

On Schur Multipliers of Pairs and Triples of Groups with Topological Approach

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ABSTRACT. In this paper, using a relation between Schur multipliers of pairs and triples of groups, the fundamental group and homology groups of a homotopy pushout of Eilenberg-MacLane spaces, we present among other things some behaviors of Schur multipliers of pairs and triples with respect to free, amalgamated free, direct products and direct limits of groups with topological approach.

Keywords: Schur multiplier of a pair of groups; Schur multiplier of a triple of groups; Homology group; Homotopy group, Eilenberg-MacLane space; Homotopy pushout.

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1. INTRODUCTION AND PRELIMINARIES

The Schur multiplier of a group G is defined to be

$$M(G) = (R \cap F')/[R, F],$$

where F/R is any free presentation of G . It is a well-known fact that $M(G)$ depends, up to isomorphism, only on G . Furthermore, it is easy to see that

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$M(-)$ is a functor from the category of groups to the category of abelian groups (see [5] for further details).

The Schur multiplier of a pair of groups is also defined in several ways [4]. As mentioned in [4], the Schur multiplier of a pair of groups is useful to obtain information on Schur multiplier of a group. Also this concept has been generalized to the schur multiplier of a triple of groups and it is proved that any result on Schur multiplier of a triple of groups yields to a result on Schur multiplier of a pair of groups. So we are interested to study on these notions in this paper. By a pair of groups (G, N) we mean a group G with a normal subgroup N . A homomorphism of pairs $(G_1, N_1) \rightarrow (G_2, N_2)$ is a group homomorphism $G_1 \rightarrow G_2$ that sends N_1 into N_2 . The Schur multiplier of a pair of groups (G, N) which was first defined by G. Ellis [4] will be a functorial abelian group $M(G, N)$ whose principal feature is a natural exact sequence

$$\begin{aligned} \cdots \rightarrow H_3(G) \rightarrow H_3(G/N) \rightarrow M(G, N) \rightarrow M(G) \rightarrow \\ M(G/N) \rightarrow N/[N, G] \rightarrow G^{ab} \rightarrow (G/N)^{ab} \rightarrow 0 \end{aligned} \quad (1.1)$$

in which $H_3(G)$ is the third homology group of G with integer coefficients.

There are several possible definitions of the Schur multiplier of a group and the Schur multiplier of a pair of groups. We are going to deal with topological one that we present in this note.

First, we note that for any group G one can construct functorially a connected CW-complex $K(G)$, called the Eilenberg-MacLane space, whose fundamental group is isomorphic to G which has all higher homotopy groups trivial [8]. By considering $H_n(X)$ as the n th singular homology group of a topological space X , with coefficients in the group \mathbf{Z} , we recall the relation $H_n(G) \cong H_n(K(G))$, for all $n \geq 0$, [1, Prop. 4.1].

By the Hopf formula for any CW-complex K with $\pi_1(K) = G$ and F/R as a free presentation for G we have the following isomorphism

$$H_2(K)/h_2(\pi_2(K)) \cong R \cap F'/[R, F],$$

where h_2 is the corresponding Hurewicz map [3]. Hence a topological definition of the Schur multiplier of a group G can be considered as the second homology group of the Eilenberg-MacLane space $K(G)$, $H_2(K(G))$. This topological interpretation of $M(G)$ can be extended to one for $M(G, N)$ as follows.

For any two group extensions $1 \rightarrow M \rightarrow P \rightarrow Q \rightarrow 1$ and $1 \rightarrow N \rightarrow P \rightarrow R \rightarrow 1$ we consider the following homotopy pushout

$$\begin{array}{ccc} K(P) & \longrightarrow & K(P/M) \\ \downarrow & & \downarrow \\ K(P/N) & \longrightarrow & X. \end{array}$$

By the Mayer-Vietoris pushout sequence, we have the following exact sequence

$$\cdots \rightarrow H_3(P) \rightarrow H_3(Q) \oplus H_3(R) \rightarrow H_3(X) \rightarrow H_2(P) \rightarrow$$

$$\begin{aligned} H_2(Q) \oplus H_2(R) &\rightarrow H_2(X) \rightarrow H_1(P) \rightarrow H_1(Q) \oplus H_1(R) \rightarrow \\ H_1(X) &\rightarrow H_0(P) \rightarrow H_0(Q) \oplus H_0(R) \rightarrow H_0(X) \rightarrow 0. \end{aligned} \quad (1.2)$$

Using [2, Corollary 3.4] we have

$$\pi_1(X) \cong P/MN \quad \text{and} \quad \pi_2(X) \cong M \cap N/[M, N].$$

If we make the assumption $P = MN$, then X is 1-connected and so by Hurewicz Theorem we have

$$H_1(X) = 0 \quad \text{and} \quad H_2(X) \cong \pi_2(X) \cong M \cap N/[M, N].$$

Hence the exact sequence (1.2) becomes as follows

$$\begin{aligned} \cdots \rightarrow H_3(P) &\rightarrow H_3(Q) \oplus H_3(R) \rightarrow H_3(X) \rightarrow H_2(P) \rightarrow \\ H_2(Q) \oplus H_2(R) &\rightarrow M \cap N/[M, N] \rightarrow P^{ab} \rightarrow Q^{ab} \oplus R^{ab} \rightarrow 0. \end{aligned} \quad (1.3)$$

If we consider the two group extensions $1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1$ and $1 \rightarrow G \rightarrow G \rightarrow 1 \rightarrow 1$ corresponding to a pair of groups (G, N) and the following homotopy pushout

$$\begin{array}{ccc} K(G) & \longrightarrow & K(G/N) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X, \end{array} \quad (1.4)$$

then we have the following natural exact sequence as (1.1)

$$\begin{aligned} H_3(G) &\rightarrow H_3(G/N) \rightarrow H_3(X) \rightarrow H_2(G) = M(G) \rightarrow \\ H_2(G/N) = M(G/N) &\rightarrow H_2(X) = N/[N, G] \rightarrow G^{ab} \rightarrow (G/N)^{ab} \rightarrow 0. \end{aligned}$$

Hence the Schur multiplier of a pair of groups (G, N) can be considered as the third homology group of a space X which is the homotopy pushout corresponding to the pair of groups (G, N) as (1.4).

Remark 1.1. As we mentioned before, the notion of the Schur multiplier of a pair of groups was introduced by G. Ellis in [4].

For a pair of groups (G, N) , the natural epimorphism $G \rightarrow G/N$ induces functorially the continuous map $f: K(G) \rightarrow K(G/N)$. Suppose that $M(f)$ is the mapping cylinder of f containing $K(G)$ as a subspace and is also homotopically equivalent to the space $K(G/N)$. Take $K(G, N)$ to be the mapping cone of the cofibration $K(G) \hookrightarrow M(f)$. By Mayer-Vietoris, the cofibration sequence $K(G) \hookrightarrow M(f) \rightarrow K(G, N)$ induces a natural long exact homology sequence

$$\cdots \rightarrow H_{n+1}(G/N) \rightarrow H_{n+1}(K(G, N)) \rightarrow H_n(G) \rightarrow H_n(G/N) \rightarrow \cdots$$

for $n \geq 0$. G. Ellis [4] showed that the Schur multiplier of the pair (G, N) can be considered as the third homology group of the cofiber space $K(G, N)$. We note that the mapping cone $K(G, N)$ of the cofiber $K(G) \rightarrow M(f)$ is homotopically equivalent to the space X which is the homotopy pushout corresponding to the pair of groups (G, N) . Therefore our topological interpretation of the Schur multiplier of a pair of groups (G, N) is equivalent to the topological definition

of G. Ellis.

In Section 2, using the topological interpretations, we present among other things some behaviors of the Schur multiplier of pairs of the free product, the amalgamated free product, and the direct product. Also, we show that the Schur multiplier of a pair commutes with direct limits in some cases.

By a triple of groups (G, M, N) we mean a group G with two normal subgroups M and N . A homomorphism of triples $(G_1, M_1, N_1) \rightarrow (G_2, M_2, N_2)$ is a group homomorphism $G_1 \rightarrow G_2$ that sends M_1 into M_2 and N_1 into N_2 . G. Ellis [4] defined the Schur multiplier of a triple (G, M, N) as a functorial abelian group $M(G, M, N)$ whose principle feature is a natural exact sequence

$$\begin{aligned} \cdots \rightarrow H_3(G, N) \rightarrow H_3(G/M, MN/M) \rightarrow M(G, M, N) \rightarrow M(G, N) \\ M(G/M, MN/M) \rightarrow M \cap N/[M \cap N, G][M, N] \\ \rightarrow N/[N, G] \rightarrow NM/M[N, G] \rightarrow 0. \end{aligned} \quad (1.5)$$

He also gave a topological interpretation for $M(G, M, N)$. In Section 3, first we give a topological definition of the Schur multiplier of a triple which is equivalent to the one of Ellis. Then using this definition we show that the Schur multiplier of a triple commutes with direct limits under some extra conditions. Nevertheless, this definition does not seem suitable to peruse the behavior of the Schur multiplier of a triple with several products of groups. So in continue, we define a new version of the Schur multiplier of a triple (G, M, N) which has some better consequences than the one of Ellis. We show that our new notion is coincide with the one of Ellis if $G = MN$. Finally, we study the behaviors of this new version of the Schur multiplier of a triple with respect to free products, amalgamated free and direct products and also a better behavior with respect to direct limits than the one of Ellis.

2. SCHUR MULTIPLIERS OF PAIRS

The following two known results obtained from the existence of the natural long exact sequence (1.1). Using the properties of Eilenberg-MacLane spaces, they can be reproved by topological viewpoints.

Theorem 2.1. *The Schur multiplier of a group G is a special case of the Schur multiplier of a pair of groups that is $M(G, G) \cong M(G)$.*

Proposition 2.2. *For any group G , the Schur multiplier of the pair $(G, 1)$ is trivial.*

Proof. First, we recall that $M(G, 1)$ is equal to the third homology group $H_3(X)$, where X is the mapping cone of the cofibration $i : K(G) \hookrightarrow K(G)$. Hence X is the quotient space of $(K(G) \times I) \sqcup K(G)$ with respect to the equivalent relation $(x, 0) \sim (x', 0)$, $(x, 1) \sim i(x)$, for all $x, x' \in K(G)$. Thus the space X is a contractible and so with trivial homology groups the proof is completed. \square

We recall that by Miller Formula [5] for any two groups G_1 and G_2 we have $M(G_1 * G_2) \cong M(G_1) \oplus M(G_2)$. The authors have also presented a topological proof for this note in [6]. In the following, we prove a new result for the structure of the Schur multiplier of pairs of the free product with topological method.

Theorem 2.3. *For any two groups G_1, G_2 and their normal subgroups $N_i \trianglelefteq G_i$ ($i = 1, 2$) we have the following isomorphism*

$$M(G_1 * G_2, \langle N_1 * N_2 \rangle^{G_1 * G_2}) \cong M(G_1, N_1) \oplus M(G_2, N_2),$$

where $\langle N_1 * N_2 \rangle^{G_1 * G_2}$ is the normal closure of $N_1 * N_2$ in $G_1 * G_2$.

Proof. Suppose that X_i ($i = 1, 2$) is the pushout of corresponding diagram

$$\begin{array}{ccc} K(G_i) & \longrightarrow & K(G_i/N_i) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_i. \end{array}$$

Using the fact that any two direct limits commute, we conclude that the space $X_1 \vee X_2$ is also a pushout for the following diagram

$$\begin{array}{ccc} K(G_1) \vee K(G_2) & \longrightarrow & K(G_1/N_1) \vee K(G_2/N_2) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_1 \vee X_2. \end{array}$$

According to Van-Kampen Theorem, we rewrite the above diagram as follows

$$\begin{array}{ccc} K(G_1 * G_2) & \longrightarrow & K(G_1/N_1 * G_2/N_2) = K(G_1 * G_2 / \langle N_1 * N_2 \rangle^{G_1 * G_2}) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_1 \vee X_2. \end{array}$$

Now by the definition and the Mayer-Vietoris sequence for the above diagram we have

$$\begin{aligned} M(G_1 * G_2, \langle N_1 * N_2 \rangle^{G_1 * G_2}) &= H_3(X_1 \vee X_2) \\ &\cong H_3(X_1) \oplus H_3(X_2) = M(G_1, N_1) \oplus M(G_2, N_2). \quad \square \end{aligned}$$

We recall that the authors in [6] proved that if G is the free amalgamated product of its two subgroups G_1 and G_2 over a subgroup H , then the following exact sequence holds

$$\begin{aligned} \cdots \rightarrow M(H) &\rightarrow M(G_1) \oplus M(G_2) \rightarrow M(G) \\ &\rightarrow H_{ab} \rightarrow G_{1ab} \oplus G_{2ab} \rightarrow G_{ab} \rightarrow \cdots \end{aligned}$$

Now we give the following result for the Schur multiplier of a pair of amalgamated free product of groups by topological argument.

Theorem 2.4. *Let $(G_1, N_1), (G_2, N_2)$ be two pairs of groups and $H \leq N_1 \cap N_2$. Then the Schur multiplier of the pair of amalgamated free products $(G_1 *_H G_2, N_1 *_H N_2)$, satisfies the following exact sequence*

$$\begin{aligned} \cdots \rightarrow H_3(K(G_1 * G_2)) &\rightarrow H_3(G_1/N_1) \oplus H_3(G_2/N_2) \rightarrow \\ M(G_1 *_H G_2, N_1 *_H N_2) &\rightarrow M(G_1 *_H G_2) \rightarrow M(G_1/N_1) \oplus M(G_2/N_2) \rightarrow \cdots \end{aligned}$$

Proof. In order to introduce the Eilenberg-MacLane space corresponding to the group $G_1 *_H G_2$, first consider the space $K(H)$ and then by attaching cells to this space in suitable ways, construct spaces $K(G_1)$ and $K(G_2)$ (for further details see [6, Theorem 2.5]). In this case, using Van-Kampen Theorem, we have the isomorphism $\pi_1(K(G_1) \cup K(G_2)) \cong G_1 *_H G_2$, and by the way of constructing the spaces $K(G_1)$ and $K(G_2)$ we can consider $K(G_1) \cup K(G_2)$ as an Eilenberg-MacLane space for the group $G_1 *_H G_2$.

Also we recall that for any $i \in \{1, 2\}$, $M(G_i, N_i)$ is the third homology group $H_3(X_i)$, where X_i is pushout of the following diagram

$$\begin{array}{ccc} K(G_i) & \longrightarrow & K(G_i/N_i) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_i. \end{array}$$

Using the fact that union preserves direct limit of spaces, the following diagram is also a pushout diagram

$$\begin{array}{ccc} K(G_1) \cup K(G_2) & \longrightarrow & K(G_1/N_1) \cup K(G_2/N_2) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_1 \cup X_2. \end{array}$$

Note that by the assumption $H \subseteq N_1 \cap N_2$, $K(G_1/N_1)$ and $K(G_2/N_2)$ has one-point intersection. Roughly speaking, $K(G_1/N_1) \cup K(G_2/N_2)$ is indeed a wedge space and so it is an Eilenberg-MacLane space corresponding to the free product $G_1/N_1 *_H G_2/N_2$. On the other hand, by the following isomorphism

$$G_1 *_H G_2/N_1 *_H N_2 \cong G_1/N_1 *_H G_2/N_2,$$

the two Eilenberg-MacLane spaces $K(G_1 *_H G_2/N_1 *_H N_2)$ and $K(G_1/N_1 *_H G_2/N_2) = K(G_1/N_1) \cup K(G_2/N_2)$ are homotopic. Hence we can rewrite the above diagram as follows

$$\begin{array}{ccc} K(G_1 *_H G_2) & \longrightarrow & K(G_1 *_H G_2/N_1 *_H N_2) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_1 \cup X_2. \end{array}$$

Now using the Mayer-Vietoris sequence for this recent pushout diagram, we obtain the following exact sequence

$$\begin{aligned} \cdots \rightarrow H_3(K(G_1 *_H G_2)) &\rightarrow H_3(K(G_1 *_H G_2/N_1 *_H N_2)) \rightarrow H_3(X_1 \cup X_2) \rightarrow \\ &H_2(K(G_1 *_H G_2)) \rightarrow H_2(K(G_1 *_H G_2/N_1 *_H N_2)) \rightarrow \cdots \end{aligned}$$

Also using Mayer-Vietoris sequence for join spaces, we have the isomorphism $H_n(K(G_1/N_1) \cup K(G_2/N_2)) \cong H_n(K(G_1/N_1)) \oplus H_n(K(G_2/N_2))$, for any $n \in \mathbf{N}$. Finally, by the topological definitions of the Schur multiplier of a group and a pair of groups, $M(G_1 *_H G_2, N_1 *_H N_2) \cong H_3(X_1 \cup X_2)$ and $M(G_1 *_H G_2) \cong H_2(K(G_1 *_H G_2))$. Hence we get the following exact sequence

$$\cdots \rightarrow H_3(K(G_1 *_H G_2)) \rightarrow H_3(G_1/N_1) \oplus H_3(G_2/N_2) \rightarrow$$

$M(G_1 *_H G_2, N_1 *_H N_2) \rightarrow M(G_1 *_H G_2) \rightarrow M(G_1/N_1) \oplus M(G_2/N_2) \rightarrow \dots$. \square

Note that the authors in [6], using topological methods, proved that for any two groups G_1 and G_2 , the following isomorphism holds

$$M(G_1 \times G_2) \cong M(G_1) \oplus M(G_2) \oplus (G_1)_{ab} \otimes (G_2)_{ab}.$$

In the following we extend this result.

Theorem 2.5. *For any two pairs of groups (G_1, N_1) , (G_2, N_2) , the Schur multiplier of a pair of the direct products $(G_1 \times G_2, N_1 \times N_2)$ satisfies the following exact sequence*

$$\begin{aligned} \dots \rightarrow H_3(G_1) \oplus (M(G_1) \otimes G_2^{ab}) \oplus (G_1^{ab} \otimes M(G_2)) \oplus \\ H_3(G_2) \oplus \text{Tor}(G_1^{ab}, G_2^{ab}) \rightarrow H_3(G_1/N_1) \oplus (M(G_1/N_1) \otimes G_2^{ab} N_2/N_2) \oplus \\ (G_1^{ab} N_1/N_1 \otimes M(G_2/N_2)) \oplus H_3(G_2/N_2) \oplus \text{Tor}(G_1^{ab} N_1/N_1, G_2^{ab} N_2/N_2) \rightarrow \\ M(G_1 \times G_2, N_1 \times N_2) \rightarrow M(G_1 \times G_2) \rightarrow M(G_1/N_1 \times G_2/N_2) \rightarrow \dots \end{aligned}$$

Proof. First, we consider the following pushout diagram

$$\begin{array}{ccc} K(G_1 \times G_2) & \longrightarrow & K(G_1 \times G_2/N_1 \times N_2) \\ & & \downarrow \quad \downarrow \\ & & 1 \longrightarrow X. \end{array}$$

By the definition, we know that $M(G_1 \times G_2, N_1 \times N_2) = H_3(X)$. On the other hand, by Mayer-Vietoris sequence for the above diagram we conclude the following exact sequence

$$\begin{aligned} \dots \rightarrow H_3(K(G_1 \times G_2)) \rightarrow H_3(K(G_1 \times G_2/N_1 \times N_2)) \rightarrow H_3(X) \rightarrow \\ H_2(K(G_1 \times G_2)) \rightarrow H_2(K(G_1 \times G_2/N_1 \times N_2)) \rightarrow \dots \end{aligned}$$

Using $K(G_1 \times G_2) = K(G_1) \times K(G_2)$, the Künneth Formula and some properties of the functor Tor and tensor product we have

$$\begin{aligned} H_3(K(G_1 \times G_2)) &= H_3(K(G_1) \times K(G_2)) \cong \\ H_3(K(G_1)) \oplus (H_2(K(G_1)) \otimes H_1(K(G_2))) \oplus (H_1(K(G_1)) \otimes H_2(K(G_2))) \oplus \\ &H_3(K(G_2)) \oplus \text{Tor}(H_1(K(G_1)), H_1(K(G_2))). \end{aligned}$$

By the similar argument for $H_3(K(G_1 \times G_2/N_1 \times N_2)) = H_3(K(G_1/N_1 \times G_2/N_2))$ and the isomorphisms $H_1(K(G)) \cong G^{ab}$, $H_2(K(G)) \cong M(G)$ we obtain the following exact sequence

$$\begin{aligned} \dots \rightarrow H_3(K(G_1)) \oplus (M(G_1) \otimes G_2^{ab}) \oplus (G_1^{ab} \otimes M(G_2)) \oplus \\ H_3(K(G_2)) \oplus \text{Tor}(G_1^{ab}, G_2^{ab}) \rightarrow H_3(K(G_1/N_1)) \oplus (M(G_1/N_1) \otimes G_2^{ab} N_2/N_2) \oplus \\ (G_1^{ab} N_1/N_1 \otimes M(G_2/N_2)) \oplus H_3(K(G_2/N_2)) \oplus \text{Tor}(G_1^{ab} N_1/N_1, G_2^{ab} N_2/N_2) \rightarrow \\ M(G_1 \times G_2, N_1 \times N_2) \rightarrow M(G_1 \times G_2) \rightarrow M(G_1/N_1 \times G_2/N_2) \rightarrow \dots \square \end{aligned}$$

Remarks 2.6. Using the isomorphism $H_3(\mathbf{Z}_m) \cong \mathbf{Z}_m [1]$, in some special cases we can rewrite the above exact sequence as follows:

(i) If G_i/N_i is a finite cyclic group of order m_i ($i = 1, 2$), then we have the following exact sequence

$$\begin{aligned} \cdots \rightarrow \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \mathbf{Z}_d \rightarrow M(G_1 \times G_2, N_1 \times N_2) \rightarrow \\ M(G_1) \oplus M(G_2) \oplus (G_1 \otimes G_2) \rightarrow \mathbf{Z}_d \rightarrow \cdots, \end{aligned}$$

where $d = \gcd(m_1, m_2)$. Moreover, if G_1 and G_2 are also cyclic, then we get the following exact sequence

$$\begin{aligned} \cdots \rightarrow \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \mathbf{Z}_d \rightarrow M(G_1 \times G_2, N_1 \times N_2) \rightarrow \\ G_1 \otimes G_2 \rightarrow \mathbf{Z}_d \rightarrow \cdots. \end{aligned}$$

(ii) If $\gcd(|G_1/N_1|, |G_2/N_2|) = 1$, then the following exact sequence holds

$$\begin{aligned} \cdots \rightarrow H_3(G_1/N_1) \oplus H_3(G_2/N_2) \rightarrow M(G_1 \times G_2, N_1 \times N_2) \rightarrow \\ M(G_1) \oplus M(G_2) \oplus (G_1 \otimes G_2) \rightarrow M(G_1/N_1) \oplus M(G_2/N_2) \rightarrow \cdots. \end{aligned}$$

Moreover, if $\gcd(|G_1|, |G_2|) = 1$, then we conclude the following exact sequence

$$\begin{aligned} \cdots \rightarrow H_3(G_1/N_1) \oplus H_3(G_2/N_2) \rightarrow M(G_1 \times G_2, N_1 \times N_2) \rightarrow \\ M(G_1) \oplus M(G_2) \rightarrow M(G_1/N_1) \oplus M(G_2/N_2) \rightarrow \cdots. \end{aligned}$$

(iii) Finally, if G_i is a finite cyclic group ($i = 1, 2$) such that $\gcd(|G_1|, |G_2|) = 1$, then we have the following exact sequence

$$\cdots \rightarrow \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \rightarrow M(G_1 \times G_2, N_1 \times N_2) \rightarrow 0.$$

The following theorem has been proved by Ellis [4]. Here we present a topological proof for this theorem.

Theorem 2.7. *If $G = N \rtimes Q$ is the semidirect product of a normal subgroup N by a subgroup Q , then*

$$M(G) \cong M(G, N) \oplus M(Q).$$

Proof. Using the corresponding pushout diagram

$$\begin{array}{ccc} K(G) & \longrightarrow & K(G/N) = K(Q) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X, \end{array}$$

and the Mayer-Vietoris sequence for this diagram, we conclude the following exact sequence

$$\begin{aligned} \cdots \rightarrow H_3(K(G)) \rightarrow H_3(K(Q)) \rightarrow H_3(X) \rightarrow \\ H_2(K(G)) \rightarrow H_2(K(Q)) \rightarrow H_2(X) \rightarrow \cdots. \quad (2.1) \end{aligned}$$

Since $G = N \rtimes Q$, there exists a homomorphism $\beta : Q \rightarrow G$ such that $\alpha \circ \beta = 1_Q$. $H_n(K(-))$ is the composition of two functors and so the induced homomorphism $\alpha_* : H_n(K(G)) \rightarrow H_n(K(Q))$ is surjection; and hence the exactness

of (2.1) implies the injectivity of the homomorphism $H_n(X) \rightarrow H_{n-1}(K(G))$. Thus we have the following exact split sequence

$$0 \rightarrow H_3(X) \rightarrow H_2(K(G)) \rightarrow H_2(K(Q)) \rightarrow 0$$

which completes the proof. \square

Theorem 2.8. *Suppose that M and N are two subgroups of a group G so that $M \cong MN$, then the following isomorphism exists:*

$$M(MN, N) \cong M(M, M \cap N).$$

Proof. By the Second Isomorphism Theorem we have $MN/N \cong M/M \cap N$. Because of the functorial property of $K(-)$, we conclude the homotopy equivalences $K(M) \approx K(MN)$ and $K(MN/N) \approx K(M/M \cap N)$. Therefore, by the uniqueness of the pushout in the category $hTop$, we can identify two following homotopy pushout diagrams in this category

$$\begin{array}{ccc} K(MN) & \longrightarrow & K(MN/N) & & K(M) & \longrightarrow & K(M/M \cap N) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & \longrightarrow & X, & & 1 & \longrightarrow & Y. \end{array}$$

Hence by the definition,

$$M(MN, N) = H_3(X) \cong H_3(Y) = M(M, M \cap N). \quad \square$$

Note that applying the five lemma and the properties of the direct limit of a directed system to an exact sequence, namely the fact that this functor is exact and it commutes with homology, one can show that for a directed system $\{(G_i, N_i)\}_{i \in I}$ of any pair of groups, the following isomorphism holds

$$M(\varinjlim G_i, \varinjlim N_i) \cong \varinjlim M(G_i, N_i).$$

However in the following we establish a topological proof for this fact in special case, where $\{(G_i, N_i)\}_{i \in I}$ is a directed system of pairs of abelian groups. First we need the following lemma.

Lemma 2.9. *The direct limit of a direct system $\{X_i\}_{i \in I}$ of Eilenberg-MacLane spaces is an Eilenberg-MacLane space. Moreover, if the system $\{X_i\}_{i \in I}$ is directed and $\pi_1(X_i)$ is abelian group, for any $i \in I$, then the Eilenberg-MacLane space $\varinjlim X_i$ is corresponding to the group $\varinjlim \pi_1(X_i)$.*

Proof. Consider the induced direct system $\{\pi_n(X_i)\}_{i \in I}$. In order to prove that $\varinjlim X_i$ is an Eilenberg-MacLane space, first we show that $\pi_1(\varinjlim X_i)$ is abelian. For any α and β in $\pi_1(\varinjlim X_i)$, there exists two paths γ and λ in $\varinjlim X_i$ such that $\alpha = [\gamma]$ and $\beta = [\lambda]$. Note that $\varinjlim X_i$ is a quotient space of the wedge space $\vee_{i \in I} X_i$, and so we can consider $\gamma \cap X_i$ and $\lambda \cap X_i$ as two paths in X_i , for any $i \in I$. Since $\pi_1(X_i)$'s are abelian groups, $\gamma \cap X_i$ and $\lambda \cap X_i$ commute with each other in the space X_i ($i \in I$) up to homotopy. Therefore two paths γ and λ commute with each other in the space $\varinjlim X_i$ up

to homotopy. Thus $\pi_1(\varinjlim X_i)$ is an abelian group. By a similar argument we can show that $\pi_n(\varinjlim X_i)$ is a trivial group, for any $n \geq 2$. If α is an element of $\pi_n(\varinjlim X_i)$, there exists an n -loop γ in $\varinjlim X_i$ such that $\alpha = [\gamma]$. Similar to the previous note, $\gamma \cap X_i$ is an n -loop in X_i ($i \in I$); and since $\pi_n(X_i)$'s are trivial groups, so $[\gamma \cap X_i]$ is trivial in $\pi_n(X_i)$, for any $i \in I$. Because of the structure of the space $\varinjlim X_i$, $[\gamma]$ is also trivial in $\pi_n(\varinjlim X_i)$. This note is true for any arbitrary $n \geq 2$ and so $\pi_n(\varinjlim X_i)$ ($n \geq 2$) is trivial. So if the space X_i , for any $i \in I$, is Eilenberg-MacLane, then the space $\varinjlim X_i$ should be also Eilenberg-MacLane.

Moreover, if $\pi_1(X_i)$'s are abelian groups, we have $\pi_1(X_i) \cong H_1(X_i)$, for any $i \in I$. Hence using the fact that homology functors commute with direct limits of directed systems, we conclude that

$$\varinjlim \pi_1(X_i) \cong \varinjlim H_1(X_i) \cong H_1(\varinjlim X_i).$$

Also, $\pi_1(\varinjlim X_i)$, as a homomorphic image of the abelian group $\varinjlim \pi_1(X_i)$, is abelian, and so we have

$$H_1(\varinjlim X_i) \cong \pi_1(\varinjlim X_i)$$

which completes the proof. \square

Theorem 2.10. *Let $\{(G_i, N_i)\}_{i \in I}$ be a given directed system of pairs of abelian groups, then $M(\varinjlim G_i, \varinjlim N_i) \cong \varinjlim M(G_i, N_i)$.*

Proof. First, for any $i \in I$, we consider the corresponding pushout diagram

$$\begin{array}{ccc} K(G_i) & \longrightarrow & K(G_i/N_i) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X_i. \end{array}$$

Using the fact that any two direct limits commute, we conclude that the following diagram is also a pushout diagram,

$$\begin{array}{ccc} \varinjlim K(G_i) & \longrightarrow & \varinjlim K(G_i/N_i) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \varinjlim X_i. \end{array}$$

Now by Lemma 2.9, we rewrite the above diagram as follow

$$\begin{array}{ccc} K(\varinjlim G_i) & \longrightarrow & K(\varinjlim G_i/N_i) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \varinjlim X_i. \end{array}$$

Finally, by the definition and the fact that homology groups commute with the direct limit of a directed system, we have

$$M(\varinjlim G_i, \varinjlim N_i) = H_3(\varinjlim X_i)$$

$$\cong \varinjlim H_3(X_i) = \varinjlim M(G_i, N_i). \quad \square$$

3. SCHUR MULTIPLIERS OF TRIPLES

Let (G, M, N) be a triple of groups. Consider the following homotopy pushout

$$\begin{array}{ccc} K(G, N) & \longrightarrow & K(G/M, MN/M) \\ & \downarrow & \downarrow \\ & 1 & \longrightarrow X. \end{array}$$

Using the Mayer-Vietoris sequence for homotopy pushout, we have the following exact sequence

$$H_4(K(G, N)) \rightarrow H_4(K(G/M, MN/M)) \rightarrow H_4(X) \rightarrow$$

$$H_3(K(G, N)) \rightarrow H_3(K(G/M, MN/M)) \rightarrow$$

$$H_3(X) \rightarrow H_2(K(G, N)) \rightarrow \dots$$

As $H_4(K(G, N)) = H_3(G, N)$, $M(G, N) = H_3(K(G, N))$, $H_3(X) = M \cap N/[M \cap N, G]$ and $H_2(K(G, N)) = 0$ (see [2]), we obtain the following exact sequence

$$H_3(G, N) \rightarrow H_3(G/M, MN/M) \rightarrow H_4(X) \rightarrow M(G, N)$$

$$M(G/M, MN/M) \rightarrow M \cap N/[M \cap N, G][M, N]$$

$$\rightarrow N/[N, G] \rightarrow NM/M[N, G] \rightarrow 0.$$

Now, the Schur multiplier of the triple (G, M, N) is defined to be the fourth homology group of the space X , $M(G, M, N) = H_4(X)$. If $K(G, M, N)$ denotes the mapping cone of the canonical map $K(G, N) \rightarrow K(G/M, MN/M)$, then it is easy to see that the space X and $K(G, M, N)$ have the same homotopy type. Therefore the above definition of the Schur multiplier of a triple coincides with the definition presented in [4] by Ellis. Using the above topological interpretation of the Schur multiplier of a triple and similar to Theorem 2.7, we can study the behavior of the Schur multiplier of a triple with respect to direct limits as follows. However, by this definition, it does not seem possible to peruse the behavior of the multiplier of a triple with several products of groups, as done for the multiplier of a pair of groups. So in continue, we intend to present a more useful definition for this notion.

Theorem 3.1. *Let $\{(G_i, M_i, N_i)\}_{i \in I}$ be a directed system of triples of abelian groups such that $M_i \cap N_i = 1$ for all $i \in I$. Then*

$$M(\varinjlim G_i, \varinjlim M_i, \varinjlim N_i) \cong \varinjlim M(G_i, M_i, N_i).$$

Proof. First, by the proof of Theorem 2.7, we recall that for any abelian group G_i , the space $\lim_{\rightarrow} K(G_i, H_i)$ can be considered as $K(\lim_{\rightarrow} G_i, \lim_{\rightarrow} H_i)$. For the group $\lim_{\rightarrow} M(G_i, M_i, N_i)$, we have the following diagram

$$\begin{array}{ccc} \lim_{\rightarrow} K(G_i, N_i) & \longrightarrow & \lim_{\rightarrow} K(G_i/M_i, M_i N_i/M_i) \\ & & \downarrow \quad \downarrow \\ & & 1 \longrightarrow \lim_{\rightarrow} X_i. \end{array}$$

But by the above facts, we can replace this diagram by the new one

$$\begin{array}{ccc} K(\lim_{\rightarrow} G_i, \lim_{\rightarrow} N_i) & \longrightarrow & K(\lim_{\rightarrow} G_i/M_i, \lim_{\rightarrow} M_i N_i/M_i) \\ & & \downarrow \quad \downarrow \\ & & 1 \longrightarrow \lim_{\rightarrow} X_i. \end{array}$$

Now by the property of direct limit which preserves exact sequences, we reform the above diagram to the following

$$\begin{array}{ccc} K(\lim_{\rightarrow} G_i, \lim_{\rightarrow} N_i) & \longrightarrow & K(\lim_{\rightarrow} G_i/\lim_{\rightarrow} M_i, \lim_{\rightarrow} M_i \lim_{\rightarrow} N_i/\lim_{\rightarrow} M_i) \\ & & \downarrow \quad \downarrow \\ & & 1 \longrightarrow \lim_{\rightarrow} X_i. \end{array}$$

The last diagram completes the proof. \square

According to our topological definition of the Schur multiplier of a pair of groups in previous section, it seems that we can generalize this notion to triples more natural than the one of Ellis. In order to define this new version of the Schur multiplier of a triple (G, M, N) consider the following homotopy pushout

$$\begin{array}{ccc} K(G) & \longrightarrow & K(G/N) \\ & \downarrow & \downarrow \\ & K(G/M) & \longrightarrow & X. \end{array}$$

Now we define the new version of the Schur multiplier of the triple (G, M, N) to be the third homology group of the space X , $H_3(X)$.

Using the Mayer-Vietoris sequence for the above diagram we conclude the following exact sequence

$$\begin{aligned} \cdots \rightarrow H_3(K(G)) \rightarrow H_3(K(G/N)) \oplus H_3(K(G/M)) \rightarrow H_3(X) \rightarrow H_2(K(G)) \rightarrow \\ H_2(K(G/N)) \oplus H_2(K(G/M)) \rightarrow H_2(X) \rightarrow \cdots. \end{aligned}$$

By Hopf Formula [7] for any group G , $H_2(K(G)) = M(G)$; so we obtain the following exact sequence

$$\begin{aligned} \cdots \rightarrow H_3(K(G)) \rightarrow H_3(K(G/N)) \oplus H_3(K(G/M)) \rightarrow \\ M(G, N, M) \rightarrow M(G) \rightarrow M(G/N) \oplus M(G/M) \rightarrow \cdots. \end{aligned}$$

Remarks 3.2.

(i) Note that if $G = MN$, then by [2] $H_3(X) \cong \ker(N \wedge M \rightarrow G)$. Also, Ellis

[4] mentioned that if $G = MN$, then $M(G, N, M) \cong \ker(N \wedge M \rightarrow G)$. Hence our definition of the Schur multiplier of a triple of groups coincide with the definition of Ellis, if $G = MN$.

(ii) Since our definition of the Schur multiplier of a triple of groups is a natural generalization of the pair's one, we can present more behaviors of this new notion with respect to free, amalgamated free, and direct products and also a better behavior with respect to direct limits than the one of Ellis.

The following results are evidence for the above claim.

Theorem 3.3. *For any two triple of groups (G_1, N_1, M_1) and (G_2, N_2, M_2) , we have the following isomorphism*

$$M(G_1 * G_2, \langle N_1 * N_2 \rangle^{G_1 * G_2}, \langle M_1 * M_2 \rangle^{G_1 * G_2}) \cong M(G_1, N_1, M_1) \oplus M(G_2, N_2, M_2).$$

Proof. Suppose that X_i ($i = 1, 2$) is the pushout of corresponding diagram

$$\begin{array}{ccc} K(G_i) & \longrightarrow & K(G_i/N_i) \\ \downarrow & & \downarrow \\ K(G_i/M_i) & \longrightarrow & X_i. \end{array}$$

Similar to the proof of the Theorem 2.3, using the fact that any two direct limits commute, we conclude that the space $X_1 \vee X_2$ is also a pushout for the following diagram

$$\begin{array}{ccc} K(G_1) \vee K(G_2) & \longrightarrow & K(G_1/N_1) \vee K(G_2/N_2) \\ \downarrow & & \downarrow \\ K(G_1/M_1) \vee K(G_2/M_2) & \longrightarrow & X_1 \vee X_2. \end{array}$$

According to Van-Kampen Theorem, we rewrite the above diagram as follows

$$\begin{array}{ccc} K(G_1 * G_2) & \longrightarrow & K(G_1/N_1 * G_2/N_2) = K(G_1 * G_2 / \langle N_1 * N_2 \rangle^{G_1 * G_2}) \\ \downarrow & & \downarrow \\ K(G_1 * G_2 / \langle M_1 * M_2 \rangle^{G_1 * G_2}) & \longrightarrow & K(G_1/M_1 * G_2/M_2) \longrightarrow X_1 \vee X_2. \end{array}$$

Now by the definition and the Mayer-Vietoris sequence for the above diagram we conclude the following isomorphism

$$\begin{aligned} M(G_1 * G_2, \langle N_1 * N_2 \rangle^{G_1 * G_2}, \langle M_1 * M_2 \rangle^{G_1 * G_2}) &= H_3(X_1 \vee X_2) \cong \\ H_3(X_1) \oplus H_3(X_2) &= M(G_1, N_1, M_1) \oplus M(G_2, N_2, M_2). \quad \square \end{aligned}$$

Also, by naturalness of our new version of the Schur multiplier of triple, we can give more results about the Schur multiplier of triples of groups deduced with the proofs similar to those of pairs as follows.

Theorem 3.4. *Let (G_1, N_1, M_1) , (G_2, N_2, M_2) be two triples of groups and $H \leq N_1 \cap N_2$, $H \leq M_1 \cap M_2$. Then the Schur multiplier of the triple of amalgamated free products $(G_1 *_H G_2, N_1 *_H N_2, M_1 *_H M_2)$ satisfies the following exact sequence*

$$\begin{aligned} \cdots \rightarrow H_3(K(G_1 *_H G_2)) &\rightarrow H_3(G_1/N_1) \oplus H_3(G_2/N_2) \oplus H_3(G_1/M_1) \oplus H_3(G_2/M_2) \rightarrow \\ M(G_1 *_H G_2, N_1 *_H N_2, M_1 *_H M_2) &\rightarrow M(G_1 *_H G_2) \rightarrow \end{aligned}$$

$$M(G_1/N_1) \oplus M(G_2/N_2) \oplus M(G_1/M_1) \oplus M(G_2/M_2) \rightarrow \cdots .$$

Theorem 3.5. For any two triples of groups (G_1, N_1, M_1) and (G_2, N_2, M_2) , the Schur multiplier of the triple of direct products $(G_1 \times G_2, N_1 \times N_2, M_1 \times M_2)$ satisfies the following exact sequence

$$\begin{aligned} H_3(K(G_1)) \oplus (M(G_1) \otimes G_2^{ab}) \oplus (G_1^{ab} \otimes M(G_2)) \oplus H_3(K(G_2)) \oplus \text{Tor}(G_1^{ab}, G_2^{ab}) \rightarrow \\ H_3(K(G_1/N_1)) \oplus (M(G_1/N_1) \otimes G_2^{ab} N_2/N_2) \oplus (G_1^{ab} N_1/N_1 \otimes M(G_2/N_2)) \oplus \\ H_3(K(G_2/N_2)) \oplus \text{Tor}(G_1^{ab} N_1/N_1, G_2^{ab} N_2/N_2) \oplus H_3(K(G_1/M_1)) \oplus \\ (M(G_1/M_1) \otimes G_2^{ab} M_2/M_2) \oplus (G_1^{ab} M_1/M_1 \otimes M(G_2/M_2)) \oplus \\ H_3(K(G_2/M_2)) \oplus \text{Tor}(G_1^{ab} M_1/M_1, G_2^{ab} M_2/M_2) \rightarrow \\ M(G_1 \times G_2, N_1 \times N_2, M_1 \times M_2) \rightarrow M(G_1 \times G_2) \\ \rightarrow M(G_1/N_1 \times G_2/N_2) \oplus M(G_1/M_1 \times G_2/M_2) \rightarrow \cdots . \end{aligned}$$

Remarks 3.6. Some special cases of the above exact sequence are as follows:

(i) If G_i/N_i and G_i/M_i are finite cyclic groups of orders m_i and l_i ($i = 1, 2$), respectively, then we have the following exact sequence

$$\begin{aligned} \cdots \rightarrow \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \mathbf{Z}_d \oplus \mathbf{Z}_{l_1} \oplus \mathbf{Z}_{l_2} \oplus \mathbf{Z}_c \rightarrow \\ M(G_1 \times G_2, N_1 \times N_2, M_1 \times M_2) \rightarrow \\ M(G_1) \oplus M(G_2) \oplus (G_1 \otimes G_2) \rightarrow \mathbf{Z}_d \oplus \mathbf{Z}_c \rightarrow \cdots , \end{aligned}$$

where $d = \gcd(m_1, m_2)$ and $c = \gcd(l_1, l_2)$. Moreover, if G_1 and G_2 are also cyclic, then we get the following exact sequence

$$\begin{aligned} \cdots \rightarrow \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \mathbf{Z}_d \oplus \mathbf{Z}_{l_1} \oplus \mathbf{Z}_{l_2} \oplus \mathbf{Z}_c \rightarrow \\ M(G_1 \times G_2, N_1 \times N_2, M_1 \times M_2) \rightarrow \\ G_1 \otimes G_2 \rightarrow \mathbf{Z}_d \oplus \mathbf{Z}_c \rightarrow \cdots . \end{aligned}$$

(ii) If $\gcd(|G_1/N_1|, |G_2/N_2|) = 1$ and $\gcd(|G_1/M_1|, |G_2/M_2|) = 1$, then the following exact sequence holds

$$\begin{aligned} \cdots \rightarrow H_3(G_1/N_1) \oplus H_3(G_2/N_2) \oplus H_3(G_1/M_1) \oplus H_3(G_2/M_2) \rightarrow \\ M(G_1 \times G_2, N_1 \times N_2, M_1 \times M_2) \rightarrow M(G_1) \oplus M(G_2) \oplus (G_1 \otimes G_2) \rightarrow \\ M(G_1/N_1) \oplus M(G_2/N_2) \oplus M(G_1/M_1) \oplus M(G_2/M_2) \rightarrow \cdots . \end{aligned}$$

Moreover, if G_i is a finite group ($i = 1, 2$) such that $\gcd(|G_1|, |G_2|) = 1$, then we conclude the following exact sequence

$$\begin{aligned} \cdots \rightarrow H_3(G_1/N_1) \oplus H_3(G_2/N_2) \oplus H_3(G_1/M_1) \oplus H_3(G_2/M_2) \rightarrow \\ M(G_1 \times G_2, N_1 \times N_2, M_1 \times M_2) \rightarrow M(G_1) \oplus M(G_2) \rightarrow \\ M(/G_1 N_1) \oplus M(G_2/N_2) \oplus M(G_1/M_1) \oplus M(G_2/M_2) \rightarrow \cdots . \end{aligned}$$

(iii) Finally, if G_i is a finite cyclic group ($i = 1, 2$) such that $\gcd(|G_1|, |G_2|) = 1$, then we have the following exact sequence

$$\cdots \rightarrow \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \mathbf{Z}_{l_1} \oplus \mathbf{Z}_{l_2} \rightarrow$$

$$M(G_1 \times G_2, N_1 \times N_2, M_1 \times M_2) \rightarrow 0.$$

Finally, we can use Lemma 2.9 and deduce a similar result for triples, as follows.

Theorem 3.7. *The Schur multiplier of a triple of groups commutes with the direct limit of a directed system of triples of abelian groups.*

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