Iranian Journal of Mathematical Sciences and Informatics
Vol. 6, No. 2 (2011), pp 33-41

Deformation of Outer Representations of Galois Group II

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Abstract. This paper is devoted to deformation theory of "anabelian" representations of the absolute Galois group landing in outer automorphism group of the algebraic fundamental group of a hyperbolic smooth curve defined over a number-field. In the first part of this paper, we obtained several universal deformations for Lie-algebra versions of the above representation using the Schlessinger criteria for functors on Artin local rings. In the second part, we use a version of Schlessinger criteria for functors on the Artinian category of nilpotent Lie algebras which is formulated by Pridham, and explore arithmetic applications.

Keywords: Deformation theory, Artin local rings, Schlessinger criteria.

2000 Mathematics subject classification: 32Gxx.

Introduction

Based on Grothendieck's anabelian philosophy, the Galois-module structure of outer automorphism group of the geometric fundamental group of a smooth hyperbolic curve should contain all the arithmetic information about X. In particular case, Grothendieck's anabelian conjecture, which was proved by Mochizuki, states that for smooth X and X' hyperbolic curves defined over a number field K, there is a natural one-to-one correspondence

$$Isom_K(X,X') \longrightarrow Out_{Gal(\bar{K}/K)}(Out(\pi_1(\bar{X})), Out(\pi_1(\bar{X}')))$$

Received 20 May 2010; Accepted 10 September 2011

where $\pi_1(\bar{X})$ denotes the geometric fundamental group [9]. Here $\pi_1(\bar{X})$ denotes the geometric fundamental group of X and $Out(\pi_1(\bar{X}))$ denotes the quotient of the automorphism group $Aut(\pi_1(\bar{X}))$ by inner automorphisms, and $Out_{Gal(\bar{K}/K)}$ denotes the set of Galois equivariant isomorphisms between the two profinite groups. The Galois structure of $Out(\pi_1(\bar{X}))$ is given by a continuous group homomorphism

$$\rho_X : Gal(\bar{K}/K) \longrightarrow Out(\pi_1(\bar{X}))$$

associated to the following short exact sequence

$$0 \longrightarrow \pi_1(\bar{X}) \longrightarrow \pi_1(X) \longrightarrow Gal(\bar{K}/K) \longrightarrow 0.$$

By a result of Grothendieck, this outer representation breaks to an outer representation of $Gal(K_S/K)$ for a finite set S of places of K, where K_S is the maximal extension of K which is unramified outside S.
One can induce a filtration on the Galois group using the induced pro-l representation

$$\rho_X^l : Gal(K_S/K) \longrightarrow Out(\pi_1(\bar{X})^{(l)})$$

and the weight filtration on the outer automorphism group. These filtrations induce graded nilpotent $\mathbb{Z}_l$-Lie algebras on both sides and a representation of graded Lie algebras

$$\mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)})$$

which is the object we are planning to deform.
In the first part of this paper, we used the classical Schlessinger criteria for deformations of functors on Artin local rings for deformation of the above representation. Here, we use a graded version of Pridham's adaptation of Schlessinger criteria for functors on finite dimensional nilpotent Lie algebras. The ultimate goal is to have the whole theory of deformations of Galois representations and its relation to modular forms, translated to the language of nilpotent Lie algebras and their representations and use its computational and conceptual advantages.
It is also natural to formulate a Lie algebra version of Grothendieck's anabelian conjectures.

1. Lie algebras associated to profinite groups

Exponential map on the tangent space of an algebraic group defined over a field k of characteristic zero, gives an equivalence of categories between nilpotent Lie algebras of finite dimension over k and unipotent algebraic groups over k. This way, one can associate a Lie algebra to the algebraic unipotent completion $\Gamma^{alg}(\mathbb{Q})$ of any profinite group Γ.
On the other hand, Malčev defines an equivalence of categories between nilpotent Lie algebras over $\mathbb{Q}$ and uniquely dividible nilpotent groups. Inclusion of

such groups in nilpotent groups has a right adjoint $\Gamma \to \Gamma_{\mathbb{Q}}$. For a nipotent group Γ, torsion elements form a subgroup T and $\Gamma_{\mathbb{Q}} = \cup(\Gamma/T)^{1/n}$. In fact we have $\Gamma_{\mathbb{Q}} = \Gamma^{alg}(\mathbb{Q})$.
Any nilpotent finite group is a product of its sylow subgroups. Therefore, the profinite completion $\Gamma^{\wedge}$ factors to pro-l completions $\Gamma_l^{\wedge}$, each one a compact open subgroup of the corresponding $\Gamma^{alg}(\mathbb{Q}_l)$ and we have the following isomorphisms of l-adic Lie groups

$$\mathrm{Lie}(\Gamma_l^{\wedge}) = \mathrm{Lie}(\Gamma^{alg}(\mathbb{Q}_l)) = \mathrm{Lie}(\Gamma^{alg}(\mathbb{Q})) \otimes \mathbb{Q}_l.$$

In fact, the adelic Lie group associated to Γ can be defined as $\mathrm{Lie}(\Gamma^{alg}(\mathbb{Q}))\otimes\mathbb{A}^f$ which is the same as $\prod \mathrm{Lie}(\Gamma_l^{\wedge})$.
Suppose we are given a nilpotent representation of Γ on a finite dimensional vector space V over k, which means that for a filtration F on V respecting the action, the induced action of $Gr_F(V)$ is trivial. The subgroup

$$\{\sigma \in GL(V) | \sigma F = F, Gr_F(\sigma) = 1\}$$

is a uniquely divisible group and one obtains a morphism

$$\mathrm{Lie}(\Gamma_{\mathbb{Q}}) \to \{\sigma \in gl(V) | \sigma F = F, Gr_F(\sigma) = 0\}$$

which is an equivalence of categories between nilpotent representations of Γ and representations of $\Gamma^{alg}(\mathbb{Q})$ and nilpotent representations of the Lie algebra $\mathrm{Lie}(\Gamma_{\mathbb{Q}})$ over the field k. The above equivalence of categories extends to an equivalence between linear representations of Γ and representations of its algebraic envelope [2].
The notion of weighted completion of a group developed by Hain and Matsumoto generalizes the concept of algebraic unipotent completion. Suppose that R is an algebraic k-group and $w : \mathbb{G}_m \to R$ is a central cocharacter. Let G be an extension of R by a unipotent group U in the category of algebraic k-groups

$$0 \longrightarrow U \longrightarrow G \longrightarrow R \longrightarrow 0.$$

The first homology of U is an R-module, and therefore an $\mathbb{G}_m$-module via w, which naturally decomposes to to direct sum of irreducible representations each isomorphic to a power of the standard character. We say that our extension is negatively weighted if only negative powers of the standard character appear in $H_1(U)$. The weighted completion of Γ with respect to the representation ρ with Zariski dense image $\rho : \Gamma \to R(\mathbb{Q}_l)$ is the universal $\mathbb{Q}_l$-proalgebraic group $\mathcal{G}$ which is a negative weighted extension of R by a prounipotent group $\mathcal{U}$ and a continuous lift of ρ to $\mathcal{G}(\mathbb{Q}_l)$ [4]. The Lie algebra of $\mathcal{G}(\mathbb{Q}_l)$ is a more sophisticated version of $\mathrm{Lie}(\Gamma_{\mathbb{Q}}) \otimes \mathbb{Q}_l$.

2. Functors on nilpotent graded Lie algebras

In this section, we review Pridham's nilpotent Lie algebra version of Schlessinger criteria [12]. The only change we impose is to consider finitely generated graded nilpotent Lie algebras with finite dimensional graded pieces, instead of finite dimensional nilpotent Lie algebras.

Fix a field k and let $\mathcal{N}_k$ denote the category of finitely generated NGLAs (nilpotent graded Lie algebras) with finite dimensional graded pieces, and $\widehat{\mathcal{N}_k}$ denote the category of pro-NGLAs with finite dimensional graded pieces which are finite dimensional in the sense that $\dim L/[L,L] < \infty$. Given $\mathcal{L} \in \widehat{\mathcal{N}_k}$ define $\mathcal{N}_{\mathcal{L},k}$ to be the category of pairs $\{N \in \mathcal{N}_k, \phi : \mathcal{L} \to N\}$ and $\widehat{\mathcal{N}_{\mathcal{L},k}}$ to be the category of pairs $\{N \in \widehat{\mathcal{N}_k}, \phi : \mathcal{L} \to N\}$.

All functors on $\mathcal{N}_{\mathcal{L},k}$ should take the 0 object to a one point set. for a functor $F : \mathcal{N}_{\mathcal{L},k} \to \text{Set}$, define $\hat{F} : \widehat{\mathcal{N}_{\mathcal{L},k}} \to \text{Set}$ by

$$\hat{F}(L) = \lim_{\leftarrow} F(L/\Gamma_n(L)),$$

where $\Gamma_n(L)$ is the n-th term in the central series of L. Then for $h_L : \mathcal{N}_{\mathcal{L},k} \to \text{Set}$ defined by $N \to Hom(L, N)$ we have an isomorphism

$$\hat{F}(L) \longrightarrow Hom(h_L, F)$$

which can be used to define the notion of a pro-representable functor.

A morphism $p \in N \to M$ in $\mathcal{N}_{\mathcal{L},k}$ is called a small section if it is surjective with a principal ideal kernel (t) such that $[N, (t)] = (0)$.

Given $F : \mathcal{N}_{\mathcal{L},k} \to \text{Set}$, and morphisms $N' \to N$ and $N'' \to N$ in $\mathcal{N}_{\mathcal{L},k}$, consider the map

$$F(N' \times_N N'') \longrightarrow F(N') \times_{F(N)} F(N'').$$

Then, by the Lie algebra analogue of the Schlessinger theorem F has a hull if and only if it satisfies the following properties

(H1) The above map is surjective whenever $N'' \to N$ is a small section.

(H2) The above map is bijective when $N = 0$ and $N'' = L(\epsilon)$.

(H3) $\dim_k(t_F < \infty$.

F is pro-representable if and only if it satisfies the following additional property

(H4) The above map is an isomorphism for any small extension $N'' \to N$.

Note that, in the case we are considering graded deformations of graded Lie algebras, only the zero grade piece of the cohomology representing the tangent space shall be checked to be finite dimensional.

3. Several deformation problems

Let X denote a hyperbolic smooth algebraic curve defined over a number field K. Let S denote the set of bad reduction places of X together with places above l. We shall construct Lie algebra versions of the pro-l outer representation of the Galois group

$$\rho_X^l : Gal(K_S/K) \longrightarrow Out(\pi_1(\bar{X})^{(l)}).$$

Let I_l denote the decreasing filtration on $Out(\pi_1(X)^{(l)})$ induced by the central series filtration of $\pi_1(\bar{X})^{(l)}$. By abuse of notation, we also denote the filtration on $Gal(K_S/K)$ by I_l. We get an injection of the associated graded $\mathbb{Z}_l$-Lie algebras on both sides

$$\mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}).$$

One can also start with the l-adic unipotent completion of the fundamental group and the outer representation of Galois group on this group.

$$\rho_X^{un,l} : Gal(K_S/K) \longrightarrow Out(\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l}).$$

By [4] 8.2 the associated Galois Lie algebra would be the same as those associated to I_l. Let U_S denote the prounipotent radical of the zariski closure of the image of $\rho_X^{un,l}$. The image of $Gal(K_S/K)$ in $Out(\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l})$ is a negatively weighted extension of $\mathbb{G}_m$ by U_S with respect to the central cocharacter $w : x \mapsto x^{-2}$. By [4] 8.4 the weight filtration induces a graded Lie algebra $\mathcal{U}_S$ which is isomorphic to $\mathcal{G}al(K_S/K) \otimes \mathbb{Q}_l$.
There are several deformation problems in this setting which are interesting. For example, the action of Galois group on unipotent completion of the fundamental group induces an action of the Galois group on the corresponding nilpotent $\mathbb{Q}_l$-Lie algebra

$$\rho_X^{ni,l} : Gal(K_S/K) \longrightarrow Aut(\mathcal{U}_S)$$

which could be deformed using the Pridham's version of Schlessinger criteria. We will explain in the following section, why this representation is completely determind by the abelianized representation of the Galois group. We will use results of Koneko mentioned in the first part of the paper. Therefore, deformation theory of this object is the same as the abelianized deformation theory. But, in this formulation we get universal deformation nilpotent Lie algebras instead of univeral deformation rings. A similar similar result will be deforming the following representation

$$Gal(K_S/K) \longrightarrow Aut(\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l}) \longrightarrow Aut(H_1(\mathcal{P}))$$

where $\mathcal{P}$ denote the nilpotent Lie algebra associated to $\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l}$. This time, the Schlessinger criteria may not help us in finding a universal representation. In the first part of this paper, we have introduced a derivation version which is a Schlessinger friendly $\mathbb{Z}_l$-Lie algebras representation

$$\mathcal{G}al(K_S/K) \longrightarrow Der(\mathcal{P})/Inn(\mathcal{P}),$$

or one could deform the following morphism, fixing its mod-l reduction

$$\mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}).$$

4. Deformation of representations of Lie algebras

The action of $Aut(\pi_1(\bar{X}))$ on $H^i(\pi_1(\bar{X}), \mathbb{Z}_l)$ is compatible with the nondegenerate alternating form defined by the cup product

$$H^1(\pi_1(\bar{X}), \mathbb{Z}_l) \times H^1(\pi_1(\bar{X}), \mathbb{Z}_l) \longrightarrow H^i(\pi_1(\bar{X}), \mathbb{Z}_l) \cong \mathbb{Z}_l$$

which is why the grade zero part of $Aut(\pi_1(\bar{X}))$ is the same as $GSp(2g, \mathbb{Z}_l)$. As a Galois module, this is exactly the Galis representation associated to the Tate module of the Jacobian variety of X. This representation completely determines the map

$$\rho_X^{ni,l} : Gal(K_S/K) \longrightarrow Aut(\mathcal{U}_S)$$

in view of the isomorphism

$$\mathcal{U}_S \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}) \otimes \mathbb{Q}_l$$

which is a convenient framework to perform different versions of restricted deformation theories appearing in the proof of Wiles using the language of Lie algebras. Our ultimate goal is to make a Lie algebra version of computations in [20] and [16].

One also has explicit information about the Malčev pro-nilpotent Lie algebra associated to $\pi_1(\bar{X})^{un}_{/\mathbb{Q}_l}$ [12] and can deform the Galois action on this Lie algebra. But let us concentrate on deforming the graded Lie algebra version

$$\mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}).$$

We could fix the mod-l representation, or fix restriction of this representation to decomposition Lie algebra $\mathcal{D}_p$ at prime p, which is induced by the same filtration as $Gal(K_S/K)$ on the decomposition group. For each prime p of K we get a map

$$\mathcal{D}_p \longrightarrow \mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)}).$$

Theorem 4.1. *For a graded $\mathbb{Z}_l$-Lie algebra L, let $D(L)$ be the set of representations of $\mathcal{G}al(K_S/K)$ to L which reduce to*

$$\bar{\rho} : \mathcal{G}al(K_S/K) \longrightarrow \mathcal{O}ut(\pi_1(\bar{X})^{(l)})/l\mathcal{O}ut(\pi_1(\bar{X})^{(l)})$$

after reduction modulo l. Assume that $\mathcal{G}al(K_S/K)$ is a free $\mathbb{Z}_l$ Lie algebra. Then, there exists a universal deformation graded $\mathbb{Z}_l$-Lie algebra L_{univ} and a universal representation

$$\mathcal{G}al(K_S/K) \longrightarrow L_{univ}$$

representing the functor D. In the case $\mathcal{G}al(K_S/K)$ is not free, then one can find a hull for the functor D.

Proof. For free $\mathcal{G}al(K_S/K)$ by theorem 2.10 in [Ras] we have

$$H^2(\mathcal{G}al(K_S/K), Ad \circ \bar{\rho}) = 0$$

which implies that D is pro-representable. In case $\mathcal{G}al(K_S/K)$ is not free, we have constructed a miniversal deformation Lie algebra for another functor in theorem 2.11 of [13], which implies that the first three Schlessinger criteria hold. By a similar argument one could prove that there exists a hull for D. Note that $\mathcal{O}ut(\pi_1(\bar{X})^{(l)})$ is pronilpotent, and for deformation of such an object one should deform the truncated object and then take a limit to obtain a universal object in pro-NGLAs. □

5. Grothendieck's anabelian conjectures

The general philosophy of Grothendieck's anabelian conjectures is to characterize invariants of a smooth hyperbolic curve by its non-abelian geometric fundamental group. In particular, Galois-module structure of the outer automorphism group of a smooth hyperbolic curve defined over a number field, should contain all the arithmetic information about X. This could not be true if we consider the pro-l completion of the fundamental group. So, translating this to the language of Lie algebras, one should work with adelic Lie algebras. On the other hand, if one considers the action of Galois group by conjugation on $\mathcal{O}ut(\pi_1(\bar{X})^{(l)})$ for all l, This can be determined by the set of l-adic representations induced by the Jacobian variety, which gives us less than what we want.

We propose to consider the Galois action on different realizations of the motivic fundamental group of a smooth hyperbolic curve. In this way we get NGLAs together with Galois actions. It is natural to expect this to carry arithmetic structure of X.

Conjecture 5.1. *Let X and X' denote smooth hyperbolic curves defined over a number field K. There exists a natural one-to-one correspondence*

$$Hom_K(X, X') \longrightarrow Hom_{Gal(\bar{K}/K)}(\pi_1^{mot}(\bar{X}), \pi_1^{mot}(\bar{X}'))$$

where $\pi_1^{mot}(\bar{X})$ denotes the motivic fundamental group of the curves over $\bar{K}$ and $Hom_{Gal(\bar{K}/K)}$ denotes the set of Galois equivariant homomorphisms between the two NGLAs.

Practically, this means that, by considering all realizations of the motivic fundamental group of a hyperbolic curve, we can grasp all arithmetic information encoded in the curve. We aslo conjecture that there should be a motivic Galois group which make the NGLA version of the above conjecture true.

Conjecture 5.2. *There exists a motivic Galois pro-NGLA $\mathcal{G}al^{mot}(\bar{K}/K)$ canonically mapping to $\pi_1^{mot}(\bar{X})$ for any hyperbolic smooth curve X defined over the number field K such that the following map is a natural one-to-one correspondence*

$$Hom_K(X, X') \longrightarrow Hom_{\mathcal{G}al^{mot}(\bar{K}/K)}(\pi_1^{mot}(\bar{X}), \pi_1^{mot}(\bar{X}')).$$

Note that, by a conjecture of Deligne, the Z_l-Lie algebras $\mathcal{G}al(\bar{K}/K)$ associated to l-adic central series filtrations on the pro-unipotent fundamental group of thrice punctured sphere $\pi_1^{un}(\mathbb{P}^1 \setminus 0, 1, \infty)$ are induced from a single $\mathbb{Z}$-Lie algebra by extension of schalars.

Acknowledgements. I would like to thank O. Gabber, M. Kontsevich, R. Mikhailov, J. Tilouine, S. Wewers for enjoyable conversations. Also, I wish to thank institute des hautes ètudes scientifiques for warm hospitality during which some parts of this work was written down.

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