

\mathfrak{R} -parts in hyperrings

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ABSTRACT. In this article, first we generalize the concept of complete parts in hyperrings by introducing the concept \mathfrak{R} -parts in hyperrings and then we study \mathfrak{R} -closures in hyperrings. Finally we characterize \mathfrak{R} -closures in hyperfields.

Keywords: hyperrings, (semi)hypergroups, complete parts.

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1. INTRODUCTION

The theory of hyperstructures was introduced in 1934 by Marty [9] at the 8th Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini and Leoreanu [1, 2], Mittas [11, 12], Stratigopoulos [16], and by various authors. Basic definitions and propositions about the hyperstructures are found in [1, 2, 17]. Krasner [8] has studied the notion of *hyperfields*, *hyperrings*, and then some researchers. Hyperrings are essentially rings with approximately modified axioms. There are different notions of hyperrings. If the addition $+$ is a hyperoperation and the multiplication is a binary operation, then the hyperring is called Krasner (additive) hyperring [8].

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Rota [14] introduced a multiplicative hyperring, where $+$ is a binary operation and the multiplication is a hyperoperation. De Salvo [15] studied hyperrings in which the additions and the multiplications were hyperoperations. In 2007, Davvaz and Leoreanu-Fotea [4] published a book titled *Hyperring Theory and Applications*. Complete parts were introduced by Koskas [7] and studied then by Miglirato [10], Corsini and Sureau [1, 2]. Mousavi et al. [13] introduced the notion of \mathfrak{R} -parts in hypergroups as a generalization of complete parts in hypergroups. In this article we generalize the notion of complete parts by introducing *left* and *right* \mathfrak{R} -parts in hyperrings and we will study \mathfrak{R} -closures in hyperrings. Finally we characterize \mathfrak{R} -closures in *hyperfields*.

2. PRELIMINARIES

A *hypergroupoid* (H, \circ) is a non-empty set H together with a hyperoperation \circ defined on H , that is a mapping of $H \times H$ into the family of non-empty subsets of H . If $(x, y) \in H \times H$, its image under \circ is denoted by $x \circ y$ and for simplicity by xy . If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup \{xy \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ (resp. $A \circ x$). A hypergroupoid (H, \circ) is called a *hypergroup* in the sense of [9] if for all $x, y, z \in H$ the following two conditions hold: (i) $x(yz) = (xy)z$, (ii) $xH = Hx = H$, means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in xu$ and $y \in vx$. If (H, \circ) satisfies only the first axiom, then it is called a *semi-hypergroup* an exhaustive review updated to 1992 of hypergroup theory appears in [1]. A recent book [2] contains a wealth of applications. A *hyperring* [17] is a triple $(R, +, \circ)$ which satisfies the ring-like axioms in the following way: (i) $(R, +)$ is a hypergroup, (ii) (R, \circ) is a semi-hypergroup, (iii) the multiplication is distributive with respect to the hyperoperation $+$. The hyperrings were studied by many authors, for example see [6], [3], [17], [5] and [19]. In [20] and [18] Vougiouklis defines the relation Γ on hyperring as follows: $x\Gamma y$ if and only if $x, y \subseteq u$, where u is a finite sum of finite products of elements of R , in fact there exist $n, k_i \in \mathbb{N}$ and $x_{ij} \in R$ such that $u = \sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij}$.

He proved that the quotient R/Γ^* , where Γ^* is the transitive closure of Γ , is a ring and also Γ^* is the smallest equivalent relation on R such that the quotient R/Γ^* is a fundamental ring. The both \oplus and \odot on R/Γ^* are defined as follow:

$$\forall z \in \Gamma^*(x) + \Gamma^*(y), 2\Gamma^*(x) \oplus \Gamma^*(y) = \Gamma^*(z);$$

$$\forall z \in \Gamma^*(x) \circ \Gamma^*(y), 2\Gamma^*(x) \odot \Gamma^*(y) = \Gamma^*(z).$$

Let M be a non-empty subset of R . We say that M is a *complete part* if for every $n \in \mathbb{N}$, $i = 1, 2, \dots, n$, $\forall k_i \in \mathbb{N}$, $\forall (z_{i1}, \dots, z_{ik_i}) \in R^{k_i}$ we have:

$$\sum_{i=1}^n \prod_{j=1}^{k_i} z_{ij} \bigcap M \neq \emptyset \Rightarrow \sum_{i=1}^n \prod_{j=1}^{k_i} z_{ij} \subseteq M.$$

3. \mathfrak{R} -PARTS

Let \mathcal{U} be the set of finite sums of finite products of elements of R and \mathfrak{R} be a relation on \mathcal{U} . In this section first we generalize the notion of complete parts by introducing the notion of \mathfrak{R} -parts and then we study \mathfrak{R} -closures.

Definition 3.1. Let R be a hyperring and \mathcal{U} be the set of finite sum of finite products of elements of R and \mathfrak{R} be a relation on \mathcal{U} . For a non-empty subset A of R we say:

(i) A is a left \mathfrak{R} -part of R with respect to \mathcal{U} (or briefly is $\mathcal{LR}_\mathcal{U}$ -part) if for all $\sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij}$ and $\sum_{i=1}^m \prod_{j=1}^{t_i} y_{ij}$ in \mathcal{U} the following implication is valid:

$$\left[\sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij} \cap A \neq \emptyset \text{ and } \sum_{i=1}^m \prod_{j=1}^{t_i} y_{ij} \in \mathfrak{R} \sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij} \right] \Rightarrow \sum_{i=1}^m \prod_{j=1}^{t_i} y_{ij} \subseteq A;$$

(ii) A is a right \mathfrak{R} -part of R with respect to \mathcal{U} (or briefly is $\mathcal{RR}_\mathcal{U}$ -part) if for all $\sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij}$ and $\sum_{i=1}^m \prod_{j=1}^{t_i} y_{ij}$ in \mathcal{U} the following implication is valid:

$$\left[\sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij} \cap A \neq \emptyset \text{ and } \sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij} \in \mathfrak{R} \sum_{i=1}^m \prod_{j=1}^{t_i} y_{ij} \right] \Rightarrow \sum_{i=1}^m \prod_{j=1}^{t_i} y_{ij} \subseteq A;$$

(iii) A is a \mathfrak{R} -part of R with respect to \mathcal{U} (or briefly is $\mathfrak{R}_\mathcal{U}$ -part) if it is $\mathcal{LR}_\mathcal{U}$ -part and $\mathcal{RR}_\mathcal{U}$ -part.

Proposition 3.2. Let \mathfrak{R} be a relation on \mathcal{U} and \mathfrak{R}^{-1} be the inverse of \mathfrak{R} then

- (i) A is $\mathcal{LR}_\mathcal{U}^{-1}$ -part if and only if it is $\mathcal{RR}_\mathcal{U}$ -part;
- (ii) A is $\mathcal{RR}_\mathcal{U}^{-1}$ -part if and only if it is $\mathcal{LR}_\mathcal{U}$ -part.

Definition 3.3. The intersection of $\mathcal{LR}_\mathcal{U}$ -parts (or $\mathcal{RR}_\mathcal{U}$ -parts, \mathfrak{R} -parts) which contain A is called $\mathcal{LR}_\mathcal{U}$ -closure (or $\mathcal{RR}_\mathcal{U}$ -closure, \mathfrak{R} -closure) of A in R and it will be denoted by $\overline{\mathcal{LR}_\mathcal{U}}(A)$ (or $\overline{\mathcal{RR}_\mathcal{U}}(A)$, $\overline{\mathfrak{R}_\mathcal{U}}(A)$).

From now on R is a hyperring, \mathcal{U} is the set of finite sum of finite products of elements of R , $u \in \mathcal{U}$ means $u = \sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij}$ and A is a non-empty subset of R .

Proposition 3.4. For a non-empty subset A of R we have:

- (i) $\overline{\mathcal{LR}_\mathcal{U}^{-1}}(A) = \overline{\mathcal{RR}_\mathcal{U}}(A)$;
- (ii) $\overline{\mathcal{RR}_\mathcal{U}^{-1}}(A) = \overline{\mathcal{LR}_\mathcal{U}}(A)$.

Proof. Follows from Proposition 3.2. □

Lemma 3.5. For a non-empty subset A of R define:

$$\sum_A^\mathcal{U} \stackrel{\text{def}}{=} \{ \mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U} \mid \overline{\mathcal{LR}_\mathcal{U}}(A) = A \} \text{ and } \sum_A^\mathcal{U} \stackrel{\text{def}}{=} \{ \mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U} \mid \overline{\mathcal{RR}_\mathcal{U}}(A) = A \}.$$

If $\sum_A \mathcal{U} \neq \emptyset$ (resp. $\sum_A \mathcal{U} \neq \emptyset$), then $(\sum_A \mathcal{U}, \circ)$ (resp. $(\sum_A \mathcal{U}, \circ)$) is a semigroup, where \circ is the operation of relation composition.

Proof. Suppose that $\mathfrak{R}, \mathfrak{R}' \in \sum_A \mathcal{U}$ and $(\sum_{i=1}^n \prod_{j=1}^{k_i} y_{ij}, \sum_{i=1}^m \prod_{j=1}^{t_i} x_{ij}) \in \mathcal{U} \times \mathcal{U}$ are given. Let $\sum_{i=1}^m \prod_{j=1}^{t_i} x_{ij} \cap A \neq \emptyset$ and $\sum_{i=1}^n \prod_{j=1}^{k_i} y_{ij} 1_{\mathfrak{R}} \circ \mathfrak{R}' 1 \sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij}$. So there exists $\sum_{i=1}^k \prod_{j=1}^{s_i} z_{ij}$ such that $\sum_{i=1}^k \prod_{j=1}^{s_i} z_{ij} 1_{\mathfrak{R}} 1 \sum_{i=1}^m \prod_{j=1}^{t_i} x_{ij}$ and $\sum_{i=1}^n \prod_{j=1}^{k_i} y_{ij} 1_{\mathfrak{R}'} 1 \sum_{i=1}^k \prod_{j=1}^{s_i} z_{ij}$. From $\sum_{i=1}^k \prod_{j=1}^{s_i} z_{ij} 1_{\mathfrak{R}} 1 \sum_{i=1}^m \prod_{j=1}^{t_i} x_{ij}$ and $\mathfrak{R} \in \sum_A \mathcal{U}$, we have $\sum_{i=1}^k \prod_{j=1}^{s_i} z_{ij} \subseteq A$. Since $\mathfrak{R}' \in \sum_A \mathcal{U}$ and $\sum_{i=1}^n \prod_{j=1}^{k_i} y_{ij} 1_{\mathfrak{R}'} 1 \sum_{i=1}^k \prod_{j=1}^{s_i} z_{ij}$, $\sum_{i=1}^n \prod_{j=1}^{k_i} y_{ij} \subseteq A$. \square

Theorem 3.6. If \mathfrak{R} is a permutation of finite order in $S_{\mathcal{U}}$ (the symmetric group on the set \mathcal{U}), then the following are equivalent:

- (i) A is $\mathcal{LR}_{\mathcal{U}}$ -part;
- (ii) A is $\mathcal{RR}_{\mathcal{U}}$ -part;
- (iii) A is $\mathcal{R}_{\mathcal{U}}$ -part.

Proof. (i) \Rightarrow (ii). For this reason suppose that A is $\mathcal{LR}_{\mathcal{U}}$ -part. So $\overline{\mathcal{LR}_{\mathcal{U}}}(A) = A$ and hence $\mathfrak{R} \in \sum_A \mathcal{U}$. Since \mathfrak{R} is a permutation of finite order in $S_{\mathcal{U}}$, $\langle \mathfrak{R} \rangle = \{\mathfrak{R}^n \mid n \in \mathbb{N}\}$ is a subgroup of $\sum_A \mathcal{U}$ and so $\mathfrak{R}^{-1} \in \sum_A \mathcal{U}$. Therefore by Proposition 3.4 we have $A = \overline{\mathcal{LR}_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{RR}_{\mathcal{U}}}(A)$, thus $\mathfrak{R} \in \sum_A \mathcal{U}$ and hence A is $\mathcal{RR}_{\mathcal{U}}$ -part. \square

Theorem 3.7. Suppose that $\mathfrak{R} 1 \subseteq 1 \mathcal{U} \times \mathcal{U}$.

- (i) We pose $K_{1, \mathfrak{R}}^{\mathcal{L}}(A) = A$ and

$$K_{n+1, \mathfrak{R}}^{\mathcal{L}}(A) = \{x \in R \mid \exists (u, v) \in \mathfrak{R}, x \in u \text{ and } v \cap K_{n, \mathfrak{R}}^{\mathcal{L}}(A) \neq \emptyset\},$$

if we consider $K_{\mathfrak{R}}^{\mathcal{L}}(A) = \bigcup_{n \geq 1} K_{n, \mathfrak{R}}^{\mathcal{L}}(A)$, then $K_{\mathfrak{R}}^{\mathcal{L}}(A) = \overline{\mathcal{LR}_{\mathcal{U}}}(A)$ and $K_{\mathfrak{R}}^{\mathcal{L}}(A)$ is the smallest $\mathcal{LR}_{\mathcal{U}}$ -part containing A ;

- (ii) We pose $K_{1, \mathfrak{R}}^{\mathcal{R}}(A) = A$ and

$$K_{n+1, \mathfrak{R}}^{\mathcal{R}}(A) = \{x \in R \mid \exists (v, u) \in \mathfrak{R}, x \in u \text{ and } v \cap K_{n, \mathfrak{R}}^{\mathcal{R}}(A) \neq \emptyset\},$$

if we consider $K_{\mathfrak{R}}^{\mathcal{R}}(A) = \bigcup_{n \geq 1} K_{n, \mathfrak{R}}^{\mathcal{R}}(A)$, then $K_{\mathfrak{R}}^{\mathcal{R}}(A) = \overline{\mathcal{RR}_{\mathcal{U}}}(A)$ and $K_{\mathfrak{R}}^{\mathcal{R}}(A)$ is the smallest $\mathcal{RR}_{\mathcal{U}}$ -part containing A ;

- (iii) We pose $K_{1, \mathfrak{R}}(A) = A$ and

$$K_{n+1, \mathfrak{R}}(A) = \{x \in R \mid \exists (u, v) \in \mathfrak{R} \cup \mathfrak{R}^{-1}, x \in u \text{ and } v \cap K_{n, \mathfrak{R}}(A) \neq \emptyset\},$$

if $K_{\mathfrak{R}}(A) = \bigcup_{n \geq 1} K_{n, \mathfrak{R}}(A)$, then $K_{\mathfrak{R}}(A) = \overline{\mathcal{R}_{\mathcal{U}}}(A)$ and $K_{\mathfrak{R}}(A)$ is the smallest $\mathcal{R}_{\mathcal{U}}$ -part containing A .

Proof. (i) It is necessary to prove:

- (1) $K_{\mathfrak{R}}^{\mathcal{L}}(A)$ is $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$ -part,
- (2) if $A \subseteq B$ and B is $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$ -part, then $K_{\mathfrak{R}}^{\mathcal{L}}(A) \subseteq B$.

For the proof (1) suppose that $v \cap K_{\mathfrak{R}}^{\mathcal{L}}(A) \neq \emptyset$ and $u1\mathfrak{R}1v$. Therefore there exists $n \in \mathbb{N}$ such that $v \cap K_{n,\mathfrak{R}}^{\mathcal{L}}(A) \neq \emptyset$, from which follows $u1 \subseteq 1K_{n+1,\mathfrak{R}}^{\mathcal{L}}(A)1 \subseteq 1K_{\mathfrak{R}}^{\mathcal{L}}(A)$. Now we prove (2) by induction on n . We have $K_{1,\mathfrak{R}}^{\mathcal{L}}(A)1 = 1A1 \subseteq 1B$. Suppose that $K_{n,\mathfrak{R}}^{\mathcal{L}}(A)1 \subseteq 1B$. We prove that $K_{n+1,\mathfrak{R}}^{\mathcal{L}}(A)1 \subseteq 1B$. If $z \in K_{n+1,\mathfrak{R}}^{\mathcal{L}}(A)$, then there exists $(u, v) \in \mathcal{U} \times \mathcal{U}$ such that $z \in u$, $u1\mathfrak{R}1v$ and $v \cap K_{n,\mathfrak{R}}^{\mathcal{L}}(A) \neq \emptyset$. Therefore $v \cap B \neq \emptyset$ and hence $z \in u1 \subseteq 1B$. So $K_{n+1,\mathfrak{R}}^{\mathcal{L}}(A)1 \subseteq 1B$.

(ii) We have

$$\begin{aligned} K_{\mathfrak{R}}^{\mathcal{R}}(A) &= K_{\mathfrak{R}_{\mathcal{U}}^{-1}}^{\mathcal{L}}(A) \\ &= \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}^{-1}}(A), \text{ 4by part (i)} \\ &= \overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(A), \text{ 4by Proposition 3.4.} \end{aligned}$$

(iii) Follows from (i) and (ii). \square

Proposition 3.8. Suppose that B is a non-empty subset of R and \mathfrak{R} is a relation on \mathcal{U} . Then we have:

- (i) $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(B) = \bigcup_{b \in B} \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(b)$;
- (ii) $\overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(B) = \bigcup_{b \in B} \overline{\mathcal{R}\mathfrak{R}_{\mathcal{U}}}(b)$;
- (iii) $\overline{\mathfrak{R}_{\mathcal{U}}}(B) = \bigcup_{b \in B} \overline{\mathfrak{R}_{\mathcal{U}}}(b)$.

Proof. (i) It is clear that for all $b \in B$, $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(b)1 \subseteq 1\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(B)$. By Theorem 3.7(i), $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(B) = \bigcup_{n \geq 1} K_{n,\mathfrak{R}}^{\mathcal{L}}(B)$. We follow the proposition by induction on n . For $n = 1$, $K_{1,\mathfrak{R}}^{\mathcal{L}}(B) = B = \bigcup_{b \in B} \{b\} = \bigcup_{b \in B} K_{1,\mathfrak{R}}^{\mathcal{L}}(b)$. Supposing it is true for n , we show that $K_{n+1,\mathfrak{R}}^{\mathcal{L}}(B)1 \subseteq 1\bigcup_{b \in B} K_{n+1,\mathfrak{R}}^{\mathcal{L}}(b)$. If $z \in K_{n+1,\mathfrak{R}}^{\mathcal{L}}(B)$, then there exists $(u, v) \in \mathfrak{R}$ such that $z \in u$ and $v \cap K_{n,\mathfrak{R}}^{\mathcal{L}}(B) \neq \emptyset$. From this it follows, by the hypothesis of induction, $v \cap (\bigcup_{b \in B} K_{n,\mathfrak{R}}^{\mathcal{L}}(b)) \neq \emptyset$ and therefore $b' \in B$ exists such that $v \cap K_{n,\mathfrak{R}}^{\mathcal{L}}(b') \neq \emptyset$. So $z \in K_{n+1,\mathfrak{R}}^{\mathcal{L}}(b')$ and hence $\overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(B)1 \subseteq 1\bigcup_{b \in B} \overline{\mathcal{L}\mathfrak{R}_{\mathcal{U}}}(b)$. \square

Theorem 3.9. Suppose that $\mathfrak{R}1 \subseteq 1\mathcal{U} \times \mathcal{U}$. The relation $K_{\mathfrak{R}}^{\mathcal{L}}$ (resp. $K_{\mathfrak{R}}^{\mathcal{R}}$) on R defined by:

$$x1K_{\mathfrak{R}}^{\mathcal{L}}1y \Leftrightarrow x \in K_{\mathfrak{R}}^{\mathcal{L}}(y)(x \in K_{\mathfrak{R}}^{\mathcal{R}}(y)),$$

where $K_{\mathfrak{R}}^{\mathcal{L}}(y) = K_{\mathfrak{R}}^{\mathcal{L}}(\{y\})$ (resp. $K_{\mathfrak{R}}^{\mathcal{R}}(y) = K_{\mathfrak{R}}^{\mathcal{R}}(\{y\})$) is a preorder. Furthermore if \mathfrak{R} is symmetric, then $K_{\mathfrak{R}}^{\mathcal{L}}$ (resp. $K_{\mathfrak{R}}^{\mathcal{R}}$) is an equivalence relation.

Proof. It is easy to see that $K_{\mathfrak{R}}^{\mathcal{L}}$ is reflexive. Now suppose that $x1K_{\mathfrak{R}}^{\mathcal{L}}1y$ and $y1K_{\mathfrak{R}}^{\mathcal{L}}1z$. So $x \in K_{\mathfrak{R}}^{\mathcal{L}}(y)$ and $y \in K_{\mathfrak{R}}^{\mathcal{L}}(z)$. By Theorem 3.7(i) we have $K_{\mathfrak{R}}^{\mathcal{L}}(z)$ is $\mathcal{LR}_{\mathcal{U}}$ -part thus $K_{\mathfrak{R}}^{\mathcal{L}}(y) \subseteq K_{\mathfrak{R}}^{\mathcal{L}}(z)$ and hence $x \in K_{\mathfrak{R}}^{\mathcal{L}}(z)$. Therefore $K_{\mathfrak{R}}^{\mathcal{L}}$ is preorder. Now let \mathfrak{R} be symmetric. We prove that $K_{\mathfrak{R}}^{\mathcal{L}}$ is symmetric as well. To this end the following is premised:

- (1) for all $n \geq 2$ and $x \in R$, $K_{n,\mathfrak{R}}^{\mathcal{L}}(K_{2,\mathfrak{R}}^{\mathcal{L}}(x)) = K_{n+1,\mathfrak{R}}^{\mathcal{L}}(x)$;
- (2) $x \in K_{n,\mathfrak{R}}^{\mathcal{L}}(y)$ if and only if $y \in K_{n,\mathfrak{R}}^{\mathcal{L}}(x)$.

We prove (1) by induction on n . Suppose that $z \in K_{2,\mathfrak{R}}^{\mathcal{L}}(K_{2,\mathfrak{R}}^{\mathcal{L}}(x))$ so there exists $(u, v) \in \mathfrak{R}$ such that $z \in u$ and $v \cap K_{2,\mathfrak{R}}^{\mathcal{L}}(x) \neq \emptyset$. Thus $z \in K_{3,\mathfrak{R}}^{\mathcal{L}}(x)$. Let $K_{n,\mathfrak{R}}^{\mathcal{L}}(K_{2,\mathfrak{R}}^{\mathcal{L}}(x)) = K_{n+1,\mathfrak{R}}^{\mathcal{L}}(x)$ so we have:

$$\begin{aligned} z \in K_{n+1,\mathfrak{R}}^{\mathcal{L}}(K_{2,\mathfrak{R}}^{\mathcal{L}}(x)) &\Leftrightarrow \exists (u, v) \in \mathfrak{R}, z \in u, v \cap K_{n,\mathfrak{R}}^{\mathcal{L}}(K_{2,\mathfrak{R}}^{\mathcal{L}}(x)) \neq \emptyset \\ &\Leftrightarrow \exists (u, v) \in \mathfrak{R}, z \in u, v \cap K_{n+1,\mathfrak{R}}^{\mathcal{L}}(x) \neq \emptyset \\ &\Leftrightarrow z \in K_{n+2,\mathfrak{R}}^{\mathcal{L}}(x). \end{aligned}$$

We also prove (2) by induction on n . It is clear that $x \in K_{2,\mathfrak{R}}^{\mathcal{L}}(y)$ if and only if $y \in K_{2,\mathfrak{R}}^{\mathcal{L}}(x)$. Suppose $x \in K_{n,\mathfrak{R}}^{\mathcal{L}}(y)$ if and only if $y \in K_{n,\mathfrak{R}}^{\mathcal{L}}(x)$. Let $x \in K_{n+1,\mathfrak{R}}^{\mathcal{L}}(y)$ be given, so there exist $(u, v) \in \mathfrak{R}$ such that $x \in u$ and $v \cap K_{n,\mathfrak{R}}^{\mathcal{L}}(y) \neq \emptyset$. Therefore there exists $b \in v \cap K_{n,\mathfrak{R}}^{\mathcal{L}}(y)$ and hence $y \in K_{n,\mathfrak{R}}^{\mathcal{L}}(b)$. Since \mathfrak{R} is symmetric and $(u, v) \in \mathfrak{R}$, $b \in v$ and $x \in u \cap K_{1,\mathfrak{R}}^{\mathcal{L}}(x)$ implies that $b \in K_{2,\mathfrak{R}}^{\mathcal{L}}(x)$ and hence $y \in K_{n,\mathfrak{R}}^{\mathcal{L}}(K_{2,\mathfrak{R}}^{\mathcal{L}}(x)) = K_{n+1,\mathfrak{R}}^{\mathcal{L}}(x)$. Similarly we can show if $y \in K_{n+1,\mathfrak{R}}^{\mathcal{L}}(x)$, then $x \in K_{n+1,\mathfrak{R}}^{\mathcal{L}}(y)$. \square

Proposition 3.10. *Let \mathfrak{R} be a relation on \mathcal{U} and A be a non-empty subset of the hyperring R . The following conditions are equivalent:*

- (i) A is a $(\mathcal{R}\mathfrak{R}_{\mathcal{U}}\text{-part}) \mathcal{LR}_{\mathcal{U}}$ -part of R ;
- (ii) $x \in A, (x1K_{\mathfrak{R}}^{\mathcal{L}}1z)z1K_{\mathfrak{R}}^{\mathcal{L}}1x \Rightarrow z \in A$.

Proof. (i) \Rightarrow (ii) If $x \in A$ and $z \in R$ such that $z1K_{\mathfrak{R}}^{\mathcal{L}}1x$, then there exists $(u, v) \in \mathfrak{R}$ such that $z \in u$ and $v \cap K_{n,\mathfrak{R}}^{\mathcal{L}}(A) \neq \emptyset$ for some $n \in \mathbb{N}$. Since A is a $\mathcal{LR}_{\mathcal{U}}$ -part by Theorem 3.7, $K_{n,\mathfrak{R}}^{\mathcal{L}}(A)1 \subseteq 1A$ and so $v \cap A \neq \emptyset$. Therefore $u1 \subseteq 1A$ and hence $z \in A$.

(ii) \Rightarrow (i) Let $u \cap A \neq \emptyset$ and $v1\mathfrak{R}1u$. So there exists $x \in A \cap u$ and $x \in u$, $u \cap K_{1,\mathfrak{R}}^{\mathcal{L}}(x) \neq \emptyset$. Now suppose that $z \in v$ is given. So

$$\begin{aligned} v1\mathfrak{R}1u &\Rightarrow z \in K_{2,\mathfrak{R}}^{\mathcal{L}}(x), \text{ 4 because } x \in u \\ &\Rightarrow z1K_{\mathfrak{R}}^{\mathcal{L}}1x \\ &\Rightarrow z \in A, \text{ 15 because } x \in A. \end{aligned}$$

Therefore $v1 \subseteq 1A$ and hence A is $\mathcal{LR}_{\mathcal{U}}$ -part of R . \square

4. RINGS DERIVED FROM HYPERRINGS

In this section we give the notion of (strongly) normal relation on \mathcal{U} and then we construct a ring from a hyperring.

Definition 4.1. Suppose that $\mathfrak{R}1 \subseteq 1\mathcal{U} \times \mathcal{U}$.

(i) for all $(x, y) \in R^2$ define the relation $\rho_{\mathcal{L}, \mathfrak{R}}$ on R by:

$$x1\rho_{\mathcal{L}, \mathfrak{R}}y \Leftrightarrow [x = y1 \text{ or } 1\exists(u, v) \in \mathfrak{R}2 \text{ such that } 1x \in u1 \text{ and } 1y \in v]$$

and $\rho_{\mathcal{L}, \mathfrak{R}}^*$ is the transitive closure of $\rho_{\mathcal{L}, \mathfrak{R}}$;

(ii) for all $(x, y) \in R^2$ define the relation $\rho_{\mathcal{R}, \mathfrak{R}}$ on R by:

$$x1\rho_{\mathcal{R}, \mathfrak{R}}y \Leftrightarrow [x = y1 \text{ or } 1\exists(v, u) \in \mathfrak{R}2 \text{ such that } 1x \in u1 \text{ and } 1y \in v]$$

and $\rho_{\mathcal{R}, \mathfrak{R}}^*$ is the transitive closure of $\rho_{\mathcal{R}, \mathfrak{R}}$;

(iii) for all $(x, y) \in R^2$ define the relation $\rho_{\mathfrak{R}}$ on R by:

$$x1\rho_{\mathfrak{R}}y \Leftrightarrow [x = y1 \text{ or } 1\exists(u, v) \in \mathfrak{R} \bigcup \mathfrak{R}^{-1}2 \text{ such that } 1x \in u1 \text{ and } 1y \in v]$$

and $\rho_{\mathfrak{R}}^*$ is the transitive closure of $\rho_{\mathfrak{R}}$.

Theorem 4.2. Suppose that $\mathfrak{R}1 \subseteq 1\mathcal{U} \times \mathcal{U}$. For all $(x, y) \in R^2$ we have:

(i) $x1K_{\mathfrak{R}}^{\mathcal{L}}1y$ if and only if $x1\rho_{\mathcal{L}, \mathfrak{R}}^*1y$;

(ii) $x1K_{\mathfrak{R}}^{\mathcal{R}}1y$ if and only if $x1\rho_{\mathcal{R}, \mathfrak{R}}^*1y$.

Proof. (i) It is easy to see that $\rho_{\mathcal{L}, \mathfrak{R}}^*1 \subseteq 1K_{\mathfrak{R}}^{\mathcal{L}}$. Conversely suppose that $x1K_{\mathfrak{R}}^{\mathcal{L}}1y$ so by Theorem 3.9 we have $x \in K_{n+1, \mathfrak{R}}^{\mathcal{L}}(y)$ for some $n \in \mathbb{N}$. So there exists $(u_1, v_1) \in \mathfrak{R}$ such that $x \in u_1$ and $v_1 \cap K_{n, \mathfrak{R}}^{\mathcal{L}}(y) \neq \emptyset$ thus there exists $x_1 \in v_1 \cap K_{n, \mathfrak{R}}^{\mathcal{L}}(y)$ and hence $x1\rho_{\mathcal{L}, \mathfrak{R}}1x_1$. Since $x_1 \in K_{n, \mathfrak{R}}^{\mathcal{L}}(y)$, there exists $(u_2, v_2) \in \mathfrak{R}$ such that $x_1 \in u_2$ and $v_2 \cap K_{n-1, \mathfrak{R}}^{\mathcal{L}}(y) \neq \emptyset$. Therefore $x_11\rho_{\mathcal{L}, \mathfrak{R}}1x_2$, where $x_2 \in v_2 \cap K_{n-1, \mathfrak{R}}^{\mathcal{L}}(y)$. As a consequence we conclude that $x_n \in v_n \cap K_{n-(n-1), \mathfrak{R}}^{\mathcal{L}}(y)$ exists such that $x_{n-1}1\rho_{\mathcal{L}, \mathfrak{R}}1x_n$. Thus we have,

$$x1\rho_{\mathcal{L}, \mathfrak{R}}1x_11\rho_{\mathcal{L}, \mathfrak{R}}1x_21\rho_{\mathcal{L}, \mathfrak{R}}1x_n1\rho_{\mathcal{L}, \mathfrak{R}}1y.$$

From this follows $K_{\mathfrak{R}}^{\mathcal{L}}1 \subseteq 1\rho_{\mathcal{L}, \mathfrak{R}}^*$ and the proof is complete.

Similarly we have (ii). □

Proposition 4.3. Suppose that \mathfrak{R} is a permutation of finite order in $S_{\mathcal{U}}$, then $\rho_{\mathcal{L}, \mathfrak{R}}^* = \rho_{\mathfrak{R}}^*$.

Proof. Since $K_{\mathfrak{R}}^{\mathcal{L}}(y)$ is $\mathcal{L}\mathfrak{R}_{\mathcal{U}}$ -part by Theorem 3.6, $K_{\mathfrak{R}}^{\mathcal{L}}(y)$ is $\mathcal{R}\mathfrak{R}_{\mathcal{U}}$ -part and hence $K_{\mathfrak{R}}^{\mathcal{R}} \subseteq K_{\mathfrak{R}}^{\mathcal{L}}$. Analogously $K_{\mathfrak{R}}^{\mathcal{L}} \subseteq K_{\mathfrak{R}}^{\mathcal{R}}$ and so $K_{\mathfrak{R}}^{\mathcal{L}} = K_{\mathfrak{R}}^{\mathcal{R}}$. From this it follows that $\rho_{\mathcal{L}, \mathfrak{R}}^* = \rho_{\mathfrak{R}}^*$. □

Definition 4.4. If $(R, +, \circ)$ is a hyperring and $\rho1 \subseteq 1R \times R$ is an equivalence, then we set:

$$A \stackrel{=}{\rho} B \Leftrightarrow a1\rho1b, \forall a \in A, \forall b \in B,$$

for all pairs (A, B) of non-empty subsets of R . The relation ρ is said to be strongly regular to the left (resp. to the right) if (i) $x1\rho1y \Rightarrow a + x \bar{\rho} a + y$ and (ii) $x1\rho1y \Rightarrow a \circ x \bar{\rho} a \circ y$ (resp. (i) $x1\rho1y \Rightarrow x + a \bar{\rho} y + a$ and (ii) $x1\rho1y \Rightarrow a \circ x \bar{\rho} a \circ y$), for all $(x, y, a) \in R^3$. ρ is called strongly regular if it is (i) strongly regular to the right and to the left and moreover (ii) there exists e in R such that: $\rho(x) = \rho(t)$, for all $t \in x \circ e \cap e \circ x$.

Definition 4.5. Let R be a hyperring, then

- (i) a relation \mathfrak{R} on \mathcal{U} is called normal if for all $x \in R$, one has $K_{\mathfrak{R}}^{\mathcal{L}}(x) = K_{\mathfrak{R}}^{\mathcal{R}}(x)$,
- (ii) a normal relation \mathfrak{R} on \mathcal{U} is called strongly normal to the left (resp. to the right) if $\rho_{\mathcal{L}, \mathfrak{R}}^*$ (resp. $\rho_{\mathcal{R}, \mathfrak{R}}^*$) is strongly regular to the left (resp. to the right),
- (iii) a normal relation \mathfrak{R} on \mathcal{U} is called strongly normal if $\rho_{\mathfrak{R}}^*$ is strongly regular.

Suppose that $\mathfrak{R}1 \subseteq \mathcal{U} \times \mathcal{U}$. For every element x of a hyperring R , set:

$$\begin{aligned} P_{\mathcal{L}, \mathfrak{R}}^n(x) &= \bigcup \{v1 \mid 1v1\mathfrak{R}1u_n, u_n = \sum_{i=1}^n \prod_{j=1}^{k_i} x_{ij}, x \in u_n\}; \\ P_{\mathcal{L}, \mathfrak{R}}(x) &= \bigcup_{n \geq 1} P_{\mathcal{L}, \mathfrak{R}}^n(x) \cup \{x\}; \\ \rho_{\mathcal{L}, \mathfrak{R}}^*(x) &= \{y \in R \mid 1y1\rho_{\mathcal{L}, \mathfrak{R}}^*1x\}. \end{aligned}$$

Theorem 4.6. Let R be a hyperring and \mathfrak{R} be a relation on \mathcal{U} . The following conditions are equivalent:

- (i) $\rho_{\mathcal{L}, \mathfrak{R}}$ is transitive;
- (ii) for every $x \in R$, $\rho_{\mathcal{L}, \mathfrak{R}}^*(x) = P_{\mathcal{L}, \mathfrak{R}}(x)$;
- (iii) for every $x \in R$, $P_{\mathcal{L}, \mathfrak{R}}(x)$ is a $\mathcal{LR}_{\mathcal{U}}$ -part of R .

Proof. (i) \Rightarrow (ii) For every pair (x, y) of elements of R we have:

$$y \in \rho_{\mathcal{L}, \mathfrak{R}}^*(x) \Leftrightarrow y1\rho_{\mathcal{L}, \mathfrak{R}}^*1x \Leftrightarrow y1\rho_{\mathcal{L}, \mathfrak{R}}1x \Leftrightarrow y \in P_{\mathcal{L}, \mathfrak{R}}(x).$$

(ii) \Rightarrow (iii) Let $(v, u) \in \mathfrak{R}$ such that $u \cap P_{\mathcal{L}, \mathfrak{R}}(x) \neq \emptyset$ be given. So $u \cap \rho_{\mathcal{L}, \mathfrak{R}}^*(x) \neq \emptyset$ and hence there exists $z \in R$ such that $z \in u$ and $z \in \rho_{\mathcal{L}, \mathfrak{R}}^*(x)$, thus $z \in K_{\mathfrak{R}}^{\mathcal{L}}(x)$, by Theorem 4.2. On the other hand, $z \in K_{\mathfrak{R}}^{\mathcal{L}}(z)$, so $u \cap K_{\mathfrak{R}}^{\mathcal{L}}(z) \neq \emptyset$ and hence $v1 \subseteq 1K_{\mathfrak{R}}^{\mathcal{L}}(z)$, because $v1\mathfrak{R}1u$ and $K_{\mathfrak{R}}^{\mathcal{L}}(z)$ is a $\mathcal{LR}_{\mathcal{U}}$ -part of R , by Theorem 3.7. Now suppose that $t \in v$ is an arbitrary element, thus $t \in K_{\mathfrak{R}}^{\mathcal{L}}(x)$ and hence $t1\rho_{\mathcal{L}, \mathfrak{R}}^*1x$. Therefore $t \in \rho_{\mathcal{L}, \mathfrak{R}}^*(x) = P_{\mathcal{L}, \mathfrak{R}}(x)$ and so $v1 \subseteq 1P_{\mathcal{L}, \mathfrak{R}}(x)$.

(iii) \Rightarrow (i) Let x, y and z in R be given such that $x1\rho_{\mathcal{L}, \mathfrak{R}}1y$ and $y1\rho_{\mathcal{L}, \mathfrak{R}}1z$. Since $x1\rho_{\mathcal{L}, \mathfrak{R}}1y$, there exists $(u, v) \in \mathfrak{R}$ such that $x \in u$ and $y \in v$. Therefore $v \cap P_{\mathcal{L}, \mathfrak{R}}(y) \neq \emptyset$ and since $P_{\mathcal{L}, \mathfrak{R}}(y)$ is a $\mathcal{LR}_{\mathcal{U}}$ -part, $u1 \subseteq 1P_{\mathcal{L}, \mathfrak{R}}(y)$ and hence $x \in P_{\mathcal{L}, \mathfrak{R}}(y)$. We can see that $P_{\mathcal{L}, \mathfrak{R}}(y)1 \subseteq 1P_{\mathcal{L}, \mathfrak{R}}(z)$, because $y1\rho_{\mathcal{L}, \mathfrak{R}}1z$ and so by above $y \in P_{\mathcal{L}, \mathfrak{R}}(z)$. Therefore $x \in P_{\mathcal{L}, \mathfrak{R}}(z)$ and hence $x1\rho_{\mathcal{L}, \mathfrak{R}}1z$. \square

Proposition 4.7. If \mathfrak{R} is a normal relation on \mathcal{U} , then:

- (i) \mathfrak{R}^{-1} is a normal relation;

(ii) $\rho_{\mathcal{L}, \mathfrak{R}}^* = \rho_{\mathfrak{R}}^*$ and $\rho_{\mathcal{L}, \mathfrak{R}}^*$ is an equivalence relation.

Proof. The proof follows from Proposition 3.4 and Theorem 4.2. \square

Theorem 4.8. Suppose that $(R, +, \circ)$ is a hyperring and \mathfrak{R} is a strongly normal relation on \mathcal{U} . A ring structure turns out to be define on $R/\rho_{\mathfrak{R}}^*$ with respect to the operations:

$$\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(y) = \rho_{\mathfrak{R}}^*(z), \text{ where } z \in x + y.$$

$$\rho_{\mathfrak{R}}^*(x) \odot \rho_{\mathfrak{R}}^*(y) = \rho_{\mathfrak{R}}^*(z), \text{ where } z \in x \circ y.$$

Proof. We will prove that the operation \oplus is well defined. Let $\rho_{\mathfrak{R}}^*(x_0) = \rho_{\mathfrak{R}}^*(x_1)$ and $\rho_{\mathfrak{R}}^*(y_0) = \rho_{\mathfrak{R}}^*(y_1)$. It is necessary to verify that $\rho_{\mathfrak{R}}^*(x_0) \oplus \rho_{\mathfrak{R}}^*(y_0) = \rho_{\mathfrak{R}}^*(x_1) \oplus \rho_{\mathfrak{R}}^*(y_1)$. By hypothesis $(m, n) \in \mathbb{N}^2$, $(z_0, z_1, \dots, z_m) \in R^{m+1}$ and $(t_0, t_1, \dots, t_n) \in R^{n+1}$ exist such that

$$x_0 = z_0 \rho_{\mathfrak{R}}^* z_1 \rho_{\mathfrak{R}}^* z_2 \rho_{\mathfrak{R}}^* \dots \rho_{\mathfrak{R}}^* z_{m-1} \rho_{\mathfrak{R}}^* z_m = x_1$$

and

$$y_0 = t_0 \rho_{\mathfrak{R}}^* t_1 \rho_{\mathfrak{R}}^* t_2 \rho_{\mathfrak{R}}^* \dots \rho_{\mathfrak{R}}^* t_{n-1} \rho_{\mathfrak{R}}^* t_n = y_1$$

Since \mathfrak{R} is normal, for all $u \in z_{s-1} + t_{s-1}$ and $v \in z_s + t_s$, where $1 \leq s \leq k$ and $k = \min\{m, n\}$, we have $u \rho_{\mathfrak{R}}^* v$. Therefore $\rho_{\mathfrak{R}}^*(x_0) \oplus \rho_{\mathfrak{R}}^*(y_0) = \rho_{\mathfrak{R}}^*(z_1) \oplus \rho_{\mathfrak{R}}^*(t_1) = \dots = \rho_{\mathfrak{R}}^*(z_k) \oplus \rho_{\mathfrak{R}}^*(t_k) = \rho_{\mathfrak{R}}^*(a_{k+i}) \oplus \rho_{\mathfrak{R}}^*(b_{k+i})$, where $k+1 \leq k+i \leq \max\{m, n\}$ and:

$$(a_{k+i}, b_{k+i}) = \begin{cases} (x_1, t_{k+i}) & \text{if } 1 \leq k+i \leq m; \\ (z_{k+i}, y_1) & \text{if } 1 \leq k+i \leq n. \end{cases}$$

Hence \oplus is well defined. Similarly the operation \odot is well defined and Theorem 31 of [2] shows that $(R/\rho_{\mathfrak{R}}^*, \oplus)$ is a group. By strongly normality of \mathfrak{R} we conclude that $(R/\rho_{\mathfrak{R}}^*, \odot)$ is a monoid with unit $\rho_{\mathfrak{R}}^*(e)$. The commutativity of \oplus is related with the existence of the unit in multiplication. Since \mathfrak{R} is strong, there exists e in R such that $\rho(x) = \rho(t)$ for all $t \in x \circ e \cap e \circ x$ which means $\rho_{\mathfrak{R}}^*(e)$ is the unit of multiplication so we have:

$$\begin{aligned} [\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(y)] \odot [\rho_{\mathfrak{R}}^*(e) \oplus \rho_{\mathfrak{R}}^*(e)] &= (\rho_{\mathfrak{R}}^*(x) \odot [\rho_{\mathfrak{R}}^*(e) \oplus \rho_{\mathfrak{R}}^*(e)]) \oplus (\rho_{\mathfrak{R}}^*(y) \odot [\rho_{\mathfrak{R}}^*(e) \oplus \rho_{\mathfrak{R}}^*(e)]) \\ &= (\rho_{\mathfrak{R}}^*(x) \odot \rho_{\mathfrak{R}}^*(e)) \oplus (\rho_{\mathfrak{R}}^*(y) \odot \rho_{\mathfrak{R}}^*(e)) \text{ and also } [\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(y)] \odot [\rho_{\mathfrak{R}}^*(e) \oplus \rho_{\mathfrak{R}}^*(e)] \\ &= ([\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(y)] \odot \rho_{\mathfrak{R}}^*(e)) \oplus ([\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(y)] \odot \rho_{\mathfrak{R}}^*(e)) \\ &= (\rho_{\mathfrak{R}}^*(x) \odot \rho_{\mathfrak{R}}^*(e)) \oplus (\rho_{\mathfrak{R}}^*(y) \odot \rho_{\mathfrak{R}}^*(e)) \\ &= (\rho_{\mathfrak{R}}^*(x) \odot \rho_{\mathfrak{R}}^*(e)) \oplus (\rho_{\mathfrak{R}}^*(y) \odot \rho_{\mathfrak{R}}^*(e)). \end{aligned}$$

So $(\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(x)) \oplus (\rho_{\mathfrak{R}}^*(y) \oplus \rho_{\mathfrak{R}}^*(y)) = (\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(y)) \oplus (\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(y))$ gives, $\rho_{\mathfrak{R}}^*(x) \oplus \rho_{\mathfrak{R}}^*(y) = \rho_{\mathfrak{R}}^*(y) \oplus \rho_{\mathfrak{R}}^*(x)$. \square

Let $(R, +, \circ)$ and $(R', +', \circ')$ be two hyperrings. We say that $f : R \rightarrow R'$ is a homomorphism if for every $(x, y) \in R^2$ we have $f(x + y) = f(x) +' f(y)$ and $f(x \circ y) = f(x) \circ' f(y)$.

Definition 4.9. Let R is a hyperring and \mathfrak{R} be a strongly normal relation on \mathcal{U} . If $\varphi_{\mathfrak{R}} : R \rightarrow R/\rho_{\mathfrak{R}}^*$ be the canonical projection, we set $\omega_{\mathfrak{R}} = \varphi_{\mathfrak{R}}^{-1}(1_{R/\rho_{\mathfrak{R}}^*})$, and called the heart of $\varphi_{\mathfrak{R}}$.

Theorem 4.10. Let $(R, +, \circ)$ is a hyperfield (i.e, $(R, +, \circ)$ be a hyperring and (R, \circ) is a hypergroup) and B is a non-empty subset of R , then we have $\omega_{\mathfrak{R}} \circ B = B \circ \omega_{\mathfrak{R}} = \varphi_{\mathfrak{R}}^{-1}(\varphi_{\mathfrak{R}}(B))$.

Proof. Clearly $\varphi_{\mathfrak{R}}^{-1}(\varphi_{\mathfrak{R}}(B)) = \{x \in R \mid \exists b \in B : \varphi_{\mathfrak{R}}(b) = \varphi_{\mathfrak{R}}(x)\}$. Let $y \in \varphi_{\mathfrak{R}}^{-1}(\varphi_{\mathfrak{R}}(B))$, thus for some $b \in B$, $\varphi_{\mathfrak{R}}(b) = \varphi_{\mathfrak{R}}(y)$. Since (R, \circ) is a hypergroup, $u \in R$ exists such that $y \in b \circ u$, so $\varphi_{\mathfrak{R}}(y) = \varphi_{\mathfrak{R}}(b) \odot \varphi_{\mathfrak{R}}(u)$. Since $(R/\rho_{\mathfrak{R}}^*, \odot)$ is a group and $\varphi_{\mathfrak{R}}(b) = \varphi_{\mathfrak{R}}(y)$, we obtain $\varphi_{\mathfrak{R}}(u) = 1_{R/\rho_{\mathfrak{R}}^*}$ and so $u \in \varphi_{\mathfrak{R}}^{-1}(1_{R/\rho_{\mathfrak{R}}^*}) = \omega_{\mathfrak{R}}$. Therefore, $\varphi_{\mathfrak{R}}^{-1}(\varphi_{\mathfrak{R}}(B)) \subseteq B \circ \omega_{\mathfrak{R}}$.

Conversely if $z \in B \circ \omega_{\mathfrak{R}}$, then $\varphi_{\mathfrak{R}}(z) \in \varphi_{\mathfrak{R}}(B)$ and so $z \in \varphi_{\mathfrak{R}}^{-1}(\varphi_{\mathfrak{R}}(B))$. It is proved that $\omega_{\mathfrak{R}} \circ B = \varphi_{\mathfrak{R}}^{-1}(\varphi_{\mathfrak{R}}(B))$ by a similar way and we obtain $\varphi_{\mathfrak{R}}^{-1}(\varphi_{\mathfrak{R}}(B)) = \omega_{\mathfrak{R}} \circ B = B \circ \omega_{\mathfrak{R}}$. \square

Theorem 4.11. If $(R, +, \circ)$ is a hyperfield and B is a non-empty subset of R , then we have $\omega_{\mathfrak{R}} \circ B = B \circ \omega_{\mathfrak{R}} = \overline{\mathfrak{R}_u}(B)$.

Proof. If $\varphi_{\mathfrak{R}}(b) = \varphi_{\mathfrak{R}}(x)$ then $x \in \overline{\mathfrak{R}_u}(b)$. Therefore $\varphi_{\mathfrak{R}}^{-1}(\varphi_{\mathfrak{R}}(B)) = \bigcup_{b \in B} \overline{\mathfrak{R}_u}(b) = \overline{\mathfrak{R}_u}(B)$. \square

5. \mathfrak{R} -PARTS AND A_R -HYPERRINGS

We recall that a K_H hypergroup is a hypergroup constructed from a hypergroup (H, \circ) and a family $\{A(x)\}_{x \in H}$ of non-empty subsets that are mutually disjoint. Put $K_H = \bigcup_{x \in H} A(x)$ and define the hyperoperation $*$ on K_H as following,

$$\forall(a, b) \in K_H^2, 2a \in A(x), b \in A(y), 3a * b \stackrel{\text{def}}{=} \bigcup_{z \in x \circ y} A(z).$$

(H, \circ) is a hypergroup if and only if $(K_H, *)$ is a hypergroup. In this case K_H is said to be a K_H -hypergroup generated by H .

Now let (R, \dagger, \star) be a commutative hyperring, S_r , $r \in R$ be a family of non-empty sets indexed in R such that for all $r_1, r_2 \in R$, $r_1 \neq r_2$, $S_{r_1} \cap S_{r_2} = \emptyset$. We set $A = \bigcup_{r \in R} S_r$ and we define the hyperoperations \uplus and \odot in A in the following way:

$$\forall(x, y) \in S_{r_1} \times S_{r_2}, \quad x \uplus y = \bigcup_{t \in r_1 \dagger r_2} S_t \quad \text{and} \quad x \odot y = \bigcup_{u \in r_1 \star r_2} S_u.$$

It is easy to see that the structure (A, \uplus, \odot) is a hyperring. The hyperring (A, \uplus, \odot) is called a A_R -hyperring with suport A or A_R -hyperring generated by

R .

For all $P \in P^*(R)$, let $S(P) = \bigcup_{x \in P} S_x$.

Theorem 5.1. *Let \mathfrak{R} be a relation on \mathcal{U} . Then P is $\mathcal{LR}_{\mathcal{U}}$ -part of R if and only if $S(P)$ is $\mathcal{L}\hat{\mathfrak{R}}_{\mathcal{U}}$ -part of A_R , where the relation $\hat{\mathfrak{R}}$ is defined as follows:*

$$\sum_{i=1}^n \prod_{j=1}^{t_i} x_{ij} \mathfrak{R} \sum_{i=1}^m \prod_{j=1}^{k_i} y_{ij} \Leftrightarrow \bigcup_{v \in \sum_{i=1}^n \prod_{j=1}^{t_i} x_{ij}} S_v 3 \hat{\mathfrak{R}} 1 \bigcup_{u \in \sum_{i=1}^m \prod_{j=1}^{k_i} y_{ij}} S_u.$$

Proof. Let $S(P)$ be a $\mathcal{L}\hat{\mathfrak{R}}_{\mathcal{U}}$ -part of A_R and $(\prod_{i=1}^n x_i, \prod_{i=1}^m y_i) \in \mathfrak{R}$ such that $\prod_{i=1}^m y_i \cap P \neq \emptyset$ be given. So $\bigcup_{v \in \prod_{i=1}^n x_i} S_v 3 \hat{\mathfrak{R}} 1 \bigcup_{u \in \prod_{i=1}^m y_i} S_u$ and we have,

$$\begin{aligned} \prod_{i=1}^m y_i \cap P \neq \emptyset &\Rightarrow \exists p \in P, \text{ such that } p \in \prod_{i=1}^m y_i \\ &\Rightarrow \exists p \in P, \text{ such that } S_p 1 \subseteq 1 \bigcup_{u \in \prod_{i=1}^m y_i} S_u \\ &\Rightarrow \bigcup_{u \in \prod_{i=1}^m y_i} S_u \cap S(P) \neq \emptyset \\ &\Rightarrow \bigcup_{v \in \prod_{i=1}^n x_i} S_v 1 \subseteq 1 S(P), \text{ because } S(P) \text{ is a } \mathcal{L}\hat{\mathfrak{R}}_{\mathcal{U}} - \text{part.} \end{aligned}$$

Now suppose that $t \in \prod_{i=1}^n x_i$ is given. Then $S_t 1 \subseteq 1 S(P)$ and so there exists $q \in P$ such that $S_t \cap S_q \neq \emptyset$. Therefore $t = q$ and hence $t \in P$, thus $\prod_{i=1}^n x_i 1 \subseteq 1 P$. For the proof of the converse implication let $\sum_{i=1}^n \prod_{j=1}^{t_i} z_{ij} \cap S(P) \neq \emptyset$

and $\sum_{i=1}^s \prod_{j=1}^{l_i} t_{ij} 1 \hat{\mathfrak{R}} 1 \sum_{i=1}^n \prod_{j=1}^{t_i} z_{ij}$ be given. Therefore there exists $x_{ij} \in A$ such that for all $1 \leq i \leq m', 1 \leq j \leq k'_i, z_{ij} \in S_{x_{ij}}$. Suppose that $u \in \bigcup_{y \in \sum_{i=1}^n \prod_{j=1}^{t_i} x_{ij}} S_y$, thus

$u \in S_{y_0}$ for some $y_0 \in \prod_{i=1}^n x_i$. Since $u \in S(P)$, then there exists $y_1 \in P$ such that $u \in S_{y_1}$. Therefore $S_{y_0} \cap S_{y_1} \neq \emptyset$, which implies $y_0 = y_1 \in \prod_{i=1}^n x_i \cap P$. Since

P is \mathcal{LR}_U -part of R and $\sum_{i=1}^s \prod_{j=1}^{l_i} x'_{ij} 1 \mathfrak{R} 1 \prod_{i=1}^n x_i$, where $t_{ij} \in S_{x'_{ij}}$ for all $1 \leq i \leq s$, then $\sum_{i=1}^s \prod_{j=1}^{l_i} x'_{ij} 1 \subseteq 1P$. So $\sum_{i=1}^s \prod_{j=1}^{l_i} t_{ij} = \bigcup_{w \in \sum_{i=1}^s \prod_{j=1}^{l_i} x'_{ij}} S_w 1 \subseteq 1 \bigcup_{u \in P} S_u = S(P)$. \square

6. CONCLUSION

In this paper we introduce and analyze a generalization of the notion of a complete part in a hyperring. We call this generalization \mathfrak{R} -part of a hyperring. Several properties are investigated, such as the structure of \mathfrak{R} -closures of a subset. This research can be continued, for instance in the study of some particular classes of hyperrings.

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