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Properties of Central Symmetric X-Form Matrices

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ABSTRACT. In this paper we introduce a special form of symmetric matrices that is called central symmetric X-form matrix and study some of their properties, the inverse eigenvalue problem and inverse singular value problem for these matrices.

Keywords: Inverse eigenvalue problem, Inverse singular value problem, eigenvalue, singular value.

2000 Mathematics subject classification: 15A29; 15A18.

. INTRODUCTION

In [3,4] H. Pickman et. studied the inverse eigenvalue problem of symmetric tridiagonal and symmetric bordered diagonal matrices. In this paper we introduce the odd and even order central symmetric X-form matrix for an integer number n respectively as below:

suppose

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For a given 2n-1 real numbers such as

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)},$$

or for a given 2n real numbers such as

$$\lambda_1^{(2n)} < \lambda_1^{(2(n-1))} < \ldots < \lambda_1^{(2)} < \lambda_2^{(2)} < \ldots < \lambda_{2(n-1)}^{(2(n-1))} < \lambda_{2n}^{(2n)},$$

we construct a matrix A_n such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the maximal and minimal eigenvalues of submatrix A_j respectively for j = 1, 2, ...n where A_j is defined by to

$$\mathbf{A}_{j} = \begin{pmatrix} a_{j} & & & b_{j} \\ & \ddots & & \ddots \\ & & a_{2} & b_{2} & & \\ & & a_{1} & & \\ & & b_{2} & a_{2} & \\ & \ddots & & & \ddots & \\ & & & & & a_{j} \end{pmatrix}_{(2j-1)\times(2j-1)},$$

matrix B_j such that $\lambda_1^{(2j)}$ and $\lambda_{2j}^{(2j)}$ are the maximal and minimal eigenvalues of submatrix B_j respectively for j = 1, 2, ...n where B_j is defined by $B_j = \begin{pmatrix} a_j & b_j \end{pmatrix}$

2. Properties of the matrices A_n and B_n

Let $p_0(\lambda) = 1$, $q_0(\lambda) = 1$, $p_j(\lambda) = det(A_j - \lambda_j)$ for j = 1, 2, ..., n and $q_j(\lambda) = det(B_j - \lambda_j)$ for j = 1, 2, ..., n.

Lemma 1. For a given matrix A_j and B_j the sequence $p_j(\lambda)$ and $q_j(\lambda)$ satisfy in the following recurrence relations:

 $\begin{array}{l} a) \ p_1(\lambda) = (a_1 - \lambda), \\ b) \ p_j(\lambda) = [(a_j - \lambda)^2 - b_j^2] p_{j-1}(\lambda), \ j = 2, 3, ..., n. \\ c) \ q_1(\lambda) = ((a_1 - \lambda)^2 - b_1^2), \\ b) \ q_j(\lambda) = [(a_j - \lambda)^2 - b_j^2] q_{j-1}(\lambda), \ j = 2, 3, ..., n. \end{array}$

Proof. The proof is clear by extending determinants of $(A_j - \lambda I_j)$ and $(B_j - \lambda I_j)$ on their first columns.

2.1. LU factorization of central symmetric X-form matrix.

Let A be a central symmetric X-form matrix in form (1) and B be a central symmetric X-form matrix in form (2), then we see that the LU Doolitel factorization of A and B are given by

$$L_A = \begin{pmatrix} 1 & & & & \\ & \ddots & & & & \\ & & 1 & & \\ & & & 1 & & \\ & & \ell_{n+1,n-1} & 1 & \\ & & \ddots & & & & 1 \end{pmatrix}_{(2n-1)\times(2n-1)}$$



where the elements $\ell_{i,j}$ and $u_{i,j}$ are as the following

$$\ell_{n+i,n-i} = \frac{b_{i+1}}{a_{i+1}} \qquad i = 1, 2, ..., n-1,$$

Also

$$\begin{pmatrix} u_{i,2n-i} = b_{n+1-i} & i = 1, 2, ..., n - 1 \\ u_{ii} = a_{n+1-i} & i = 1, 2, ..., n \\ u_{n+i,n+i} = \frac{a_{i+1}^2 - b_{i+1}^2}{a_{i+1}} & i = 1, 2, ..., n - 1. \end{cases}$$

and L_B and U_B in factorization of $B = L_B U_B$ are as below



Remark 1. We observe that the matrices L_A and L_B in LU factorization of central symmetric X-form matrix has a unit λ -matrix.

Corollary 1. If A and B are odd-order and even-order of a central symmetric X-form matrices in form (1) and (2) respectively, then

$$det(A) = a_1 \prod_{i=2}^n (a_i^2 - b_i^2), det(B) = \prod_{i=1}^n (a_i^2 - b_i^2).$$

2.2. **Inverse of** A_n and B_n . It is clear that the necessary and sufficient conditions for invertibility of A_n are $a_1 \neq 0$ and $a_i \neq \pm b_i$ for i = 2, 3, ..., n. If the matrix Φ be the inverse of A_n , then we have $A_n \Phi = I$. If the elements of column j of Φ be $(\Phi_{1j}, \Phi_{2j}, ..., \Phi_{nj})^T$ then we have the following linear system of equations,

$$\begin{pmatrix} a_n & & & & b_n \\ a_{n-1} & & & b_{n-1} \\ & \ddots & & \ddots & & \\ & & a_2 & b_2 & & & \\ & & a_1 & & & \\ & & b_2 & a_2 & & \\ & & \ddots & & \ddots & \\ & & b_{n-1} & & & a_{n-1} \\ b_n & & & & & a_n \end{pmatrix} \begin{pmatrix} \Phi_{1,j} \\ \Phi_{2,j} \\ \vdots \\ \Phi_{j,j} \\ \Phi_{j,j} \\ \vdots \\ \Phi_{j+1,j} \\ \vdots \\ \Phi_{2n-2,j} \\ \Phi_{2n-1,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

With solving the above linear system for all column of Φ we have

$$\begin{cases} \Phi_{ii} = \Phi_{2n-i,2n-i} = -\frac{a_{n+1-i}}{b_{n+1-i}^2 - a_{n+1-i}^2} & i = 1, 2, ..., n-1, \\ \Phi_{nn} = \frac{1}{a_1} & i = 1, 2, ..., n-1, \\ \Phi_{2n-i,i} = \Phi_{i,2n-i} = \frac{b_{n+1-i}}{b_{n+1-i}^2 - a_{n+1-i}^2} & i = 1, 2, ..., n-1. \end{cases}$$

and this shows that Φ is also the central symmetric X-form matrix. For the inverse of B_n we also have similar relations.

3. Inverse eigenvalue problem

Theorem 1. Assume $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ for j = 1, ..., n are the 2n - 1 distinct real numbers, then there exist a central symmetric X-form matrix in form (1) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of submatrix A_j respectively in form (3) if and only if

(4)
$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}.$$

Proof. Existence of matrices A_n such that $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are the its maximal and minimal eigenvalues respectively of its submatrix for j = 1, 2, ..., n is equivalence to finding the solution for the following linear system of equations:

(5)
$$p_j(\lambda_1^{(j)}) = \left[(a_j - \lambda_1^{(j)})^2 - b_j^2 \right] p_{j-1}(\lambda_1^{(j)}) = 0,$$

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(6)
$$p_j(\lambda_j^{(j)}) = [(a_j - \lambda_j^{(j)})^2 - b_j^2] p_{j-1}(\lambda_j^{(j)}) = 0,$$

(5)
$$\Longrightarrow$$
 $[(a_j - \lambda_1^{(j)})^2 - b_j^2] [(a_{j-1} - \lambda_1^{(j)})^2 - b_{j-1}^2] \cdots [(a_2 - \lambda_1^{(j)})^2 - b_2^2] [a_1 - \lambda_1^{(j)}] = 0,$

(6)
$$\implies [(a_j - \lambda_j^{(j)})^2 - b_j^2] [(a_{j-1} - \lambda_j^{(j)})^2 - b_{j-1}^2] \cdots [(a_2 - \lambda_j^{(j)})^2 - b_2^2] [a_1 - \lambda_j^{(j)}] = 0.$$

Thus
 $(a_j - \lambda_j^{(j)})^2 - b_j^2 = 0 \text{ for } j = 2, 3, ..., n$

$$(a_j - \lambda_1^{(j)})^2 - b_j^2 = 0$$
 for $j = 2, 3, ..., n$
 $(a_j - \lambda_j^{(j)})^2 - b_j^2 = 0$ for $j = 2, 3, ..., n$.

Then $a_1 = \lambda_1^{(1)}$ and whereas $\lambda_1^{(j)} \neq \lambda_j^{(j)}$, we have $a_j = \frac{\lambda_1^{(j)} + \lambda_j^{(j)}}{2}$ and $b_j^2 = (\frac{\lambda_j^{(j)} - \lambda_1^{(j)}}{2})^2$ for j = 2, 3, ..., n, therefore we can find all entries of matrix A_n .

Conversely since $p_j(\lambda) = [(a_j - \lambda)^2 - b_j^2] p_{j-1}(\lambda)$, then each root of p_{j-1} is a root of p_j , and we know that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of submatrix A_j in form (3) respectively, thus $\lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)}$ are in between $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, i.e

(7)
$$\lambda_1^{(j)} < \lambda_1^{(j-1)} < \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}.$$

and so on we can write

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}.$$

So the proof is completed. \Box

Theorem 2. Assume $\lambda_1^{(2j)}$, $\lambda_{2j}^{(2j)}$ for j = 1, ..., n are the 2n distinct real numbers, then there exist an even-order central symmetric X-form matrix in form (2) such that $\lambda_1^{(2j)}$ and $\lambda_{2j}^{(2j)}$ are the minimal and maximal eigenvalues of submatrix B_j respectively, if and only if

(8)
$$\lambda_1^{(2n)} < \lambda_1^{(2n-2)} < \ldots < \lambda_1^{(2)} < \lambda_2^{(2)} < \ldots < \lambda_{2n}^{(2n)}$$

Proof. Proof is similar to proof of Theorem 1. \Box

Remark 2. If b_j for j = 2, 3, ..., n, are positive, then the matrix A_n is unique. **Remark 3.** Whereas all eigenvalues of A_{j-1} are the subset of eigenvalues A_j then all eigenvalues relation (4) are all eigenvalues of A_n .

Lemma 2. If A_n is a central symmetric X-form matrix in form (1), then we have (a) $\lambda_1^{(j)} < a_j < \lambda_j^{(j)}$ where $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of submatrix of A_n , for j = 2, 3, ..., n.

(b) If $b_i > 0$ for i = 2, ..., j, then $||A_j||_{\infty} = ||A_j||_1 = \lambda_j^{(j)}$, and if $b_i < 0$ for i = 2, ..., j, then $||A_j||_{\infty} = ||A_j||_1 = \lambda_1^{(j)}$.

Proof. (a) According to the previous theorem 2, we have $a_j = \frac{\lambda_1^{(j)} + \lambda_j^{(j)}}{2}$ and also we have $\lambda_1^{(j)} < \lambda_j^{(j)}$ for j = 2, ..., n, then

$$\frac{\lambda_1^{(j)} + \lambda_1^{(j)}}{2} < \frac{\lambda_1^{(j)} + \lambda_j^{(j)}}{2} < \frac{\lambda_j^{(j)} + \lambda_j^{(j)}}{2} \implies \lambda_1^{(j)} < a_j < \lambda_j^{(j)}.$$

(b) Case (I): $b_i > 0$ for i = 2, ..., n, then

$$||A_j||_{\infty} = ||A_j||_1 = \max\{a_i + b_i = \frac{\lambda_1^{(i)} + \lambda_i^{(i)}}{2} + \frac{\lambda_i^{(i)} - \lambda_1^{(i)}}{2} = \lambda_i^{(i)} \quad i = 2, ..., j\} = \lambda_j^{(j)}$$

for j=2,...,n. Case (II): $b_i < 0$ for i = 2,...,n, then

$$||A_{j}||_{\infty} = ||A_{j}||_{1} = \max \{a_{i} + b_{i} = \frac{\lambda_{1}^{(i)} + \lambda_{i}^{(i)}}{2} + \frac{\lambda_{1}^{(i)} - \lambda_{i}^{(i)}}{2} = \lambda_{1}^{(i)} \quad i = 2, ..., j\} = \lambda_{1}^{(j)}$$

for j = 2, ..., n.

so that proof is completed. \Box

4. Inverse singular value problem

In this section we study two inverse singular value problems as below: **problem I.** Given 2n - 1 nonnegative real numbers $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n. We find $(2n - 1) \times (2n - 1)$ central symmetric X-form matrix A_n in form (1), such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n, are minimal and maximal singular value of submatrix A_j of A_n in form (3), and for given 2n nonnegative real numbers $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for j = 1, 2, ..., n, similarly we find $(2n) \times (2n)$ central symmetric X-form matrix B_n in form (2), such that $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for j = 1, 2, ..., n, are minimal and maximal singular value of submatrix B_j of B_n .

problem II. Given 2n - 1 nonnegative real numbers $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n, we find the λ -matrix Λ_n in form (9) such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n, are the minimal and maximal singular values of submatrix Λ_j from Λ_n , where



Furthermore for 2n given nonnegative real numbers $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for j = 1, 2, ..., n, we find the λ -matrix Γ_n in form (10) such that $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for j = 1, 2, ..., n, are the minimal and maximal singular values of submatrix Γ_j from Γ_n , where



Theorem 3. Assume $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n are the (2n-1) real nonnegative numbers, then there exist a central symmetric X-form matrix in form (1) such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ are the minimal and maximal singular values of submatrix A_j respectively in form (3) if and only if $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n satisfy in the following

relation:

(11)
$$\sigma_1^{(n)} < \sigma_1^{(n-1)} < \dots < \sigma_1^{(2)} < \sigma_1^{(1)} < \sigma_2^{(2)} < \dots < \sigma_n^{(n)}$$

Proof. Let $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n be the real nonnegative number that satisfy in (11). It is clear that $(\sigma_1^{(j)})^2$ and $(\sigma_1^{(j)})^2$ for j = 1, 2, ..., n, satisfy in there relations, this means

(12)
$$(\sigma_1^{(n)})^2 < (\sigma_1^{(n-1)})^2 < \dots < (\sigma_1^{(2)})^2 < (\sigma_1^{(1)})^2 < (\sigma_2^{(2)})^2 < \dots < (\sigma_n^{(n)})^2.$$

By Theorem 1 there exist an odd-order central symmetric X-form matrix that $(\sigma_1^{(j)})^2$ and $(\sigma_j^{(j)})^2$ are the minimal and maximal eigenvalues of its submatrices respectively. We show this matrix by A_n as follows:



On the other hand if C_n be an odd-order central symmetric X-form matrix as follows



then

Since $A_n = C_n C_n^T$, we can find the all elements of matrix $C_n C_n^T$ as the following

$$a_{1} = \alpha_{1}^{2}, a_{i} = \alpha_{i}^{2} + \beta_{i}^{2}, \quad i = 2, 3, ..., n b_{i} = 2\alpha_{i}\beta_{i}, \qquad i = 2, 3, ..., n$$

by combination of the above relations we have

$$\begin{cases} (\alpha_i + \beta_i)^2 = a_i + b_i \\ (\alpha_i - \beta_i)^2 = a_i - b_i \end{cases} \implies \begin{cases} \alpha_i = \frac{\sqrt{(a_i + b_i)} + \sqrt{(a_i - b_i)}}{2} \\ \beta_i = \frac{\sqrt{(a_i + b_i)} - \sqrt{(a_i - b_i)}}{2} \end{cases} i = 2, 3, \dots, n,$$

Therefore the matrix C_n is solution of our problem.

Conversely at first, assume C_n is a matrix of form (1) of order $(2n-1) \times (2n-1)$ such that σ_1^j and σ_j^j are the minimal and maximal singular values of submatrix C_j in form (3) respectively. Then $(\sigma_1^j)^2$ and $(\sigma_j^j)^2$ are the minimal and maximal eigenvalues of submatrices $(C_n C_n^T)_j$ of $C_n C_n^T$ respectively. By Theorem 1 we have

(15)
$$(\sigma_1^{(n)})^2 < (\sigma_1^{(n-1)})^2 < \dots < (\sigma_1^{(2)})^2 < (\sigma_1^{(1)})^2 < (\sigma_2^{(2)})^2 < \dots < (\sigma_n^{(n)})^2$$
,
consequently we have

$$\sigma_1^{(n)} < \sigma_1^{(n-1)} < \dots < \sigma_1^{(2)} < \sigma_1^{(1)} < \sigma_2^{(2)} < \dots < \sigma_n^{(n)},$$

and proof will be completed. \Box

Remark 4. There is a similar result for even-order of above Theorem.

Theorem 4. Assume $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n are the (2n - 1) positive real numbers, then there exist a matrix in form (9) such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ are the minimal and maximal singular values of submatrix Λ_j of Λ_n respectively, if and only if $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, ..., n satisfy in the following relation

(16)
$$\sigma_1^{(n)} < \sigma_1^{(n-1)} < \dots < \sigma_1^{(2)} < \sigma_1^{(1)} < \sigma_2^{(2)} < \dots < \sigma_n^{(n)}$$

Proof. Assume $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$, are 2n-1 positive real numbers which satisfy in the relation (11), consider the squares $(\sigma_1^{(j)})^2$ and $(\sigma_j^{(j)})^2$ for j = 1, ..., n, it is clear that

these satisfy in relation (15), then from Theorem 1 there exist a central symmetric X-form matrix



such that $(\sigma_1^{(j)})^2$ and $(\sigma_1^{(j)})^2$ for j = 1, 2, ..., n, are the minimal and maximal eigenvalues of A_j from A respectively. We observe that if matrix Λ has form (9) then $\Lambda\Lambda^T$ has form (1) as follows



Now we set $\alpha_j^2 = a_j$ j = 1, ..., n and $\beta_j \alpha_j = b_j$, j = 2, ..., n, to compute the entries of an $(2n-1) \times (2n-1)$ matrix Λ of the form (9) with the prescribed extremal singular values for the submatrices Λ_j

The proof of the second part is similar to the proof of inverse Theorem 3 . \Box **Remark 5.** There is a similar result for even-order of above Theorem.

5. Examples

Example 1. Assume n = 5 and given 9 real numbers as below

$$\lambda_1^{(5)} \quad \lambda_1^{(4)} \quad \lambda_1^{(3)} \quad \lambda_1^{(2)} \quad \lambda_1^{(1)} \quad \lambda_2^{(2)} \quad \lambda_3^{(3)} \quad \lambda_4^{(4)} \quad \lambda_5^{(5)},$$

-5 -3 0 2 6 9 10 12 23,

find the central symmetric X-form matrix such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for j = 1, 2, 3, 4, 5 are the eigenvalues of submatrix A_j respectively.

Solution. By theorem 1 and some simple calculations, the solution of problem obtain

in the following form

9	0	0	0	0	0	0	0	14	
0	4.5	0	0	0	0	0	7.5	0	
0	0	5	0	0	0	5	0	0	
0	0	0	5.5	0	3.5	0	0	0	
0	0	0	0	6	0	0	0	0	
0	0	0	3.5	0	5.5	0	0	0	
0	0	5	0	0	0	5	0	0	
0	7.5	0	0	0	0	0	4.5	0	
14	0	0	0	0	0	0	0	9	Ϊ

Example 2. Assume n = 5, given 9 real numbers as below $\sigma_1^{(5)} \quad \sigma_1^{(4)} \quad \sigma_1^{(3)} \quad \sigma_1^{(2)} \quad \sigma_1^{(1)} \quad \sigma_2^{(2)} \quad \sigma_3^{(3)} \quad \sigma_4^{(4)} \quad \sigma_4^{(5)}$

 $0.51338 \quad 0.56793 \quad 0.6448 \quad 0.76537 \quad 1 \quad 1.8478 \quad 2.5080 \quad 3.065 \quad 3.554$

find the central symmetric X-form matrix C_n and λ -matrix Λ_n such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, 3, 4, 5 are the singular values of submatrices Λ_j for j = 1, 2, 3, 4, 5 respectively such that Λ_j has form (9) and C_j has form (3).

Solution. At first we find X-form matrix A by Theorem 1 as below

	6.452214	0	0	0	0	0	0	0	6.188655	\
	0	4.858691	0	0	0	0	0	4.536147	0	
	0	0	3.352916	0	0	0	2.937148	0	0	
	0	0	0	2.000078	0	1.414287	0	0	0	
A =	0	0	0	0	1	0	0	0	0	,
	0	0	0	1.414287	0	2.000078	0	0	0	
	0	0	2.937148	0	0	0	3.352916	0	0	
	0	4.536147	0	0	0	0	0	4.858691	0	
	6.188655	0	0	0	0	0	0	0	6.452214	/
					-					

such that $(\sigma_1^{(j)})^2$ and $(\sigma_j^{(j)})^2$ for j = 1, 2, 3, 4, 5 are the minimal and maximal eigenvalues of submatrices A respectively in form (3). Then by Theorem 3 we find X-form matrix C_n , such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, 3, 4, 5 are the minimal and maximal singular values of submatrices C_n respectively in form (3)

	(-2.03369)	0	0	0	0	0	0	0	-1.52101
	0	1.816515	0	0	0	0	0	1.248585	0
	0	0	-1.5764	0	0	0	-0.9316	0	0
	0	0	0	-1.306585	0	-0.541215	0	0	0
$C_n =$	0	0	0	0	1	0	0	0	0
	0	0	0	-0.541215	0	-1.306585	0	0	0
	0	0	-0.9316	0	0	0	-1.5764	0	0
	0	-1.248585	0	0	0	0	0	1.8165155	0
	-1.52101	0	0	0	0	0	0	0	-2.03369 /

Then by Theorem 4 we find the λ -form matrix Λ_n such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for j = 1, 2, 3, 4, 5 are the minimal and maximal singular values of submatrices Λ_n respectively in form (9)

	1	2.540121	0	0	0	0	0	0	0	0	\
	(0	2.204244	Õ	0	õ	0	0	0	0	
		0	0	1.831097	0	0	0	0	0	0	
		0	0	0	1.414241	0	0	0	0	0	
$\Lambda_n =$		0	0	0	0	1	0	0	0	0	.
		0	0	0	1.000032	0	1.000001	0	0	0	
		0	0	1.604038	0	0	0	0.883164	0	0	
		0	2.057915	0	0	0	0	0	0.789732	0	
	ſ	2.436362	0	0	0	0	0	0	0	0.718577)

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