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Properties of Central Symmetric X-Form Matrices

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Abstract. In this paper we introduce a special form of symmetric matrices that is called central symmetric X-form matrix and study some of their properties, the inverse eigenvalue problem and inverse singular value problem for these matrices.

Keywords: Inverse eigenvalue problem, Inverse singular value problem, eigenvalue, singular value.

2000 Mathematics subject classification: 15A29; 15A18.

1. Introduction

In [3,4] H. Pickman et. studied the inverse eigenvalue problem of symmetric tridiagonal and symmetric bordered diagonal matrices. In this paper we introduce the odd and even order central symmetric X-form matrix for an integer number n respectively as below:
suppose

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$$A_n = \begin{pmatrix} a_n & & & & & & b_n \\ & \ddots & & & & ⋰ & \\ & & a_2 & & b_2 & & \\ & & & a_1 & & & \\ & & b_2 & & a_2 & & \\ & ⋰ & & & & \ddots & \\ b_n & & & & & & a_n \end{pmatrix}_{(2n-1)\times(2n-1)}, \tag{1}$$

$$B_n = \begin{pmatrix} a_n & & & & & & & b_n \\ & \ddots & & & & & ⋰ & \\ & & a_2 & & & b_2 & & \\ & & & a_1 & b_1 & & & \\ & & & b_1 & a_1 & & & \\ & & b_2 & & & a_2 & & \\ & ⋰ & & & & & \ddots & \\ b_n & & & & & & & a_n \end{pmatrix}_{(2n)\times(2n)}. \tag{2}$$

For a given $2n-1$ real numbers such as

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)},$$

or for a given $2n$ real numbers such as

$$\lambda_1^{(2n)} < \lambda_1^{(2(n-1))} < \ldots < \lambda_1^{(2)} < \lambda_2^{(2)} < \ldots < \lambda_{2(n-1)}^{(2(n-1))} < \lambda_{2n}^{(2n)},$$

we construct a matrix A_n such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the maximal and minimal eigenvalues of submatrix A_j respectively for $j = 1, 2, ...n$ where $\mathrm{A}_j isdefinedbyto$

$$\mathrm{A}_j = \begin{pmatrix} a_j & & & & & & b_j \\ & \ddots & & & & ⋰ & \\ & & a_2 & & b_2 & & \\ & & & a_1 & & & \\ & & b_2 & & a_2 & & \\ & ⋰ & & & & \ddots & \\ b_j & & & & & & a_j \end{pmatrix}_{(2j-1)\times(2j-1)}, \text{ and} \tag{3}$$

matrix B_j such that $\lambda_1^{(2j)}$ and $\lambda_{2j}^{(2j)}$ are the maximal and minimal eigenvalues of submatrix B_j respectively for $j = 1, 2, ...n$ where $\mathrm{B}_j isdefinedby B_j =$

$$\begin{pmatrix} a_j & & & & & & & b_j \\ & \ddots & & & & & \iddots & \\ & & a_2 & & & b_2 & & \\ & & & a_1 & b_1 & & & \\ & & & b_1 & a_1 & & & \\ & & b_2 & & & a_2 & & \\ & \iddots & & & & & \ddots & \\ b_j & & & & & & & a_j \end{pmatrix}_{(2j)\times(2j)}.$$

2. Properties of the matrices A_n and B_n

Let $p_0(\lambda) = 1$, $q_0(\lambda) = 1$, $p_j(\lambda) = det(A_j - \lambda_j)$ for $j = 1, 2, ..., n$ and $q_j(\lambda) = det(B_j - \lambda_j)$ for $j = 1, 2, ..., n$.

Lemma 1. *For a given matrix A_j and B_j the sequence $p_j(\lambda)$ and $q_j(\lambda)$ satisfy in the following recurrence relations:*

a) $p_1(\lambda) = (a_1 - \lambda)$,
b) $p_j(\lambda) = [(a_j - \lambda)^2 - b_j^2]p_{j-1}(\lambda),\ j = 2, 3, ..., n.$
c) $q_1(\lambda) = ((a_1 - \lambda)^2 - b_1^2)$,
b) $q_j(\lambda) = [(a_j - \lambda)^2 - b_j^2]q_{j-1}(\lambda),\ j = 2, 3, ..., n.$

Proof. The proof is clear by extending determinants of $(A_j - \lambda I_j)$ and $(B_j - \lambda I_j)$ on their first columns.

2.1. LU factorization of central symmetric X-form matrix.

Let A be a central symmetric X-form matrix in form (1) and B be a central symmetric X-form matrix in form (2), then we see that the LU Doolitel factorization of A and B are given by

$$L_A = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & \ell_{n+1,n-1} & & 1 & \\ & \iddots & & & & \\ \ell_{2n-1,1} & & & & & 1 \end{pmatrix}_{(2n-1)\times(2n-1)},$$

$$U_A = \begin{pmatrix} u_{1,1} & & & & & & u_{1,2n-1} \\ & \ddots & & & & \iddots & \\ & & u_{n-1,n-1} & & u_{n-1,n+1} & & \\ & & & u_{n,n} & & & \\ & & & & u_{n+1,n+1} & & \\ & & & & & \ddots & \\ & & & & & & u_{2n-1,2n-1} \end{pmatrix}_{(2n-1)\times(2n-1)},$$

where the elements $\ell_{i,j}$ and $u_{i,j}$ are as the follwing

$$\ell_{n+i,n-i} = \frac{b_{i+1}}{a_{i+1}} \qquad i = 1, 2, ..., n-1,$$

Also

$$\begin{cases} u_{i,2n-i} = b_{n+1-i} & i = 1, 2, .., n-1 \\ u_{ii} = a_{n+1-i} & i = 1, 2, ..., n \\ u_{n+i,n+i} = \frac{a_{i+1}^2 - b_{i+1}^2}{a_{i+1}} & i = 1, 2, ..., n-1. \end{cases}$$

and L_B and U_B in factorization of $B = L_B U_B$ are as below

$$L_B = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & \frac{b_1}{a_1} & & 1 & \\ & \iddots & & & & \ddots \\ \frac{b_n}{a_n} & & & & & 1 \end{pmatrix}_{(2n)\times(2n)},$$

$$U_B = \begin{pmatrix} a_n & & & & & & b_n \\ & \ddots & & & & \iddots & \\ & & a_2 & & b_2 & & \\ & & & a_1 \;\; b_1 & & & \\ & & & \frac{a_1^2 - b_1^2}{a_1} & & & \\ & & & & \frac{a_2^2 - b_2^2}{a_2} & & \\ & & & & & \ddots & \\ & & & & & & \frac{a_n^2 - b_n^2}{a_n} \end{pmatrix}_{(2n)\times(2n)},$$

Remark 1. We observe that the matrices L_A and L_B in LU factorization of central symmetric X-form matrix has a unit λ-matrix.

Corollary 1. *If A and B are odd-order and even-order of a central symmetric X-form matrices in form* (1) *and (2) respectively, then*

$$\det(A) = a_1 \prod_{i=2}^{n} (a_i^2 - b_i^2),$$
$$\det(B) = \prod_{i=1}^{n} (a_i^2 - b_i^2).$$

2.2. **Inverse of A_n and B_n.** It is clear that the necessary and sufficient conditions for invertibility of A_n are $a_1 \neq 0$ and $a_i \neq \pm b_i$ for $i = 2, 3, ..., n$. If the matrix Φ be the inverse of A_n, then we have $A_n\Phi = I$. If the elements of column j of Φ be $(\Phi_{1j}, \Phi_{2j}, ..., \Phi_{nj})^T$ then we have the following linear system of equations,

$$\begin{pmatrix} a_n & & & & & & b_n \\ & a_{n-1} & & & & b_{n-1} & \\ & & \ddots & & \iddots & & \\ & & a_2 & & b_2 & & \\ & & & a_1 & & & \\ & & b_2 & & a_2 & & \\ & & \iddots & & \ddots & & \\ & b_{n-1} & & & & a_{n-1} & \\ b_n & & & & & & a_n \end{pmatrix} \begin{pmatrix} \Phi_{1,j} \\ \Phi_{2,j} \\ \vdots \\ \Phi_{j-1,j} \\ \Phi_{j,j} \\ \Phi_{j+1,j} \\ \vdots \\ \Phi_{2n-2,j} \\ \Phi_{2n-1,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

With solving the above linear system for all column of Φ we have

$$\begin{cases} \Phi_{ii} = \Phi_{2n-i,2n-i} = -\frac{a_{n+1-i}}{b^2_{n+1-i} - a^2_{n+1-i}} & i = 1, 2, .., n-1, \\ \Phi_{nn} = \frac{1}{a_1} & \\ \Phi_{2n-i,i} = \Phi_{i,2n-i} = \frac{b_{n+1-i}}{b^2_{n+1-i} - a^2_{n+1-i}} & i = 1, 2, ..., n-1. \end{cases},$$

and this shows that Φ is also the central symmetric X-form matrix. For the inverse of B_n we also have similar relations.

3. Inverse eigenvalue problem

Theorem 1. *Assume $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ for $j = 1, ..., n$ are the $2n-1$ distinct real numbers, then there exist a central symmetric X-form matrix in form (1) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of submatrix A_j respectively in form (3) if and only if*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}. \tag{4}$$

Proof. Existence of matrices A_n such that $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are the its maximal and minimal eigenvalues respectively of its submatrix for $j = 1, 2, ..., n$ is equivalence to finding the solution for the following linear system of equations:

$$p_j(\lambda_1^{(j)}) = [(a_j - \lambda_1^{(j)})^2 - b_j^2]\, p_{j-1}(\lambda_1^{(j)}) = 0, \tag{5}$$

$$p_j(\lambda_j^{(j)}) = [(a_j - \lambda_j^{(j)})^2 - b_j^2]\, p_{j-1}(\lambda_j^{(j)}) = 0, \tag{6}$$

$$(5) \Longrightarrow \quad [(a_j - \lambda_1^{(j)})^2 - b_j^2]\, [(a_{j-1} - \lambda_1^{(j)})^2 - b_{j-1}^2]\, \cdots\, [(a_2 - \lambda_1^{(j)})^2 - b_2^2]\, [a_1 - \lambda_1^{(j)}] = 0,$$

$$(6) \Longrightarrow \quad [(a_j - \lambda_j^{(j)})^2 - b_j^2]\, [(a_{j-1} - \lambda_j^{(j)})^2 - b_{j-1}^2]\, \cdots\, [(a_2 - \lambda_j^{(j)})^2 - b_2^2]\, [a_1 - \lambda_j^{(j)}] = 0.$$

Thus

$$\begin{array}{lll} (a_j - \lambda_1^{(j)})^2 - b_j^2 = 0 & \text{for} & j = 2, 3, ..., n \\ (a_j - \lambda_j^{(j)})^2 - b_j^2 = 0 & \text{for} & j = 2, 3, ..., n. \end{array}$$

Then $a_1 = \lambda_1^{(1)}$ and whereas $\lambda_1^{(j)} \neq \lambda_j^{(j)}$, we have $a_j = \frac{\lambda_1^{(j)}+\lambda_j^{(j)}}{2}$ and $b_j^2 = (\frac{\lambda_j^{(j)}-\lambda_1^{(j)}}{2})^2$ for $j = 2, 3, ..., n$, therefore we can find all entries of matrix A_n.
Conversely since $p_j(\lambda) = [(a_j - \lambda)^2 - b_j^2]\ p_{j-1}(\lambda)$, then each root of p_{j-1} is a root of p_j, and we know that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of submatrix A_j in form (3) respectively, thus $\lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)}$ are in between $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, i.e

$$\lambda_1^{(j)} < \lambda_1^{(j-1)} < \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}. \tag{7}$$

and so on we can write

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}.$$

So the proof is completed. □

Theorem 2. *Assume $\lambda_1^{(2j)}$, $\lambda_{2j}^{(2j)}$ for $j = 1, ..., n$ are the $2n$ distinct real numbers, then there exist an even-order central symmetric X-form matrix in form (2) such that $\lambda_1^{(2j)}$ and $\lambda_{2j}^{(2j)}$ are the minimal and maximal eigenvalues of submatrix B_j respectively, if and only if*

$$\lambda_1^{(2n)} < \lambda_1^{(2n-2)} < \ldots < \lambda_1^{(2)} < \lambda_2^{(2)} < \ldots < \lambda_{2n}^{(2n)}. \tag{8}$$

Proof. Proof is similar to proof of Theorem 1. □
Remark 2. If b_j for $j = 2, 3, ..., n$, are positive, then the matrix A_n is unique.
Remark 3. Whereas all eigenvalues of A_{j-1} are the subset of eigenvalues A_j then all eigenvalues relation (4) are all eigenvalues of A_n.

Lemma 2. *If A_n is a central symmetric X-form matrix in form (1), then we have*
(a) $\lambda_1^{(j)} < a_j < \lambda_j^{(j)}$ where $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of submatrix of A_n, for $j = 2, 3, ..., n$.
(b) If $b_i > 0$ for $i = 2, ..., j$, then $\| A_j \|_\infty = \| A_j \|_1 = \lambda_j^{(j)}$, and if $b_i < 0$ for $i = 2, ..., j$, then $\| A_j \|_\infty = \| A_j \|_1 = \lambda_1^{(j)}$.

Proof. (a) According to the previous theorem 2, we have $a_j = \frac{\lambda_1^{(j)}+\lambda_j^{(j)}}{2}$ and also we have $\lambda_1^{(j)} < \lambda_j^{(j)}$ for $j = 2, ..., n$, then

$$\frac{\lambda_1^{(j)} + \lambda_1^{(j)}}{2} < \frac{\lambda_1^{(j)} + \lambda_j^{(j)}}{2} < \frac{\lambda_j^{(j)} + \lambda_j^{(j)}}{2} \quad \Longrightarrow \quad \lambda_1^{(j)} < a_j < \lambda_j^{(j)}.$$

(b) *Case (I):* $b_i > 0$ for $i = 2, ..., n$, then

$$\| A_j \|_\infty = \| A_j \|_1 = \max\{a_i + b_i = \frac{\lambda_1^{(i)} + \lambda_i^{(i)}}{2} + \frac{\lambda_i^{(i)} - \lambda_1^{(i)}}{2} = \lambda_i^{(i)} \quad i = 2, ..., j\} = \lambda_j^{(j)}$$

for j=*2,…,n.*
Case (II): $b_i < 0$ for $i = 2, ..., n$, then

$$\| A_j \|_\infty = \| A_j \|_1 = \max\{a_i + b_i = \frac{\lambda_1^{(i)} + \lambda_i^{(i)}}{2} + \frac{\lambda_1^{(i)} - \lambda_i^{(i)}}{2} = \lambda_1^{(i)} \quad i = 2, ..., j\} = \lambda_1^{(j)}$$

for j=*2,…,n.*

so that proof is completed. □

4. Inverse singular value problem

In this section we study two inverse singular value problems as below:
problem I. Given $2n-1$ nonnegative real numbers $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$. We find $(2n-1) \times (2n-1)$ central symmetric X-form matrix A_n in form (1), such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$, are minimal and maximal singular value of submatrix A_j of A_n in form (3), and for given $2n$ nonnegative real numbers $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for $j = 1, 2, ..., n$, similarly we find $(2n) \times (2n)$ central symmetric X-form matrix B_n in form (2), such that $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for $j = 1, 2, ..., n$, are minimal and maximal singular value of submatrix B_j of B_n.

problem II. Given $2n-1$ nonnegative real numbers $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$, we find the λ-matrix Λ_n in form (9) such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$, are the minimal and maximal singular values of submatrix Λ_j from Λ_n, where

$$(9)\quad \Lambda_n = \begin{pmatrix} \alpha_n & & & & & 0 \\ & \ddots & & & \iddots & \\ & & \alpha_2 & 0 & & \\ & & & \alpha_1 & & \\ & & \beta_2 & \sqrt{\alpha_2^2-\beta_2^2} & & \\ & \iddots & & & \ddots & \\ \beta_n & & & & & \sqrt{\alpha_n^2-\beta_n^2} \end{pmatrix}_{(2n-1)\times(2n-1)}.$$

Furthermore for $2n$ given nonnegative real numbers $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for $j = 1, 2, ..., n$, we find the λ-matrix Γ_n in form (10) such that $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for $j = 1, 2, ..., n$, are the minimal and maximal singular values of submatrix Γ_j from Γ_n, where

$$(10)\quad \Gamma_n = \begin{pmatrix} \alpha_n & & & & & & 0 \\ & \ddots & & & & \iddots & \\ & & \alpha_2 & & & & \\ & & & \alpha_1 & 0 & & \\ & & & 0 & \alpha_1 & & \\ & & \beta_2 & & & \sqrt{\alpha_2^2-\beta_2^2} & \\ & \iddots & & & & \ddots & \\ \beta_n & & & & & & \sqrt{\alpha_n^2-\beta_n^2} \end{pmatrix}_{(2n)\times(2n)}.$$

Theorem 3. *Assume $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$ are the $(2n-1)$ real nonnegative numbers, then there exist a central symmetric X-form matrix in form (1) such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ are the minimal and maximal singular values of submatrix A_j respectively in form (3) if and only if $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$ satisfy in the following* *relation:*

$$\sigma_1^{(n)} < \sigma_1^{(n-1)} < \cdots < \sigma_1^{(2)} < \sigma_1^{(1)} < \sigma_2^{(2)} < \ldots < \sigma_n^{(n)} \tag{11}$$

Proof. Let $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$ be the real nonnegative number that satisfy in (11). It is clear that $(\sigma_1^{(j)})^2$ and $(\sigma_1^{(j)})^2$ for $j = 1, 2, ..., n$, satisfy in there relations, this means

$$(\sigma_1^{(n)})^2 < (\sigma_1^{(n-1)})^2 < \cdots < (\sigma_1^{(2)})^2 < (\sigma_1^{(1)})^2 < (\sigma_2^{(2)})^2 < \ldots < (\sigma_n^{(n)})^2. \tag{12}$$

By Theorem 1 there exist an odd-order central symmetric X-form matrix that $(\sigma_1^{(j)})^2$ and $(\sigma_j^{(j)})^2$ are the minimal and maximal eigenvalues of its submatrices respectively. We show this matrix by A_n as follows:

$$A_n = \begin{pmatrix} a_n & & & & & & b_n \\ & \ddots & & & & \iddots & \\ & & a_2 & & b_2 & & \\ & & & a_1 & & & \\ & & b_2 & & a_2 & & \\ & \iddots & & & & \ddots & \\ b_n & & & & & & a_n \end{pmatrix}_{(2n-1)\times(2n-1)}, \tag{13}$$

where

$$a_i = \frac{(\sigma_1^{(i)})^2 + (\sigma_i^{(i)})^2}{2}, \qquad i = 1, 2, ..., n$$

and

$$b_i = \frac{((\sigma_i^{(i)})^2 - (\sigma_1^{(i)})^2)^2}{2}. \qquad i = 2, 3, ..., n$$

On the other hand if C_n be an odd-order central symmetric X-form matrix as follows

$$C_n = \begin{pmatrix} \alpha_n & & & & & & \beta_n \\ & \ddots & & & & \iddots & \\ & & \alpha_2 & & \beta_2 & & \\ & & & \alpha_1 & & & \\ & & \beta_2 & & \alpha_2 & & \\ & \iddots & & & & \ddots & \\ \beta_n & & & & & & \alpha_n \end{pmatrix}_{(2n-1)\times(2n-1)},$$

then

$$
(14)\quad C_nC_n^T = \begin{pmatrix}
\alpha_n^2+\beta_n^2 & & & & & & 2\alpha_n\beta_n \\
& \ddots & & & & \iddots & \\
& & \alpha_2^2+\beta_2^2 & & 2\alpha_2\beta_2 & & \\
& & & \alpha_1^2 & & & \\
& & 2\alpha_2\beta_2 & & \alpha_2^2+\beta_2^2 & & \\
& \iddots & & & & \ddots & \\
2\alpha_n\beta_n & & & & & & \alpha_n^2+\beta_n^2
\end{pmatrix}_{(2n-1)\times(2n-1)}.
$$

Since $A_n = C_nC_n^T$, we can find the all elements of matrix $C_nC_n^T$ as the following

$$
\begin{aligned}
&a_1 = \alpha_1^2,\\
&a_i = \alpha_i^2+\beta_i^2, \quad i = 2,3,...,n\\
&b_i = 2\alpha_i\beta_i, \qquad i = 2,3,...,n
\end{aligned}
$$

by combination of the above relations we have

$$
\begin{cases} (\alpha_i+\beta_i)^2 = a_i+b_i \\ \\ (\alpha_i-\beta_i)^2 = a_i-b_i \end{cases} \Longrightarrow \qquad \begin{cases} \alpha_i = \frac{\sqrt{(a_i+b_i)}+\sqrt{(a_i-b_i)}}{2} \\ \beta_i = \frac{\sqrt{(a_i+b_i)}-\sqrt{(a_i-b_i)}}{2} \end{cases} \quad i = 2,3,\dots,n,
$$

Therefore the matrix C_n is solution of our problem.
Conversely at first, assume C_n is a matrix of form (1) of order $(2n-1)\times(2n-1)$ such that σ_1^j and σ_j^j are the minimal and maximal singular values of submatrix C_j in form (3) respectively. Then $(\sigma_1^j)^2$ and $(\sigma_j^j)^2$ are the minimal and maximal eigenvalues of submatrices $(C_nC_n^T)_j$ of $C_nC_n^T$ respectively. By Theorem 1 we have

$$
(15)\ (\sigma_1^{(n)})^2 < (\sigma_1^{(n-1)})^2 < \cdots < (\sigma_1^{(2)})^2 < (\sigma_1^{(1)})^2 < (\sigma_2^{(2)})^2 < \ldots < (\sigma_n^{(n)})^2,
$$

consequently we have

$$
\sigma_1^{(n)} < \sigma_1^{(n-1)} < \cdots < \sigma_1^{(2)} < \sigma_1^{(1)} < \sigma_2^{(2)} < \ldots < \sigma_n^{(n)},
$$

and proof will be completed. □

Remark 4. There is a similar result for even-order of above Theorem.

Theorem 4. *Assume $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1,2,...,n$ are the $(2n-1)$ positive real numbers, then there exist a matrix in form (9) such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ are the minimal and maximal singular values of submatrix Λ_j of Λ_n respectively, if and only if $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1,2,...,n$ satisfy in the following relation*

$$
(16)\qquad \sigma_1^{(n)} < \sigma_1^{(n-1)} < \cdots < \sigma_1^{(2)} < \sigma_1^{(1)} < \sigma_2^{(2)} < \ldots < \sigma_n^{(n)}
$$

Proof. Assume $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$, are $2n-1$ positive real numbers which satisfy in the relation (11), consider the squares $(\sigma_1^{(j)})^2$ and $(\sigma_j^{(j)})^2$ for $j = 1,...,n$, it is clear that these satisfy in relation (15), then from Theorem 1 there exist a central symmetric X-form matrix

$$A = \begin{pmatrix} a_n & & & & & & b_n \\ & \ddots & & & & \iddots & \\ & & a_2 & & b_2 & & \\ & & & a_1 & & & \\ & & b_2 & & a_2 & & \\ & \iddots & & & & \ddots & \\ b_n & & & & & & a_n \end{pmatrix}_{(2n-1)\times(2n-1)}.$$

such that $(\sigma_1^{(j)})^2$ and $(\sigma_1^{(j)})^2$ for $j = 1, 2, ..., n$, are the minimal and maximal eigenvalues of A_j from A respectively. We observe that if matrix Λ has form (9) then $\Lambda\Lambda^T$ has form (1) as follows

$$\Lambda\Lambda^T = \begin{pmatrix} \alpha_n^2 & & & & & & & & \alpha_n\beta_n \\ & \alpha_{n-1}^2 & & & & & & \alpha_{n-1}\beta_{n-1} & \\ & & \ddots & & & & \iddots & & \\ & & & \alpha_2^2 & & \alpha_2\beta_2 & & & \\ & & & & \alpha_1^2 & & & & \\ & & & \alpha_2\beta_2 & & \alpha_2^2 & & & \\ & & \iddots & & & & \ddots & & \\ & \alpha_{n-1}\beta_{n-1} & & & & & & \alpha_{n-1}^2 & \\ \alpha_n\beta_n & & & & & & & & \alpha_n^2 \end{pmatrix}_{(2n-1)\times(2n-1)}.$$

Now we set $\alpha_j^2 = a_j \quad j = 1, ..., n$ and $\beta_j\alpha_j = b_j, \quad j = 2, ..., n$, to compute the entries of an $(2n-1)\times(2n-1)$ matrix Λ of the form (9) with the prescribed extremal singular values for the submatrices Λ_j

The proof of the second part is similar to the proof of inverse Theorem 3 . □

Remark 5. There is a similar result for even-order of above Theorem.

5. Examples

Example 1. Assume $n = 5$ and given 9 real numbers as below

$$\begin{array}{ccccccccc} \lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} & \lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)}, \\ -5 & -3 & 0 & 2 & 6 & 9 & 10 & 12 & 23, \end{array}$$

find the central symmetric X-form matrix such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for $j = 1, 2, 3, 4, 5$ are the eigenvalues of submatrix A_j respectively.

Solution. By theorem 1 and some simple calculations, the solution of problem obtain in the following form

$$\begin{pmatrix} 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \\ 0 & 4.5 & 0 & 0 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5.5 & 0 & 3.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.5 & 0 & 5.5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 7.5 & 0 & 0 & 0 & 0 & 0 & 4.5 & 0 \\ 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}.$$

Example 2. Assume $n = 5$, given 9 real numbers as below

$\sigma_1^{(5)}$	$\sigma_1^{(4)}$	$\sigma_1^{(3)}$	$\sigma_1^{(2)}$	$\sigma_1^{(1)}$	$\sigma_2^{(2)}$	$\sigma_3^{(3)}$	$\sigma_4^{(4)}$	$\sigma_4^{(5)}$
0.51338	0.56793	0.6448	0.76537	1	1.8478	2.5080	3.065	3.554

find the central symmetric X-form matrix C_n and λ-matrix Λ_n such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, 3, 4, 5$ are the singular values of submatrices Λ_j for $j = 1, 2, 3, 4, 5$ respectively such that Λ_j has form (9) and C_j has form (3).

Solution. At first we find X-form matrix A by Theorem 1 as below

$$A = \begin{pmatrix} 6.452214 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.188655 \\ 0 & 4.858691 & 0 & 0 & 0 & 0 & 0 & 4.536147 & 0 \\ 0 & 0 & 3.352916 & 0 & 0 & 0 & 2.937148 & 0 & 0 \\ 0 & 0 & 0 & 2.000078 & 0 & 1.414287 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.414287 & 0 & 2.000078 & 0 & 0 & 0 \\ 0 & 0 & 2.937148 & 0 & 0 & 0 & 3.352916 & 0 & 0 \\ 0 & 4.536147 & 0 & 0 & 0 & 0 & 0 & 4.858691 & 0 \\ 6.188655 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.452214 \end{pmatrix},$$

such that $(\sigma_1^{(j)})^2$ and $(\sigma_j^{(j)})^2$ for $j = 1, 2, 3, 4, 5$ are the minimal and maximal eigenvalues of submatrices A respectively in form (3). Then by Theorem 3 we find X-form matrix C_n, such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, 3, 4, 5$ are the minimal and maximal singular values of submatrices C_n respectively in form (3)

$$C_n = \begin{pmatrix} -2.03369 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.52101 \\ 0 & 1.816515 & 0 & 0 & 0 & 0 & 0 & 1.248585 & 0 \\ 0 & 0 & -1.5764 & 0 & 0 & 0 & -0.9316 & 0 & 0 \\ 0 & 0 & 0 & -1.306585 & 0 & -0.541215 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.541215 & 0 & -1.306585 & 0 & 0 & 0 \\ 0 & 0 & -0.9316 & 0 & 0 & 0 & -1.5764 & 0 & 0 \\ 0 & 1.248585 & 0 & 0 & 0 & 0 & 0 & 1.8165155 & 0 \\ -1.52101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.03369 \end{pmatrix}.$$

Then by Theorem 4 we find the λ-form matrix Λ_n such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, 3, 4, 5$ are the minimal and maximal singular values of submatrices Λ_n respectively in form (9)

$$\Lambda_n = \begin{pmatrix} 2.540121 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.204244 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.831097 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.414241 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.000032 & 0 & 1.000001 & 0 & 0 & 0 \\ 0 & 0 & 1.604038 & 0 & 0 & 0 & 0.883164 & 0 & 0 \\ 0 & 2.057915 & 0 & 0 & 0 & 0 & 0 & 0.789732 & 0 \\ 2.436362 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.718577 \end{pmatrix}.$$

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