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Constructing Finite Frames via Platonic Solids

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Abstract. Finite tight frames have many applications and some interesting physical interpretations. One of the important subjects in this area is the ways for constructing such frames. In this article we give a concrete method for constructing finite normalized frames using Platonic solids.

Keywords: finite frame, tight frame, normalized frame, platonic solids.

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1. Introduction

Frames were first introduced in 1952 by Duffin and Sheafer [5] in the context of nonharmonic Fourier series. Theory of frames have developed very fast in last two decades. This is because they provide powerful tools in various area such as signal processing, data compression, wireless communications etc.[7,9,10]. Also they have been studied from pure settings [8]. Frames are systems of vectors in Hilbert spaces that provide robust, stable and mostly non-unique representations of vectors. Recently, frames in finite - dimensional Hilbert spaces have become of interests for many of researchers because of their nice interpretations and useful applications[1,2,4]. One of the important subjects in this area is the ways for constructing such frames. A few methods are introduced by some authors as in [3] and [6]. In this article we give an explicit

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and concrete method for constructing such frames using regular polyhedrons known as "Platonic solids". For this purpose, in the next section we introduce the basis of frame theory and some related topics. In the last section, we give our main results which is the method for constructing finite normalized tight frames using Platonic solids.

2. Frames

Let H be a Hilbert space. A sequence $\{f_i\}_{i\in I}$ in H is said to be a *frame* for H if there exist constants A and B such that $0 < A \leq B < \infty$ and the inequalities

$$A \parallel f \parallel^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B \parallel f \parallel^2,$$

holds for every f in H. If only the right side of the above inequalities holds, then $\{f_i\}_{i\in I}$ is called a *Bessel sequence*. Constants A and B are called *lower* and *upper* frame bounds, respectively. The frame $\{f_i\}_{i\in I}$ is said to be *tight* (or A-tight)if $A = B$, and it is a *Parseval* frame if $A = B = 1$. In this case, A is said to be the *frame constant*. When the index set I is a finite set, the frame will be called *finite*. A *normalized frame* is the one whose elements have norm one.

To each Bessel sequence $\{f_i\}_{i\in I}$, corresponds an operator

$$F : H \to l^2(I),\ F(f) = \{\langle f, f_i\rangle\}_{i\in I}$$

called *analysis operator*, where $l^2(I)$ is the space of all complex sequences $\{c_i\}_{i\in I}$ such that $\sum_{i\in I} |c_i|^2 < \infty$. This is a well-defined and bounded operator. Its adjoint is the operator

$$F^* : l^2(I) \to H,\ F^*(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i,$$

called *the synthesis operator*. If $\{f_i\}_{i\in I}$ is a frame with frame bounds A and B, then the operator

$$F^*F : H \to H,\ F^*F(f) = \sum_{i\in I} \langle f, f_i\rangle f_i$$

is called *the frame operator* of the frame $\{f_i\}_{i\in I}$. It is a positive, self-adjoint, bounded and hence invertible operator with the inverse $(F^*F)^{-1}$. In fact, $AI \leq F^*F \leq BI$ and $B^{-1}I \leq (F^*F)^{-1} \leq A^{-1}I$.

If $\{g_i\}$ is another sequence in H such that each $f \in H$ can be represented as $f = \sum_{i\in I}\langle f, f_i\rangle g_i$, then $\{g_i\}_{i\in I}$ is called a *dual frame* for $\{f_i\}_{i\in I}$. It can be shown that $\{(F^*F)^{-1}(f_i)\}_{i\in I}$ is a dual frame for the frame $\{f_i\}_{i\in I}$, called the *canonical dual* of $\{f_i\}_{i\in I}$. Having this dual, we get the following reconstruction formula:

$$f = F^*F(F^*F)^{-1}(f) = \sum_{i\in I} \langle f, (F^*F)^{-1} f_i\rangle f_i.$$

If $\{f_i\}_{i\in I}$ is a tight frame, i.e. $A = B$, then $F^*F = AI$ and hence we have $f = \frac{1}{A}\sum_i \langle f, f_i\rangle f_i$, for every $f \in H$.
For the rest of this article, we suppose that $H = H_N$ is a finite-dimensional Hilbert space. According to this, our frame will be of the form $\{f_i\}_{i=1}^M$, where M is some positive integer. Also we will replace $l^2(I)$ by K^M, where $K = \mathbb{R}$, or $K = \mathbb{C}$.

Lemma 2.1. *Every finite sequence $\{f_i\}_{i=1}^M$ in the Hilbert space H_N is a Bessel sequence.*

Proof. Put $B = \sum_{i=1}^M \| f_i \|^2$. Since $|\langle f, f_i\rangle|^2 \leq \| f \| \| f_i \|^2$, so

$$\sum_i^M |\langle f, f_i\rangle|^2 \leq \sum_{i=1}^M \| f \|^2 \| f_i \|^2 \leq B \| f \|^2 . \square$$

The above lemma guarantees the existence of the analysis and synthesis operator in the finite case. In fact,

$$F : H_N \to K^M,\ F^* : K^M \to H_N,\ F^*F : H_N \to H_N.$$

These operators, from left to right, can be showed by $M \times N$, $N \times M$, and $N \times N$ matrices, respectively. By considering an orthonormal basis for H_N, we get an explicit structure for analysis and synthesis operators when dealing with a frame. Let $\{e_n\}_{n=1}^N$ be an orthonormal basis for H_N. The coordinates of a vector $h \in H_N$ with respect to this basis is the column vector $[h]$ so that $[h] \in K^N$ and $[h](n) = \langle h, e_n\rangle$. So, when $\{f_m\}_{m=1}^M$ is a frame, the matrix representation of the synthesis and analysis operators with respect to the basis $\{e_n\}_{n=1}^N$ and the standard basis for K^N,will be as below:

$$[F] = \begin{pmatrix} [f_1]^* \\ [f_2]^* \\ \vdots \\ [f_M]^* \end{pmatrix},\ [F^*] = [F]^* = \begin{pmatrix} [f_1] & [f_2] & \dots & [f_M] \end{pmatrix}.$$

So,

$$[F] = \begin{pmatrix} \langle e_1, f_1\rangle, & \dots & \langle e_N, f_1\rangle \\ \langle e_1, f_2\rangle, & \dots & \langle e_n, f_2\rangle \\ \vdots & & \vdots \\ \langle e_1, f_M\rangle, & \dots & \langle e_N, f_M\rangle \end{pmatrix} \text{ and } [F]^* = \begin{pmatrix} \langle f_1, e_1\rangle & \dots & \langle f_M, e_1\rangle \\ \langle f_1, e_2\rangle & \dots & \langle f_M, e_2\rangle \\ \vdots & & \vdots \\ \langle f_1, e_N\rangle & \dots & \langle f_m, e_n\rangle \end{pmatrix}.$$

A finite normalized tight frame with the frame constant A will be called an A-FNTF. Benedetto and Fickus [1] proved the following result.

Theorem 2.2. *If $\{x_n\}_{n=1}^N$ is an A-FNTF for a d-dimensional Hilbert space H, then $A = \frac{N}{d}$.*

Examples 2.3. *a) Consider the following four vectors in* $\mathbb{R}^3$*:*

$$f_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, f_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, f_4 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Clearly they form a frame for $\mathbb{R}^3$*. Its synthesis, analysis and frame operators are:*

$$[F]^* = \begin{pmatrix} 2 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, [F] = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}, [F^*F] = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Since F^*F *is not a multiple of identity, this is not a tight frame.*

b) Consider the following set of vectors in $\mathbb{R}^3$*:*

$$\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\}$$

Again this set of vectors is a frame for $\mathbb{R}^3$ *whose corresponding operators are*

$$[F^*] = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, [F] = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, [F^*F] = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Since $[F^*F] = 4I$*, this is an A-tight frame with* $A = 4$*, but by the above theorem it is not anA-FNTF because* $A \neq \frac{4}{3}$*.*

3. The Platonic Solids

A Platonic solid is a convex polyhedron that is regular, in the sense of a regular polygon. Specifically, the faces of a Platonic solid are congruent regular polygons, with the same number of faces meeting at each vertex; thus, all its edges are congruent, as are its vertices and angles. There are precisely five Platonic solids: Tetrahedron, Cube (or hexahedron), Octahedron, Dodecahedron and Icosahedron. The following theorem is our main result in this article.

Theorem 3.1. *Vertices of each of the Platonic solids form an A-FNTF for* $\mathbb{R}^3$*.*

Proof. (i) (Tetrahedron):To show that vertices of tetrahedron form an A-FNTF for $\mathbb{R}^3$, first we consider the third roots of the unity

$$(1,0),\ ,(-\frac{1}{2}, \frac{\sqrt{3}}{2}),\ (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

These three points are vertices of an equilateral triangle. Now we translate $\sqrt{x}$ times of these point as $\sqrt{1-x}$ into the space to get three new points.

These points are located on the surface of S^2, the unit sphere in $\mathbb{R}^3$. To have a tetrahedron, we need another point. Let this point be shown as (y_1, y_2, y_3). Now according the discussions after Lemma 2.1,

$$F^* = \begin{pmatrix} \sqrt{x} & -\frac{\sqrt{x}}{2} & -\frac{\sqrt{x}}{2} & y_1 \\ 0 & \frac{\sqrt{3x}}{2} & -\frac{\sqrt{3x}}{2} & y_2 \\ \sqrt{1-x} & \sqrt{1-x} & \sqrt{1-x} & y_3 \end{pmatrix}.$$

In order that these four vectors form a tight frame for $\mathbb{R}^3$, we should have $F^*F = AI$ and Theorem 2.3 forces that $A = \frac{4}{3}$. For this happens, the following equations should hold:

$$\begin{aligned} x + \tfrac{x}{4} + \tfrac{x}{4} + y_1^2 &= \tfrac{4}{3}, \\ \tfrac{3x}{4} + \tfrac{3x}{4} + y_2^2 &= \tfrac{4}{3}, \\ 3 - 3x + y_3^2 &= \tfrac{4}{3}, \end{aligned}$$

which imply that

$$\begin{aligned} \tfrac{3}{2}x + y_1^2 &= \tfrac{4}{3}, \\ \tfrac{3}{2}x + y_2^2 &= \tfrac{4}{3}, \\ 3 - 3x + y_3^2 &= \tfrac{4}{3}. \end{aligned}$$

From the first and the second equations it follows that $y_1^2 = y_2^2$, and also from the equation $F^*F = \frac{4}{3}I$ it follows that

$$y_1y_2 = 0, \ y_1y_3 = 0, \ y_3y_2 = 0.$$

These equations together imply that $y_1 = y_2 = 0$. Hence $\frac{3}{2}x = \frac{4}{3}$ which implies $x = \frac{8}{9}$. So $y_3^2 = 1$ and by choosing $y_3 = -1$ we will get the following four vectors:

$$\begin{pmatrix} \frac{\sqrt{8}}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ \sqrt{\frac{2}{3}} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ -\sqrt{\frac{2}{3}} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

It can be checked easily that this points are the vertices of a tetrahedron and form an A-FNTF for $\mathbb{R}^3$.

(ii)(Hexahedron): Consider the fourth roots of the unity which are $(1,0),(0,1),(-1,0),(0,-1)$. Then translate $\sqrt{x}$ times of these points first as $\sqrt{1-x}$ and next as $-\sqrt{1-x}$ into the space. With these operations, we get eight vectors in $\mathbb{R}^3$. To be a tight frame, this set of vectors should be so that $F^*F = \frac{8}{3}I$ where

$$F^* = \begin{pmatrix} \sqrt{x} & 0 & 0 & -\sqrt{x} & \sqrt{x} & 0 & 0 & -\sqrt{x} \\ 0 & \sqrt{x} & -\sqrt{x} & 0 & 0 & \sqrt{x} & -\sqrt{x} & 0 \\ \sqrt{1-x} & \sqrt{1-x} & \sqrt{1-x} & \sqrt{1-x} & -\sqrt{1-x} & -\sqrt{1-x} & -\sqrt{1-x} & -\sqrt{1-x} \end{pmatrix}.$$

A simple calculation as in the case (i) shows that $x = \frac{2}{3}$. By this value of x, these eight vectors become vertices of a hexahedron and form an $A - FNTF$ for $\mathbb{R}^3$ with $A = \frac{8}{3}$.

(iii) (Octahedron): As the previous case, first consider the fourth roots of the unity and then by adding the third component 0 to each of them, assume them as points in $\mathbb{R}^3$: $(1,0,0),(0,1,0),(-1,0,0),(0,-1,0)$. These are four vertices of the octahedron. the two other vertices are $(0,0,1)$ and $(0,0,-1)$. It can easily be checked that this set of vectors form an A-FNTF for $\mathbb{R}^3$ with $A=2I$, that is $F^*F=2I$ where

$$F^*=\begin{pmatrix}1&0&-1&0&0&0\\0&1&0&-1&0&0\\0&0&0&0&1&-1\end{pmatrix}.$$

(iv)(Dodecahedron): This polygon has twenty vertices. To find these points in $\mathbb{R}^3$, we consider the fifth roots of the unity in the plane: These are points as $e^{\frac{2\pi ik}{5}}$ where $k=0,1,2,3,4$. In fact they are points with the coordinates $(1,0),(cos\frac{2\pi i}{5},sin\frac{2\pi i}{5}),(cos\frac{4\pi i}{5},sin\frac{4\pi i}{5}),(cos\frac{6\pi i}{5},sin\frac{6\pi i}{5}),(cos\frac{8\pi i}{5},sin\frac{8\pi i}{5})$. At the first step, once we translate $\sqrt{x}$ times of these points as $\sqrt{1-x}$ and again translate $\sqrt{y}$ times of them as $\sqrt{1-y}$. After finding values of x and y, ten vertices of the dodecahedron will denote. At the second step,first we rotate those roots as $\frac{\pi}{5}$ on the plane.Then, as before, once we translate $\sqrt{x}$ times of these recent points as $-\sqrt{1-x}$ and again $\sqrt{y}$ times of them as $-\sqrt{1-y}$. So we get the following set of twenty vectors in $\mathbb{R}^3$: $\begin{pmatrix}\sqrt{x}\\0\\\sqrt{1-x}\end{pmatrix},\begin{pmatrix}\sqrt{x}\cos\frac{2\pi}{5}\\\sqrt{x}\sin\frac{2\pi}{5}\\\sqrt{1-x}\end{pmatrix},\begin{pmatrix}-\sqrt{x}\cos\frac{\pi}{5}\\\sqrt{x}\sin\frac{\pi}{5}\\\sqrt{1-x}\end{pmatrix},\begin{pmatrix}-\sqrt{x}\cos\frac{\pi}{5}\\-\sqrt{x}\sin\frac{\pi}{5}\\\sqrt{1-x}\end{pmatrix},\begin{pmatrix}\sqrt{x}\cos\frac{2\pi}{5}\\-\sqrt{x}\sin\frac{2\pi}{5}\\\sqrt{1-x}\end{pmatrix},$
$\begin{pmatrix}\sqrt{y}\\0\\\sqrt{1-y}\end{pmatrix},\begin{pmatrix}\sqrt{y}\cos\frac{2\pi}{5}\\\sqrt{y}\sin\frac{2\pi}{5}\\\sqrt{1-y}\end{pmatrix},\begin{pmatrix}-\sqrt{y}\cos\frac{\pi}{5}\\\sqrt{y}\sin\frac{\pi}{5}\\\sqrt{1-y}\end{pmatrix},\begin{pmatrix}-\sqrt{y}\cos\frac{\pi}{5}\\-\sqrt{y}\sin\frac{\pi}{5}\\\sqrt{1-y}\end{pmatrix},\begin{pmatrix}\sqrt{y}\cos\frac{2\pi}{5}\\-\sqrt{y}\sin\frac{2\pi}{5}\\\sqrt{1-y}\end{pmatrix},$
$\begin{pmatrix}\sqrt{x}\cos\frac{\pi}{5}\\\sqrt{x}\sin\frac{\pi}{5}\\-\sqrt{1-x}\end{pmatrix},\begin{pmatrix}-\sqrt{x}\cos\frac{2\pi}{5}\\\sqrt{x}\sin\frac{2\pi}{5}\\-\sqrt{1-x}\end{pmatrix},\begin{pmatrix}-\sqrt{x}\\0\\-\sqrt{1-x}\end{pmatrix},\begin{pmatrix}-\sqrt{x}\cos\frac{2\pi}{5}\\-\sqrt{x}\sin\frac{2\pi}{5}\\-\sqrt{1-x}\end{pmatrix},\begin{pmatrix}\sqrt{x}\cos\frac{\pi}{5}\\-\sqrt{x}\sin\frac{\pi}{5}\\-\sqrt{1-x}\end{pmatrix},$
$\begin{pmatrix}\sqrt{y}\cos\frac{\pi}{5}\\\sqrt{y}\sin\frac{\pi}{5}\\-\sqrt{1-y}\end{pmatrix},\begin{pmatrix}-\sqrt{y}\cos\frac{2\pi}{5}\\\sqrt{y}\sin\frac{2\pi}{5}\\-\sqrt{1-y}\end{pmatrix},\begin{pmatrix}-\sqrt{y}\\0\\-\sqrt{1-y}\end{pmatrix},\begin{pmatrix}-\sqrt{y}\cos\frac{2\pi}{5}\\-\sqrt{y}\sin\frac{2\pi}{5}\\-\sqrt{1-y}\end{pmatrix},\begin{pmatrix}\sqrt{y}\cos\frac{\pi}{5}\\-\sqrt{y}\sin\frac{\pi}{5}\\-\sqrt{1-y}\end{pmatrix}.$
These vectors form the columns of the matrix F^*. To be an A- FNTF, it is necesary that $F^*F=\frac{20}{3}I$. This will happen if

$$x+y=\frac{4}{3}. \tag{1}$$

. On the other hand, for these points to be vertices of a dodecahedron, the distance between two points $(\sqrt{x},0,\sqrt{1-x})$ and $(\sqrt{x}\cos\frac{\pi}{5},\sqrt{x}\sin\frac{\pi}{5},-\sqrt{1-x})$ should be equal to that of the points $(\sqrt{y}\cos\frac{\pi}{5},\sqrt{y}\sin\frac{\pi}{5},-\sqrt{1-y})$ and $(-\sqrt{y}\cos\frac{2\pi}{5},\sqrt{y}\sin\frac{2\pi}{5},-\sqrt{1-y})$. This causes

$$-x(1+\cos\frac{\pi}{5})+2=y(1+\cos\frac{3\pi}{5}). \tag{2}$$

By putting the value of x from (1) into (2), we get

$$(y-\frac{4}{3})(1+\cos\frac{\pi}{5})+2=y+y\cos\frac{3\pi}{5},$$

which implies

$$y = 0.368524268,\ x = 0.964809064.$$

By substituting these values of x and y in the components of the desired vectors, it can easily be checked that these points are vertices of a dodecahedron.

(v) (Icosahedron) As in the case of dodecahedron, we start with the fifth roots of the unity. We translate $\sqrt{x}$ times of these points as $\sqrt{1-x}$ into the space. Again, after rotating those roots as $\frac{\pi}{5}$ on the plane, translating $\sqrt{x}$ times of them as $-\sqrt{1-x}$ gives us five other points. By adding two extra points $(0,0,1)$ and $(0,0,-1)$, we get the following set of twelve vectors in $\mathbb{R}^3$:

$$\begin{pmatrix}\sqrt{x}\\0\\\sqrt{1-x}\end{pmatrix}, \begin{pmatrix}\sqrt{x}\cos\frac{2\pi}{5}\\\sqrt{x}\sin\frac{2\pi}{5}\\\sqrt{1-x}\end{pmatrix}, \begin{pmatrix}-\sqrt{x}\cos\frac{\pi}{5}\\\sqrt{x}\sin\frac{\pi}{5}\\\sqrt{1-x}\end{pmatrix}, \begin{pmatrix}-\sqrt{x}\cos\frac{\pi}{5}\\-\sqrt{x}\sin\frac{\pi}{5}\\\sqrt{1-x}\end{pmatrix},$$

$$\begin{pmatrix}\sqrt{x}\cos\frac{2\pi}{5}\\-\sqrt{x}\sin\frac{2\pi}{5}\\\sqrt{1-x}\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}\sqrt{x}\cos\frac{\pi}{5}\\\sqrt{x}\sin\frac{\pi}{5}\\-\sqrt{1-x}\end{pmatrix}, \begin{pmatrix}-\sqrt{x}\cos\frac{2\pi}{5}\\\sqrt{x}\sin\frac{2\pi}{5}\\-\sqrt{1-x}\end{pmatrix},$$

$$\begin{pmatrix}-\sqrt{x}\\0\\-\sqrt{1-x}\end{pmatrix}, \begin{pmatrix}-\sqrt{x}\cos\frac{2\pi}{5}\\-\sqrt{x}\sin\frac{\pi}{5}\\-\sqrt{1-x}\end{pmatrix}, \begin{pmatrix}\sqrt{x}\cos\frac{2\pi}{5}\\-\sqrt{x}\sin\frac{2\pi}{5}\\-\sqrt{1-x}\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}$$

These vectors form a frame for $\mathbb{R}^3$ for suitable values of x. To be an A-FNTF, we should have $F^*F = 4I$. This implies the equation $10(1-x)+2 = 4$ which leads to $x = \frac{4}{5}$. By this value of x, it is easy to check that the desired points are vertices of the icosahedron.

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