the proof is completed. □

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Theorem 2.9. *Let S be a 3-similarity on a set U, and let $\cong_{S,\lambda}$ be the λ-cut of S, for any $\lambda \in [0,1]$. Then $\{\cong_{S,\lambda}: \lambda \in [0,1]\}$ is a family of 3-equivalence relations such that,*

(i) *$\lambda \leq \mu$ implies that $\cong_{S,\mu} \subseteq \cong_{S,\lambda}$, for any μ and λ in $[0,1]$.*
(ii) *$\cap_{\lambda<\mu} \cong_{S,\lambda} = \cong_{S,\mu}$, for any μ in $[0,1]$.*

Conversely, let $\{\cong_\lambda: \lambda \in [0,1]\}$ be a family of 3-equivalence relations satisfying conditions (i), (ii). Then the relation S defined by setting $S(x,y,z) = Sup\{\lambda \in [0,1] : (x,y,z) \in \cong_\lambda\}$ is a 3-similarity whose family of $\lambda-$cuts is equal to the family $\{\cong_\lambda\}_{\lambda\in[0,1]}$.

Proof. Let S be a 3-similarity on U, then Lemma 2.8 implies that $\cong_{S,\lambda}$ is a 3-equivalence relation, for all λ in $[0,1]$ and clearly (i) and (ii) are satisfied (note that its proof is similar to the proof of Proposition 2.3 in [7]).
Conversely, let $\{\cong_\lambda: \lambda \in [0,1]\}$ be a family of 3-equivalence relations on U satisfying conditions (i) and (ii), and

$$S(x,y,z) = Sup\{\lambda \in [0,1] : (x,y,z) \in \cong_\lambda\}. \qquad (3)$$

We will show that S is a 3-similarity on U. The reflexivity and symmetry properties of S are obvious. For the transitivity, let $x,y,z,t \in U$, and:
$A = \{\lambda \in [0,1] : (x,y,z) \in \cong_\lambda\}$,
$B = \{\lambda \in [0,1] : (t,y,z) \in \cong_\lambda\}$,
$C = \{\lambda \in [0,1] : (x,t,z) \in \cong_\lambda\}$,
$D = \{\lambda \in [0,1] : (x,y,t) \in \cong_\lambda\}$.

Then by (3), $S(x,y,z) = SupA = \alpha$, $S(t,y,z) = SupB$, $S(x,t,z) = SupC$, $S(x,y,t) = SupD$. We must show that:

$$S(x,y,z) \geq \min\{S(t,y,z), S(x,t,z), S(x,y,t)\}. \qquad (4)$$

On the contrary, suppose that (4) does not hold, then $\alpha < SupB$, $\alpha < SupC$, and $\alpha < SupD$. Then, there are $\lambda_1 \in B$, $\lambda_2 \in C$, and $\lambda_3 \in D$ such that $\alpha < \lambda_i$ for $i = 1,2,3$. Put $\gamma = \min\{\lambda_1, \lambda_2, \lambda_3\}$. Since $(t,y,z) \in \cong_{\lambda_1} \subseteq \cong_\gamma$, we have $(t,y,z) \in \cong_\gamma$. By similarity $(x,t,z) \in \cong_\gamma$ and $(x,y,t) \in \cong_\gamma$. Therefore, by transitivity property of $\cong_\gamma$ we have $(x,y,z) \in \cong_\gamma$ and hence $\gamma \in A$. This is a contradiction to $SupA = \alpha < \gamma$. Therefore, S is a 3-similarity and (4) holds.

Now let $0 \leq \mu \leq 1$ and consider $\cong_{S,\mu}$, as the $\mu - cut$ of S given in Definition 2.7. Then we have to show that $\cong_{S,\mu}$ is equal to $\cong_\mu$. Clearly $\cong_\mu \subseteq \cong_{S,\mu}$. On the other hand, let $(x,y,z) \in \cong_{S,\mu}$. We show that $(x,y,z) \subseteq \cong_\mu$. Let $SupA > \mu$. Then $SupA = S(x,y,z) \geq \mu$, so, there is $\lambda \in A$ such that $\mu < \lambda \leq SupA$. Since $\lambda \in A$, $(x,y,z) \in \cong_\lambda$, and $\mu < \lambda$, $\cong_\lambda \subseteq \cong_\mu$, therefore, $(x,y,z) \in \cong_\mu$ and $\cong_{S,\mu} \subseteq \cong_\mu$. The case $SupA \leq \mu$ is similar. □

It's worth mentioning that Novak and Novotny in [10] have given another definition of n-equivalence relation. If we compare their definition to our proposed

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definition (Definition 2.5), we will notice that the reflexive and the symmetric properties are the same, but their definition of transitive property differs. They have defined n-transitive property as follows: If $(x_1, ..., x_n) \in R$ and $(y_1, ..., y_n) \in R$ hold; also, if there exist natural numbers $i_0 > j_0$ such that $1 < i_0 \leq n$, $1 \leq j_0 < n$, and $x_{i0} = y_{j0}$, then the n-tuple $(x_{i_1}, ..., x_{i_k}, y_{j_{k+1}}, ..., y_{j_n}) \in R$ for any natural number $1 \leq k < n$ and $i_1, ..., i_k, j_{k+1}, ..., j_n$ such that $1 \leq i_1 < ... < i_k < i_0, j_0 < j_{k+1} < ... < j_n \leq n$.

The following example shows that the Definition 2.5 for 3-equivalence relation is different from the definition in [10].

Example 2.10. If $x = \{1, 2, 3, 4, 5\}$ and n=3, then the 3-equivalence relation is as follows:

$R = \{(1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (2,3,5), (1,2,5),$
$(1,3,2), (1,3,5), (3,2,5), (3,5,2), (5,3,2), (5,2,3), (2,5,3),$
$(2,1,5), (2,5,1), (5,1,2), (5,2,1), (1,5,2), (2,1,3), (2,3,1),$
$(3,1,2), (3,2,1), (1,2,3), (3,1,5), (3,5,1), (5,1,3), (5,3,1), (1,5,3)\}.$

It is obvious that R satisfies the Definition 2.5, but R does not satisfy the n-transitive property. To see this fact, let $k = 1$, $i_0 = 2$, $j_0 = 1$, $(x_1, x_2, x_3) = (2, 3, 5)$, and $(y_1, y_2, y_3) = (3, 2, 1)$. Now $x_{i0} = y_{j0}$ and $1 \leq i_1 < 2$ implies that $i_1 = 1$. Also $1 < j_2 < j_3 \leq 3$ implies that $j_0 = 2$ and $j_3 = 3$. Then, $(x_{i_1}, y_{j_2}, y_3) = (2, 2, 1) \notin R$, whereas, both $(x_1, x_2, x_3) = (2, 3, 5) \in R$ and $(y_1, y_2, y_3) = (3, 2, 1) \in R$.

Obviously, if R has the 3-transitive property, then the 3-equvalence relation satisfies. Therefore our definition of 3-equivalence relation, Definition 2.5, covers a wider range of 3-ary relations. Consequently, on the issues of 3-equivalence relations and 3-similarities, our proposed definition, Definition 2.5, have preference over the Novak-Novotny's definition.

Stefanesco in [18], on the basis of the definition of relation and its properties has found the corresponding hypergroups. Cristea in [4] has defined hypergroups with corresponding n-ary's. Considering their reflexivity, symmetry, and n-transitivity properties, sufficient conditions for existence of "total hypergroup" , "semi hypergroup", and "associative hypergroup" have been obtained. As a future work, we are currently studying the connection between hypergroups and the third property of Definition 2.5.

For 2-similarity, Sessa in the paper [16] said that in a set of three numbers like $\{S(x, y), S(x, z), S(y, z)\}$, two of the members are equal and the third member is either equal or greater than the other two. We generalize this fact for 3-similarity in the following proposition.

Proposition 2.11. *Let S be a 3-similarity, a=S(x,y,z), b=S(t,y,z), c=S(x,t,z), and d=S(x,y,t). Then two of the a, b, c, and d, are equal, and the others are either equal or greater than to the first two.*

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Proof. Using Definition 2.2 (*iii*), any of the following could be holding:

(i) $a \geq b \wedge c \wedge d$,
(ii) $b \geq a \wedge c \wedge d$,
(iii) $c \geq a \wedge b \wedge d$,
(iv) $d \geq a \wedge b \wedge c$.

Case 1 : Suppose the equality hold for all (i), (ii), (iii), and (iv) above. Then $a = b \wedge c \wedge d$ and (i) implies that $a = a \wedge b \wedge c \wedge d$. Likewise for (ii), (iii), and (iv), we have:
$b = b \wedge b = b \wedge a \wedge c \wedge d$,
$c = c \wedge c = c \wedge a \wedge b \wedge d$,
$d = d \wedge d = d \wedge a \wedge b \wedge c$.
Consequently, we have $a = b = c = d$.
Case 2: Suppose for at least one of the (i), (ii), (iii), or (iv), the equality does not hold.
Without loss of generality, consider the first one, that is:

$$a > b \wedge c \wedge d.$$

Case 2.1: $a > b$. Then $a \wedge b = b$, from (iii) and (iv) we conclude $c \geq b \wedge d$ and $d \geq b \wedge c$. Now, either $c \geq b$, or $c \geq d$, and also either $d \geq b$, or $d \geq c$. We investigate one of the cases, the others are similar.

Case 2.1.1: $c \geq b$ and $d \geq b$. If on the contrary $c \neq b$ and $d \neq b$, then $b < c$ and $b < d$, we also had $b < a$, consequently, $b < a \wedge d \wedge c$ which contradicts to (ii), that is, either $c = b$, or $b = d$.

Case 2.2: $a = b$, $a > b \wedge c \wedge d$ implies that $a > c \wedge d$.

Case 2.2.1: $c < d$, $a > c \wedge d$ implies that $a > c$ and hence $c < a \wedge b \wedge d$. This contradicts (iii). The case $c > d$ is similar. □

Lemma 2.12. *Suppose, $S : U \times U \times U \longrightarrow [0, 1]$ is a map with reflexive and symmetric properties as in Definition 2.2. For $x, y, z, t \in U$, set*

$$R = \{S(x, y, z), S(t, y, z), S(x, t, z), S(x, y, t)\}.$$

If any two of the members of R are equal, and the other two are either equal or greater than the first two, then S is a 3-similarity.

Proof. Case 1: If $S(x, y, z)$ is equal to any of the other 3, then the transitivity condition holds.

Case 2: If two members of R are equal and $S(x, y, z)$ is greater than those two, then the minimum would be one of those two, and $S(x, y, z)$ is equal or greater than the minimum. □

2.1. **T-norms.** We can use T-norms to generalize the concept of 2-similarity to 3-similarity. In the following, after giving the definition of T-norms [17], we will show how they can be utilized for this purpose.

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Definition 2.13. A T-norm (triangular norm) is a function $T : [0,1]\times[0,1] \longrightarrow [0,1]$ which satisfies the following properties for all $a, b, c, d \in [0,1]$

(1) T(a,b)=T(b,a) (commutativity)
(2) $T(a,b) \le T(c,d)$, if $a \le c$, and $b \le d$ (monotonicity)
(3) T(a,T(b,c))=T(T(a,b),c) (associativity)
(4) T(a,1)=a (boundary condition)

Minimum T-norm, $T_{min}(a,b) = min\{a,b\}$, is one of the prominent examples of T-norms which we will use in Theorem 2.14.

Theorem 2.14. *Let U be a set, T be a T-norm of $T : [0,1] \times [0,1] \longrightarrow [0,1]$, and S_2 be a 2-similarity on U, then for all $x, y, z \in U$ consider,*

$$S_3(x,y,z) = T(T(S_2(x,y), S_2(y,z)), S_2(x,z)).$$

Then S_3 is a 3-similarity on U.

Proof. $S_3(x,x,x) = T(T(S_2(x,x), S_2(x,x)), S_2(x,x) = T(T(1,1),1) = T(1,1) = 1$. Also, it is obvious that by changing the positions of x, y, and z, the result would not change.

Now, for the transitivity property, we have to show that for all $x, y, z, t \in U$,

$$S_3(x,y,z) \ge \{S_3(t,y,z) \wedge S_3(x,t,z) \wedge S_3(x,y,t)\}.$$

W.L.G. let

(i) $S_3(t,y,z) \le S_3(x,t,z)$, and
(ii) $S_3(t,y,z) \le S_3(x,y,t)$.

We have to show $S_3(t,y,z) \le S_3(x,y,z)$. From (i) we have,
$T(T(S_2(t,y), S_2(y,z)), S_2(t,z)) \le T(T(S_2(x,t), S_2(t,z)), S_2(x,z))$, hence,
$S_2(t,y) \le S_2(x,t)$, $S_2(y,z) \le S_2(t,z)$, and $S_2(t,z) \le S_2(x,z)$.
From (ii) we have, $S_2(t,y) \le S_2(x,y)$, $S_2(y,z) \le S_2(y,t)$, and $S_2(t,z) \le S_2(x,t)$. Now, the nonequalities $S_2(t,y) \le S_2(x,y), S_2(t,z) \le S_2(x,z)$, and $S_2(y,z) \le S_2(y,z)$ implies that
$T(T(S_2(t,y), S_2(y,z)), S_2(t,z)) \le T(T(S_2(x,y), S_2(y,z)), S_2(x,z))$, and hence,
$S_3(t,y,z) \le S_3(x,y,z)$.

□

If S_2 is a 2-similarity on U and $A \subset U$, then $\mu(A)$ has been introduced in [5] as

$$\mu(A) = \bigwedge_{x,x' \in A} S_2(x,x') \qquad (5)$$

If we consider A as a set with 3 elements, then we can interpret $\mu(A)$ as a special case of 3-similarity. This case has been brought into the following corollary.

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Corollary 2.15. *Let $S_2 : U \times U \longrightarrow [0,1]$ be a 2-similarity on U. For all $x, y, z \in U$, define $S_3 : U \times U \times U \longrightarrow [0,1]$ as follows,*

$$S_3(x,y,z) = \min\{S_2(x,y), S_2(y,z), S_2(x,z)\}. \qquad (6)$$

Then for every $x, y \in U$ we have,

(a) *S_3 is a 3-similarity on U.*
(b) *$S_3(x,x,y) = S_2(x,y)$.*
(c) *$S_3(x,x,y) \geq S_3(x,y,z)$.*
(d) *$S_3(x,x,y) = S_3(x,y,y)$.*

Proof. (a) is obtained using Theorem 2.14 and the logical product as a T-norm, i.e., $T(a,b) = min(a,b)$
For (b), we have,

$$\begin{aligned} S_3(x,x,y) &= \min\{S_2(x,x), S_2(x,y), S_2(x,y)\} \\ &= \min\{1, S_2(x,y), S_2(x,y)\} \\ &= S_2(x,y). \end{aligned}$$

(c) and (d) obviously could be obtained from (b) and (6). □

The following remark shows that our definition of 3-similarity is a generalization for definition of similarity of A in (5).
Remark: If $A = \{x,y,z\} \subset U$, then $\mu_(A) = S_3(x,y,z)$.

2.2. **Induced similarities.**

Definition 2.16. If S_2 is a 2-similarity, and S_3 is derived from S_2 according to Corollary 2.15, then we call S_3, a 3-similarity induced by S_2.

If S_3 is a 3-similarity on U, it seems that the relation S_2 on $U \times U$ which can be defined as

$$S_2(x,y) = S_3(x,x,y). \qquad (7)$$

is a 2-similarity on U. That is, in a 3-similarity, if two members are identical, the 2-similarity concludes. For instance, the 3-similarity of *rabbit*, *rabbit*, and *lion* are the same as the 2-similarity between *rabbit* and *lion*. However, as the next example shows, we may introduce a 3-similarity like S_3, for which a 2-similarity holding (7) dose not exist. Consequently, every 3-similarity does not give us a 2-similarity as indicated in (7). In Example 2.17, S_3 does not satisfy the following condition

$$S_3(x,x,y) = S_3(x,y,y), \quad \forall x, y \in U. \qquad (8)$$

In Example 2.18 one can see that (8) also holds, even the following

$$S_3'(x,x,y) \geq S_3'(x,y,z), \quad \forall x, y, z \in U. \qquad (9)$$

holds too, but still S_2 obtained from (7) cannot be a 2-similarity on U. So, Example 2.18 indicates that even a 3-similarity satisfies (8) and (9), it cannot

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give us a 2-similarity that satisfies (7).

In the following examples we will try to make our ideas clear.

Example 2.17. Let $U = \{R, M, P, W\}$, where the letters in the set stand for: $M = mouce, P = pigeon, R = rabit, W = wolf$. Define $S_3 : U\times U\times U \longrightarrow [0, 1]$ be a function as in Table 1. Also, let the commutative condition for S_3 holds.

Table 1. The level values of 3-similarity S_3

$S_3(x, y, z)$	(x, y, z)
1	(W,W,W) (P,P,P) (R,R,R) (M,M,M)
0.7	(R,P,P) (R,R,P) (R,R,W)
0.4	(R,M,M) (R,W,W) (W,M,M) (P,M,M)
0.3	(R,R,M) (W,P,P) (M,P,P) (W,W,P) (W,W,M)
0.2	(W,M,P) (R,W,P)
0.1	(R,M,W) (R,M,P)

It is clear that S_3 is a 3-similarity. If we define $S_2 : U \times U \longrightarrow [0, 1]$ as $S_2(x, y) = S_3(x, x, y)$, then $S_2(R, M) = S_3(R, R, M) = 0.3$, and $S_2(M, R) = S_3(M, M, R) = 0.4$. Consequently, $S_2(R, M)$ and $S_2(M, R)$ are not equal. Hence, S_2 is not a 2-similarity.

Now, consider the following example:

Example 2.18. Let U be the set $U = \{R, M, P, W\}$, where these letters stand as in Example 2.17. Let $S'_3 : U \times U \times U \longrightarrow [0, 1]$ be defined as a 3-similarity on U as in Table 2, and assume that the commutative condition for S'_3 holds. Then S'_3 is a 3-similarity on U and we have $S'_3(x, x, y) = S'_3(x, y, y)$ for each $x, y \in U$.

Table 2. The level values of 3-similarity of S'_3.

$S_3(x, y, z)$	(x, y, z)
1	(W,W,W) (P,P,P) (R,R,R) (M,M,M)
0.7	(R,P,P) (R,R,P) (R,R,W) (R,W,W)
0.4	(R,M,M) (R,R,M)
0.3	(M,M,P) (W,M,M) (W,P,P) (M,P,P) (W,W,P) (W,W,M)
0.2	(W,M,P) (R,W,P)
0.1	(R,M,W) (R,M,P)

If we get S_2 from the equality $S_2(x, y) = S'_3(x, x, y)$, then S_2 is obtained as shown in Table 3. The transitivity condition in the Definition 2.1,

$$S_2(P, M) \geq \min\{S_2(R, M), S_2(P, R)\}$$

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TABLE 3. The level values of S_2.

$S_2(x,y)$	(x,y)
1	(W,W) (P,P) (R,R) (M,M)
0.7	(R,P) (R,W) (P,R) (W,R)
0.4	(M,R) (R,M)
0.3	(P,M)(W,M)(W,P)(M,P)(M,W)(P,W)
0.2	
0.1	

does not hold, since $S_2(P,M) = 0.3$, $S_2(R,M) = 0.4$, and $S_2(P,R) = 0.7$. So S_2 is not a 2-similarity.
At any case Corollary 2.15 guarantees the existence of a 3-similarity induced by a 2-similarity.

Now, we pose another open question,
Open Question 3: What conditions on a 3-similarity can create a 2-similarity such that (6) holds?

2.3. **n-similarities.** Using the definitions of 2-similarity and 3-similarity, we can define an n-similarity.

Definition 2.19. $S : U \times U \times U...U \longrightarrow [0,1]$ is an $n - similarity$, if

(i) $S(x,x,,...,x) = 1$, (reflexivity).
(ii) $S(x_1,x_2,...,x_n) = S(x_{i_1},x_{i_2},...,x_{i_n})$ for all permutations $(i_1,i_2,i_3,...i_n)$ of $(1,2,...n)$, (symmetry).
(iii) $S(x_1,x_2,...,x_n) \geq \min\{S(z,x_2,...,x_n),...,S(x_1,x_2,...,x_{n-1},z)\}$ for all $x_1,x_2,...x_n,z \in U$, (transitivity).

Similarly, we can obtain an $n - similarity$ from an $(n-1) - similarity$. The proof is similar to the proof of Corollary 2.15. Also, we can prove it by induction.

Proposition 2.20. *Let S_{n-1} be an (n-1) similarity on U and for $x_1,x_2,...x_n$ in U define,*
$S_n(x_1,...x_n) =$
$\min\{S_{n-1}(x_2,x_3,...,x_n),S_{n-1}(x_1,x_3,...,x_n),...,S_{n-1}(x_1,x_2,...,x_{n-1}\}.$

(a) *S_n is an n-similarity on U,*
(b) $S_n(x_1,x_1,x_2,x_3...x_{n-1}) = S_{n-1}(x_1,x_2,...,x_{n-1})$,
(c) $S_n(x_1,x_1,x_2,x_3...x_{n-1}) \geq S_n(x_1,x_2,...,x_n)$,
(d) $S_n(x_1,x_1,x_2,x_3...x_{n-1}) = S_n(x_1,x_2,...,x_{n-1},x_{n-1})$.

The n-similarity, S_n, which is obtained from S_{n-1} as in Corollary 2.15 is called the n-similarity induced by S_{n-1}. Example 2.4 could be generalized for the case of n-similarity. That is:

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Example 2.21. Let Z be the set of integer numbers and let m_1, m_2,..... $m_n \in Z$, such that $m_j \equiv (i_j \text{ mod n})$ where $0 \leq i_j \leq n-1$, j=1,2,...,n
Set $A = \{i_1, i_2, i_3, ..., i_n\}$. Define

$$S(m_1, m_2, ..., m_n) = \frac{n - |A|}{n-1},$$

then S is an n-similarity.

3. Conclusion

In this paper we showed how it is possible to form 3-similarities. We proved that all the definitions and propositions related to 2-similarity relations could also be used for 3-similarity relations. We showed under certain conditions we can obtain a 3-similarity if we have a 2-similarity. Also in some examples we saw that not all 3-similarities can produce 2-similarities. As an open question, we are interested in obtaining the conditions that could be imposed on a 3-similarity to produce an induced 2-similarity. We also showed that the idea can be generalized towards the n-similarities.

The possibility of constructing a set X based on U, such that all 3-equivalence relations on X correspond to a subset of all partitions on X, and the possibility of redefining 3-equivalence relations in such a way that the set of all the 3-equivalence relations correspond to the set of all partitions on $U \times U$ are the matters that should be investigated. Another matter that we posed as an open question was defining the conditions on which a 3-similarity can create a 2-similarity. As future work, we are currently working on extending our propositions of 3-equivalence relations and 3-similarities to fuzzy logic programming.

Also, we should mention that using the discussion of dissimilarities between two objects [7], 3-dissimilarity (or n-dissimilarity) can be defined. Also the relation between 3-similarity and 3-dissimilarity can be investigated. Of course to avoid a lengthy paper, the authors will present these cases somewhere else.

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