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Linear Preservers of Majorization

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Abstract. For vectors $X, Y \in \mathbb{R}^n$, we say X is left matrix majorized by Y and write $X \prec_\ell Y$ if for some row stochastic matrix R, $X = RY$. Also, we write $X \sim_\ell Y$, when $X \prec_\ell Y \prec_\ell X$. A linear operator $T\colon \mathbb{R}^p \to \mathbb{R}^n$ is said to be a linear preserver of a given relation $\prec$ if $X \prec Y$ on $\mathbb{R}^p$ implies that $TX \prec TY$ on $\mathbb{R}^n$. In this note we study linear preservers of $\sim_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$. In particular, we characterize all linear preservers of $\sim_\ell$ from $\mathbb{R}^2$ to $\mathbb{R}^n$, and also, all linear preservers of $\sim_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^p$.

Keywords: Linear preservers, Row stochastic matrix, Matrix majorization.

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1. Introduction

Let M_{nm} be the algebra of all $n \times m$ real matrices, and the usual notation $\mathbb{R}^n$, for $n \times 1$ real vectors. A matrix $R = [r_{ij}] \in M_{nm}$ is called a *row stochastic* matrix if $r_{ij} \geq 0$ and $\Sigma_{k=1}^m r_{ik} = 1$ for all i, j. For vectors $X, Y \in \mathbb{R}^n$, we say X is left (resp. right) matrix majorized by Y and write $X \prec_\ell Y$ (resp. $X \prec_r Y$) if for some row stochastic matrix R, $X = RY$ (resp. $X = YR$). For more information about right and left matrix majorization and some other majorizations, we refer to [1], [5] and [11]. Also for $X, Y \in \mathbb{R}^n$, we write $X \sim_\ell Y$, if $X \prec_\ell Y \prec_\ell X$.

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A linear operator $T\colon \mathbb{R}^p \to \mathbb{R}^n$ is said to be a linear preserver of a given relation $\prec$ if $X \prec Y$ on $\mathbb{R}^p$ implies that $TX \prec TY$ on $\mathbb{R}^n$. Linear preservers of $\prec_\ell$ and $\prec_r$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ are fully characterized in [6] and [7]. For more information about linear preservers of majorization we refer the reader to [1]-[4] and [10]. [8] introduced an extension of this preservers which is the characterization of linear preservers of $\prec_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$, such that p and n are not necessarily equal. Also [8] characterized the structure of theses linear preservers of $\prec_\ell$ for $p \leq n \leq p(p-1)$. In [9], by a geometric approach one can see the characterization of linear preservers of $\prec_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$ without any additional conditions on p and n. Here we focus on this method. And in the final section we characterize linear preserver of $\sim_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$, with some restrictions on p and n.
We shall use the following conventions throughout the paper, Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a nonzero linear operator and let $[T] = [t_{ij}]$ denotes the matrix representation of T with respect to the standard bases $\{e_1, e_2, \ldots, e_p\}$ of $\mathbb{R}^p$ and $\{f_1, f_2, \ldots, f_n\}$ of $\mathbb{R}^n$. If $p = 1$, then all linear operators on $\mathbb{R}^1$ are preservers of $\prec_\ell$. Thus, we assume $p \geq 2$. Let A_i be $m_i \times p$ matrices, $i = 1, \ldots, k$. We use the notation $[A_1/A_2/\ldots/A_k]$ to denote the corresponding $(m_1+m_2+\ldots+m_k)\times p$ matrix. Denote

$$\begin{aligned} \mathbf{a} : &= \max\{t_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq p\}, \\ \mathbf{b} : &= \min\{t_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq p\}. \end{aligned} \tag{1}$$

We also use the notation P for the permutation matrix such that $P(e_i) = e_{i+1}$, $1 \leq i \leq p-1$, $P(e_p) = e_1$. Let I denote the $p \times p$ identity matrix, and let $r, s \in \mathbb{R}$ be such that $rs < 0$. Define the $p(p-1) \times p$ matrix $\mathcal{P}_p(r,s) = [P_1/P_2/\ldots/P_{p-1}]$ where $P_j = rI + sP^j$, for all $j = 1, 2, \ldots, p-1$. It is clear that up to a row permutation the matrices $\mathcal{P}_p(r,s)$ and $\mathcal{P}_p(s,r)$ are equal. Also define $\mathcal{P}_p(r,0) := rI$, $\mathcal{P}_p(0,s) := sI$ and $\mathcal{P}_p(0,0)$ as $1 \times p$ zero matrix.
Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator and let $[T] = [T_1/\ldots/T_n]$, where $T_i = [t_{i1}, t_{i2}]$, $1 \leq i \leq n$. Let

$$\Delta := \mathrm{Conv}(\{(t_{i1}, t_{i2}), (t_{i2}, t_{i1}), 1 \leq i \leq n\}) \subseteq \mathbb{R}^2, \tag{2}$$

where $\mathrm{Conv}(A)$ denotes the convex hull of a set A. Also, let $C(T)$ denotes the set of all corners of Δ.
Now, we study the characterization of linear preservers of $\prec_\ell$ from R^p to R^n.

Theorem 1.1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator. Then, T is a linear preserver of $\prec_\ell$ if and only if $\mathcal{P}_2(x,y)$ is a sub-matrix of $[T]$ and $xy \leq 0$ for all $(x,y) \in C(T)$.*

Now let $p \geq 3$, we study all linear preservers $T : \mathbb{R}^p \to \mathbb{R}^n$ of $\prec_\ell$. First we need some definitions.

Definition 1.2. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator and let $[T] = [T_1/\dots/T_n]$. Define*

$$\Omega := \mathrm{Conv}(\{T_i = (t_{i1}, \dots, t_{ip}), 1 \le i \le n\}) \subseteq \mathbb{R}^p.$$

Also, let $C(T)$ be the set of all corners of Ω.

Definition 1.3. *Let $T\colon \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. We denote by P_i* (resp. *N_i) the sum of the non negative* (resp. *nonpositive) entries in the i^{th} row of $[T]$. If all the entries in the i^{th} row are positive* (resp. *negative), we define $N_i = 0$* (resp. *$P_i = 0$).*

Definition 1.4. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. Define*

$$\begin{aligned} \Delta : \ &= \ \mathrm{Conv}(\{(P_i, N_i), (N_i, P_i) : 1 \le i \le n\}), \\ E(T) : \ &= \ \{(P_i, N_i) : (P_i, N_i) \text{ is a corner of } \Delta\}, \end{aligned}$$

where P_i, N_i are as in Definition 1.3.

Theorem 1.5. [7, *Theorem* 4.6] *Let T and $E(T)$ be as in Definition 1.4. Then T preserves $\prec_\ell$ if and only if $\mathcal{P}_p(\alpha, \beta)$ is a sub-matrix of $[T]$ for all $(\alpha, \beta) \in E(T)$.*

For $X, Y \in \mathbb{R}^p$, we define $X \sim_\ell Y$, when $X \prec_\ell Y \prec_\ell X$. This paper consists of two sections. First section characterizes linear preservers of $\sim_\ell$ from $\mathbb{R}^2$ to $\mathbb{R}^n$. In the second section we obtain a key necessary condition for $T\colon \mathbb{R}^p \to \mathbb{R}^n (p \ge 3)$, to be a linear preserver of $\sim_\ell$, in particular we prove $p \le n$, when $p \ge 3$. At first we have some lemmas.

Lemma 1.6. *Let $X, Y \in \mathbb{R}^p$, $X \sim_\ell Y$ if and only if $\max X = \max Y$ and $\min X = \min Y$ where the maximum and minimum are taken over the entries of X and Y.*

Proof. By [9, *Remark* 3.1] we know $X \prec_\ell Y$ if and only if $\min Y \le \min X \le \max X \le \max Y$. Hence $X \prec_\ell Y \prec_\ell X$ if and only if $\max X = \max Y$ and $\min X = \min Y$.

Lemma 1.7. *Let $T\colon \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator such that $\min TX = \min TY$ for all $X \sim_\ell Y$. Then T is a linear preserver of $\sim_\ell$.*

Proof. If $X \sim_\ell Y$ then $-X \sim_\ell -Y$ and hence $\min -TX = \min -TY$, which implies $\max TX = \max TY$.

Lemma 1.8. *If $T\colon \mathbb{R}^p \to \mathbb{R}^n$ is a linear preserver of $\prec_\ell$, then T is a linear preserver of $\sim_\ell$.*

Proof. Let T be a linear preserver of $\prec_\ell$ and $X \prec_\ell Y \prec_\ell X$, for some $X, Y \in \mathbb{R}^p$. Hence $TX \prec_\ell TY \prec_\ell TX$ which is $TX \sim_\ell TY$.
The converse of Lemma 1.8 is not true, we will show it in the next section .

2. Linear Preservers on $\mathbb{R}^2$

We know that T is a linear preserver of $\sim_\ell$ if and only if αT is a linear preserver of $\sim_\ell$ for any nonzero real number α. Without loss of generality we can assume $|\mathbf{b}| \leq \mathbf{a}$. Also throughout the remainder of this paper we fix the notation $\mathcal{P}(n)$ for all $n \times n$ permutation matrices. In this section we shall characterize all linear preservers $T\colon \mathbb{R}^2 \to \mathbb{R}^n$ of $\sim_\ell$. We have the following lemma.

Lemma 2.1. *(i) Let $X, Y \in \mathbb{R}^2$, then $X \sim_\ell Y$ if and only if $X = PY$ for some $P \in \mathcal{P}(2)$.*
(ii) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^n$ $(n \leq 2)$ be a linear operator then T is a preserver of $\sim_\ell$ if and only if for all $X \in \mathbb{R}^2$ and $P \in \mathcal{P}(2)$, there exists a permutation matrix $Q \in \mathcal{P}(n)$ such that $T(PX) = QT(X)$.

Proof. (i) Let $X, Y \in \mathbb{R}^2$, by Lemma 1.6 $X \sim_\ell Y$ if and only if $\max X = \max Y$ and $\min X = \min Y$. Hence $X = PY$ for some $P \in \mathcal{P}(2)$.
(ii) Let $X \in \mathbb{R}^2$, by part (i) T is a linear preserver of $\sim_\ell$ if and only if $T(X) \sim_\ell T(PX)$ for every permutation matrix $P \in \mathcal{P}(2)$. Now by part (i), if $T(X) \sim_\ell T(PX)$ then $T(PX) = QT(X)$ for some suitable permutation matrix Q.

Proposition 2.2. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear operator. Then T preserves $\sim_\ell$ if and only if $[T] = [a\ a]$, for some $a \in R$.*

Proof. Let T preserve $\sim_\ell$, since $e_1 \sim_\ell e_2$ then $\max Te_1 = \max Te_2$ and $\min Te_1 = \min Te_2$. Hence $[T] = [a\ a]$, $a \in R$. Conversely, let $[T] = [a\ a]$ for some $a \in \mathbb{R}$ hence $T(X) = ax_1 + ax_2 = ax_2 + ax_1 = T(PX)$, for all $X \in \mathbb{R}^2$ and for all $P \in \mathcal{P}(2)$. Therefore T is a linear preserver of $\sim_\ell$.

Theorem 2.3. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. T preserves $\sim_\ell$, if and only if $[T] = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ or $[T] = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$, for some $a, b \in R$.*

Proof. Let T be a linear preserver of $\sim_\ell$. Since $e_1 \sim_\ell e_2$, then $Te_1 \sim_\ell Te_2$ and hence $Te_1 = PTe_2$ for some permutation matrix P. Therefore $Te_1 = Te_2$ or $Te_1 = PTe_2$, $(P \neq I)$. Which implies $[T] = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$ or $[T] = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.
Conversely it is easy to check that $T(X) = T(PX)$, for all $X \in \mathbb{R}^2$ and every 2×2 permutation matrix P. Hence T preserves $\sim_\ell$.
The following example shows that the converse of Lemma 1.8 is not true.

Example 2.4. *Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, by Theorem 2.3, A is a linear preserver of $\sim_\ell$, but by Theorem 1.1, A is not a linear preserver of $\prec_\ell$.*

Definition 2.5. *Let $x, y \in R$, define $\mathcal{Q}_2(x, y) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$, if $x \neq y$ and $\mathcal{Q}_2(x, x) = [x\ x]$.*

Theorem 2.6. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator. Then T is a linear preserver of $\sim_\ell$ if and only if $\mathcal{Q}_2(x,y)$ is a sub-matrix of $[T]$, for all $(x,y) \in C(T)$, where $C(T)$ denote the set of all corners of Δ as in (2).*

Proof. Let T be a linear preserver of $\sim_\ell$ and $(x,y) \in C(T)$, $x \neq y$. Let $T_i = (t_{i1}, t_{i2}) = (x,y)$. Then there exist real numbers m, M such that $mt_{i1} + Mt_{i2} < mt_{j1} + Mt_{j2}, j \neq i$. Choose $\varepsilon_0 > 0$ small enough so that $(m-\varepsilon)t_{i1} + (M+\varepsilon)t_{i2} < (m-\varepsilon)t_{j1} + (M+\varepsilon)t_{j2}, j \neq i, 0 < \varepsilon \leq \varepsilon_0$. Since $(M+\varepsilon, m-\varepsilon)^t \sim_\ell (m-\varepsilon, M+\varepsilon)^t$, $T(M+\varepsilon, m-\varepsilon)^t \sim_\ell T(m-\varepsilon, M+\varepsilon)^t$. Hence, for all $0 < \varepsilon \leq \varepsilon_0$, there exist $1 \leq k \leq n$ such that $T_k = (t_{k1}, t_{k2}) \in C(T)$ and $(m-\varepsilon)t_{i1} + (M+\varepsilon)t_{i2} = \min T(m-\varepsilon, M+\varepsilon)^t = \min T(M+\varepsilon, m-\varepsilon)^t = (M+\varepsilon)t_{k1} + (m-\varepsilon)t_{k2}$. Since $k \in \{1, 2, \ldots, n\}$ is a finite set, there exists k such that $t_{k1} = t_{i2}$ and $t_{k2} = t_{i1}$. Therefore, $\mathcal{Q}_2(x,y)$ is a sub-matrix of $[T]$, .
Conversely, let $\mathcal{Q}_2(x,y)$ be a sub-matrix of $[T]$ for all $(x,y) \in C(T)$. Define the linear operator $\widehat{T}$ on R^2 such that $[\widehat{T}] = [\mathcal{Q}_2(x_1,y_1)/\cdots/\mathcal{Q}_2(x_r,y_r)]$, where $(x_i,y_i) \in C(T), 1 \leq i \leq r$. By elementary convex analysis, we know that $\max T(X) = \max \widehat{T}(X)$ and $\min T(X) = \min \widehat{T}(X)$ for all $X \in \mathbb{R}^2$. Hence it is enough to show that $\widehat{T}$ is a linear preserver of $\prec_\ell$. By Theorems 2.2 and 2.3, each $\mathcal{Q}_2(x_i,y_i)$ is a linear preserver of $\sim_\ell$. Thus, $\widehat{T}$ is a linear preserver of $\sim_\ell$.

3. Linear Preservers on $\mathbb{R}^p$

In this section we consider linear operators $T\colon \mathbb{R}^p \to \mathbb{R}^n$ for $p \geq 3$.

Lemma 3.1. *Let $T\colon \mathbb{R}^p \to \mathbb{R}^n$, $p \geq 3$ be a linear preserver of $\sim_\ell$ and let $\mathbf{a}, \mathbf{b}$ are as in* (1). *Then the following assertions hold,*
(i) $\mathbf{a} = \max Te_i$ *and* $\mathbf{b} = \min Te_i$ *for all* $i = 1, \ldots, n$. *In particular every column of* $[T]$ *contains at least one entry equal to* $\mathbf{a}$ *and at least one entry equal to* $\mathbf{b}$.
(ii) *If* $t_{ij} = \mathbf{a}$ *for some* i, j *then* $t_{ik} \leq 0$, *for all* $k \neq j$. *Also if* $t_{ij} = \mathbf{b}$ *for some* i, j *then* $t_{ik} \geq 0$, *for all* $k \neq j$.
(iii) $p \leq n$.

Proof. (i) We have $e_i \sim_\ell e_j$ for all $1 \leq i, j \leq p$, hence $Te_i \sim_\ell Te_j$ which implies that $\max Te_i = \max Te_j$ and $\min Te_i = \min Te_j$, for all $1 \leq i, j \leq p$. By (1), $\mathbf{a} = \max Te_i$ and $\mathbf{b} = \min Te_i$, for all $1 \leq i \leq p$.
(ii) For $p \geq 3$, and for all $r, s \in \{1, \ldots, p\}$, $r \neq s$, we know $(e_r + e_s) \sim_\ell e_r$ so $(Te_r + Te_s) \sim_\ell Te_r$. Therefore, if the i^{th} of $[T]$ contains an entry equal to $\mathbf{a}$ (resp. $\mathbf{b}$), then all other entries of the i^{th} row of $[T]$ are nonpositive (resp. nonnegative).
(iii) By parts (i) and (ii) we know that each column of $[T]$ has at least one entry equal to $\mathbf{a}$ and each row of $[T]$ has at most one entry equal to $\mathbf{a}$, hence $p \leq n$.
Now, we state the key theorem of this section.

Theorem 3.2. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\sim_\ell$ and let* $\mathbf{a}$ *and* $\mathbf{b}$ *be as in* (1). *Then there exist* $0 \le \alpha \le \mathbf{a}$ *and* $\mathbf{b} \le \beta \le 0$ *such that* $\mathcal{P}_p(\mathbf{a}, \beta)$ *and* $\mathcal{P}_p(\alpha, \mathbf{b})$ *are sub-matrices of* $[T]$.

Proof. Let $[T] = [t_{ij}]$, by Lemma 3.1 we know that in each column of $[T]$ there is at least one entry equal to $\mathbf{a}$ and at least one entry equal to $\mathbf{b}$. Let $1 \le k \le p$, define $I_k = \{i\colon 1 \le i \le n, t_{ik} = \mathbf{a}\}$ and $J_k = \{j\colon 1 \le j \le n, t_{jk} = \mathbf{b}.\}$ Since T is a linear preserver of $\sim_\ell$, the sets I_k and J_k are nonempty. Also, Lemma 3.1 follows that a row containing an entry $\mathbf{a}$ (resp. $\mathbf{b}$) contains other positive (resp. negative) entries. That is $t_{il} \le 0$ and $t_{jl} \ge 0$ whenever $i \in I_k$, $j \in J_k$ and $l \ne k$. For $i \in I_k$ and $j \in J_k$ we set $\beta_k^i = \sum_{l \ne k} t_{il} \le 0$, $\alpha_k^j = \sum_{l \ne k} t_{jl} \ge 0$, and

$$\beta_k := \min\{\beta_k^i,\ i \in I_k\}, \quad \alpha_k := \max\{\alpha_k^j,\ j \in J_k\}. \tag{3}$$

Define $X_k = -(N+1)e_k + e$. Choose N_0 large enough such that for all $N \ge N_0$ and $1 \le i \le n$,

$$\min T(X_k) = -N\mathbf{a} + \beta_k \le -Nt_{ik} + \sum_{l \ne k} t_{il} \le -N\mathbf{b} + \alpha_k = \max T(X_k). \tag{4}$$

We know that $X_k \sim_\ell X_r = -(N+1)e_r + e$, $1 \le r \le p$ and T is a linear preserver of $\sim_\ell$. Hence by (4), $\alpha := \alpha_k = \alpha_r$ and $\beta := \beta_k = \beta_r, 1 \le r \le p$. Also, $X_k \sim_\ell -Ne_i + e_j$, $i \ne j$. For each $N \ge N_0$, there exists $1 \le h \le n$ such that $-Nt_{hi} + t_{hj} = \min T(-Ne_i + e_j) = \min T(X_k) = -N\mathbf{a} + \beta$ and for each $1 \le i \le p$, $1 \le j \le p$ and $N \ge N_0$, there exists $1 \le h \le n$ such that $-N(\mathbf{a} - t_{hi}) = t_{hj} - \beta$. It follows that $t_{hi} = a$, $t_{hj} = \beta$. Hence $\mathcal{P}_p(\mathbf{a}, \beta)$ is a sub-matrix of $[T]$. Similarly, there exists N_1, such that for each $N \ge N_1$, there exists $1 \le h \le n$ such that $-Nt_{hi} + t_{hj} = \max T(-Ne_i + e_j) = \max T(X_k) = -N\mathbf{b} + \alpha$ and $-N(\mathbf{b} - t_{hi}) = t_{hj} - \alpha$. Thus, $t_{hi} = \mathbf{b}$ and $t_{hj} = \alpha$. Since $1 \le i \ne j \le p$ was arbitrary, $\mathcal{P}_p(\mathbf{b}, \alpha)$ is a sub-matrix of $[T]$. Therefore, $\mathcal{P}_p(\mathbf{a}, \beta)$ and $\mathcal{P}_p(\mathbf{b}, \alpha)$ are sub-matrices of $[T]$.

In the following example we will show that for all $n \ge p$ there exists $T\colon \mathbb{R}^p \to \mathbb{R}^n$ which preservers $\sim_\ell$.

Example 3.3. *Let $p \le n$. Assume I is the $p \times p$ identity matrix and E is an $(n-p) \times p$ row stochastic matrix. Define $T\colon \mathbb{R}^p \to \mathbb{R}^n$ by $[T] = [I/E]$. By Theorem 1.5, we know that T is a linear preserver of $\prec_\ell$. Hence, by Lemma 1.8, T is a linear preserver of $\sim_\ell$*

Theorem 3.4. *$T\colon \mathbb{R}^p \to \mathbb{R}^p$, $p \ge 3$ is a linear preserver of $\sim_\ell$ if and only if $TX = cPX$, for some $p \times p$ permutation matrix P, $c \in \mathbb{R}$ and for all $X \in \mathbb{R}^p$.*

Proof. Let T be a linear preserver of $\sim_\ell$, by Lemma 3.1 each column of $[T]$ has at least, one entry equal to $\mathbf{a}$ and one entry equal to $\mathbf{b}$. Also each row of $[T]$ has at most, one entry equal to $\mathbf{a}$ and one entry equal to $\mathbf{b}$. Since $[T]$ is $p \times p$, then all rows and all columns of $[T]$ have exactly one entry equal to $\mathbf{a}$ and one entry equal to $\mathbf{b}$ with all other entries equal to zero. Without loss of generality let $t_{11} = \mathbf{a}$, $t_{12} = \mathbf{b}$ and $t_{1k} = 0$ for all $k \neq 1, 2$. Therefore $\max T(e_1 - e_2) = \mathbf{a} - \mathbf{b}$, but $\max T(e_1 - e_3) = a$. We know that $T(e_1 - e_2) \sim_\ell T(e_1 - e_3)$, since T is a linear preserver of $\sim_\ell$ and $(e_1 - e_2) \sim_\ell (e_1 - e_3)$. Hence $\max T(e_1 - e_2) = \max T(e_1 - e_3)$ which is $\mathbf{a} - \mathbf{b} = \mathbf{a}$. Therefore $\mathbf{b} = 0$ and $[T] = \mathbf{a}P$ for some $P \in \mathcal{P}(p)$.

Conversely, Let $Tx = cPx$ for all $x \in \mathbb{R}^p$ and $P \in \mathcal{P}(p)$. Since T is a linear preserver of $\sim_\ell$ if and only if αT for $\alpha \in \mathbb{R}$ is a linear preserver of $\sim_\ell$, we can assume $c \geq 0$. Let $x \sim_\ell y$ and $m = \min x = \min y$ and $M = \max x = \max y$. Obviously $cm = \min Tx = \min Ty$ and $cM = \max Tx = \max Ty$. Therefore $Tx \sim_\ell Ty$.

By [6] we know that for $p \geq 3$, $T \colon \mathbb{R}^p \to \mathbb{R}^p$is a linear preserver of $\prec_\ell$ if and only if T has the form $x \rightarrowtail aPx$, for some $a \in \mathbb{R}$ and some $P \in \mathcal{P}(p)$.

Corollary 3.5. *$T \colon \mathbb{R}^p \to \mathbb{R}^p$, $p \geq 3$ is a linear preserver of $\sim_\ell$ if and only if T is a linear preserver of $\prec_\ell$.*

Problem. Let $3 \leq p < n$ be given. It will be nice to characterize all linear preservers of $\sim_\ell$ from $\mathbb{R}^p$ to $\mathbb{R}^n$.

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