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$\Re$-parts in hyperrings

H. Babaei[a], M. Jafarpour[b] and S. Sh. Mousavi[c,*]

[a]Mathematics Department, Azad University of Kerman, Kerman, Iran
[b]Mathematics Department, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran
[c]Mathematics Department, Shahid Bahonar University of Kerman, Kerman, Iran

Abstract. In this article, first we generalize the concept of complete parts in hyperrings by introducing the concept $\Re$-parts in hyperrings and then we study $\Re$-closures in hyperrings. Finally we characterize $\Re$-closures in hyperfields.

Keywords: hyperrings, (semi)hypergroups, complete parts.

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1. Introduction

The theory of hyperstructures was introduced in 1934 by Marty [9] at the 8th Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini and Leoreanu [1, 2] , Mittas [11, 12], Stratigopoulos [16], and by various authors. Basic definitions and propositions about the hyperstructures are found in [1, 2, 17]. Krasner [8] has studied the notion of *hyperfields, hyperrings*, and then some researchers. Hyperrings are essentially rings with approximately modified axioms. There are different notions of hyperrings . If the addition + is a hyperoperation and the multiplication is a binary operation, then the hyperring is called Krasner (additive) hyperring [8].

*Corresponding Author

E-mail addresses: babaei@iauk.ac.ir (H. Babaei), m.j@mail.vru.ac.ir (M. Jafarpour), smousavi@mail.uk.ac.ir (S. Sh. Mousavi)

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Rota [14] introduced a multiplicative hyperring, where + is a binary operation and the multiplication is a hyperoperation. De Salvo [15] studied hyperrings in which the additions and the multiplications were hyperoperations. In 2007, Davvaz and Leoreanu-Fotea [4] published a book titled Hyperring Theory and Applications. Complete parts were introduced by Koskas [7] and studied then by Miglirato [10], Corsini and Sureau [1, 2]. Mousavi et al. [13] introduced the notion of $\Re$-parts in hypergroups as a generalization of complete parts in hypergroups. In this article we generalize the notion of complete parts by introducing *left* and *right* $\Re$-*parts* in hyperrings and we will study $\Re$-*closures* in hyperrings. Finally we characterize $\Re$-closures in *hyperfields*.

2. Preliminaries

A *hypergroupoid* $(H, \circ)$ is a non-empty set H together with a hyperoperation $\circ$ defined on H, that is a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$ and for simplicity by xy. If A , B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup\{xy \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ (resp. $A \circ x$). A hypergroupoid $(H, \circ)$ is called a *hypergroup* in the sense of [9] if for all $x, y, z \in H$ the following two conditions hold: (i) $x(yz) = (xy)z$, (ii) $xH = Hx = H$, means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in xu$ and $y \in vx$. If $(H, \circ)$ satisfies only the first axiom, then it is called a *semi-hypergroup* an exhaustive review updated to 1992 of hypergroup theory appears in [1]. A recent book [2] contains a wealth of applications. A *hyperring* [17] is a triple $(R, +, \circ)$ which satisfies the ring-like axioms in the following way:(i) $(R, +)$ is a hypergroup , (ii) $(R, \circ)$ is a semi-hypergroup, (iii) the multiplication is distributive with respect to the hyperoperation $+$. The hyperrings were studied by many authors, for example see [6], [3], [17], [5] and [19]. In [20] and [18] Vougiouklis defines the relation Γ on hyperring as follows: $x\Gamma y$ if and only if $x, y \subseteq u$, where u is a finite sum of finite products of elements of R, in fact there exist $n, k_i \in \mathbb{N}$ and $x_{ij} \in R$ such that $u = \sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} x_{ij}$. He proved that the quotient R/Γ^*, where Γ^* is the transitive closure of Γ, is a ring and also Γ^* is the smallest equivalent relation on R such that the quotient R/Γ^* is a fundamental ring. The both $\oplus$ and $\odot$ on R/Γ^* are defined as follow:
$\forall z \in \Gamma^*(x) + \Gamma^*(y), 2\Gamma^*(x) \oplus \Gamma^*(y) = \Gamma^*(z);$

$$\forall z \in \Gamma^*(x) \circ \Gamma^*(y), 2\Gamma^*(x) \odot \Gamma^*(y) = \Gamma^*(z).$$

Let M be a non-empty subset of R. We say that M is a *complete part* if for every $n \in \mathbb{N}$, $i = 1, 2, ..., n$, $\forall k_i \in \mathbb{N}$, $\forall (z_{i1}, ..., z_{ik_i}) \in R^{k_i}$ we have:

$$\sum_{i=1}^{n} \prod_{j=1}^{k_i} z_{ij} \bigcap M \neq \emptyset \Rightarrow \sum_{i=1}^{n} \prod_{j=1}^{k_i} z_{ij} \subseteq M.$$

3. $\Re$-PARTS

Let $\mathcal{U}$ be the set of finite sums of finite products of elements of R and $\Re$ be a relation on $\mathcal{U}$. In this section first we generalize the notion of complete parts by introducing the notion of $\Re$-parts and then we study $\Re$-closures.

Definition 3.1. *Let R be a hyperring and $\mathcal{U}$ be the set of finite sum of finite products of elements of R and $\Re$ be a relation on $\mathcal{U}$. For a non-empty subset A of R we say:*

(i) A is a left $\Re$-part of R with respect to $\mathcal{U}$ (or briefly is $\mathcal{L}\Re_{\mathcal{U}}$-part) if for all $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} x_{ij}$ and $\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} y_{ij}$ in $\mathcal{U}$ the following implication is valid:

$$[\sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij} \bigcap A \neq \emptyset 2 and 2 \sum_{i=1}^{m}\prod_{j=1}^{t_i} y_{ij} 2 \Re 1 \sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij}] \Rightarrow \sum_{i=1}^{m}\prod_{j=1}^{t_i} y_{ij} 1 \subseteq 1 A;$$

(ii) A is a right $\Re$-part of R with respect to $\mathcal{U}$ (or briefly is $\mathcal{R}\Re_{\mathcal{U}}$-part) if for all $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} x_{ij}$ and $\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} y_{ij}$ in $\mathcal{U}$ the following implication is valid:

$$[\sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij} \bigcap A \neq \emptyset 2 and 2 \sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij} 2 \Re 1 \sum_{i=1}^{m}\prod_{j=1}^{t_i} y_{ij}] \Rightarrow \sum_{i=1}^{m}\prod_{j=1}^{t_i} y_{ij} 1 \subseteq 1 A;$$

(iii) A is a $\Re$-part of R with respect to $\mathcal{U}$ (or briefly is $\Re_{\mathcal{U}}$-part) if it is $\mathcal{L}\Re_{\mathcal{U}}$-part and $\mathcal{R}\Re_{\mathcal{U}}$-part.

Proposition 3.2. *Let $\Re$ be a relation on $\mathcal{U}$ and $\Re^{-1}$ be the inverse of $\Re$ then*
(i) A is $\mathcal{L}\Re_{\mathcal{U}}^{-1}$-part if and only if it is $\mathcal{R}\Re_{\mathcal{U}}$-part;
(ii) A is $\mathcal{R}\Re_{\mathcal{U}}^{-1}$-part if and only if it is $\mathcal{L}\Re_{\mathcal{U}}$-part.

Definition 3.3. *The intersection of $\mathcal{L}\Re_{\mathcal{U}}$-parts (or $\mathcal{R}\Re_{\mathcal{U}}$-parts, $\Re$-parts) which contain A is called $\mathcal{L}\Re_{\mathcal{U}}$-closure (or $\mathcal{R}\Re_{\mathcal{U}}$-closure, $\Re$-closure) of A in R and it will be denoted by $\overline{\mathcal{L}\Re_{\mathcal{U}}}(A)$ (or $\overline{\mathcal{R}\Re_{\mathcal{U}}}(A)$, $\overline{\Re_{\mathcal{U}}}(A)$).*

From now on R is a hyperring, $\mathcal{U}$ is the set of finite sum of finite products of elements of R, $u \in \mathcal{U}$ means $u = \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} x_{ij}$ and A is a non-empty subset of R.

Proposition 3.4. *For a non-empty subset A of R we have:*
(i) $\overline{\mathcal{L}\Re_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{R}\Re_{\mathcal{U}}}(A)$;
(ii) $\overline{\mathcal{R}\Re_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{L}\Re_{\mathcal{U}}}(A)$.

Proof. Follows from Proposition 3.2. □

Lemma 3.5. *For a non-empty subset A of R define:*

$${}_{A}\sum^{\mathcal{U}} \overset{def}{:=} \{\Re 1 \subseteq 1 \mathcal{U} \times \mathcal{U} \mid \overline{\mathcal{L}\Re_{\mathcal{U}}}(A) = A\} and \sum_{A}^{\mathcal{U}} \overset{def}{:=} \{\Re 1 \subseteq 1 \mathcal{U} \times \mathcal{U} \mid \overline{\mathcal{R}\Re_{\mathcal{U}}}(A) = A\}.$$

If ${}_A\sum^{\mathcal{U}} \neq \emptyset$ *(resp.* $\sum_A^{\mathcal{U}} \neq \emptyset$*), then* $({}_A\sum^{\mathcal{U}}, \circ)$ *(resp.* $(\sum_A^{\mathcal{U}}, \circ)$*) is a semigroup, where* $\circ$ *is the operation of relation composition.*

Proof. Suppose that $\Re, \Re' \in {}_A\sum^{\mathcal{U}}$ and $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij}, \sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} x_{ij}) \in \mathcal{U}\times\mathcal{U}$ are given. Let $\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} x_{ij} \bigcap A \neq \emptyset$ and $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij} 1 \Re \circ \Re' 1 \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} x_{ij}$. So there exists $\sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij}$ such that $\sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij} 1\Re 1 \sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} x_{ij}$ and $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij} 1\Re' 1 \sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij}$. From $\sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij} 1\Re 1 \sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} x_{ij}$ and $\Re \in {}_A\sum^{\mathcal{U}}$, we have $\sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij} \subseteq A$. Since $\Re' \in {}_A\sum^{\mathcal{U}}$ and $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij} 1\Re' 1 \sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij}$, $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij} \subseteq A$. □

Theorem 3.6. *If* $\Re$ *is a permutation of finite order in* $S_{\mathcal{U}}$ *(the symmetric group on the set* $\mathcal{U}$*), then the following are equivalent:*

(i) A *is* $\mathcal{L}\Re_{\mathcal{U}}$*-part;*
(ii) A *is* $\mathcal{R}\Re_{\mathcal{U}}$*-part;*
(iii) A *is* $\Re_{\mathcal{U}}$*-part.*

Proof. (i)⇒ (ii). For this reason suppose that A is $\mathcal{L}\Re_{\mathcal{U}}$-part. So $\overline{\mathcal{L}\Re_{\mathcal{U}}}(A) = A$ and hence $\Re \in {}_A\sum^{\mathcal{U}}$. Since $\Re$ is a permutation of finite order in $S_{\mathcal{U}}$, $\langle\Re\rangle = \{\Re^n \mid n \in \mathbb{N}\}$ is a subgroup of ${}_A\sum^{\mathcal{U}}$ and so $\Re^{-1} \in {}_A\sum^{\mathcal{U}}$. Therefore by Proposition 3.4 we have $A = \overline{\mathcal{L}\Re^{-1}_{\mathcal{U}}}(A) = \overline{\mathcal{R}\Re_{\mathcal{U}}}(A)$, thus $\Re \in \sum_A^{\mathcal{U}}$ and hence A is $\mathcal{R}\Re_{\mathcal{U}}$-part. □

Theorem 3.7. *Suppose that* $\Re 1 \subseteq 1 \mathcal{U}\times\mathcal{U}$.

(i) We pose $K^{\mathcal{L}}_{1,\Re}(A) = A$ *and*

$$K^{\mathcal{L}}_{n+1,\Re}(A) = \{x \in R \mid \exists (u,v) \in \Re, x \in u \text{ and } v \cap K^{\mathcal{L}}_{n,\Re}(A) \neq \emptyset\},$$

if we consider $K^{\mathcal{L}}_{\Re}(A) = \cup_{n\geq 1} K^{\mathcal{L}}_{n,\Re}(A)$*, then* $K^{\mathcal{L}}_{\Re}(A) = \overline{\mathcal{L}\Re_{\mathcal{U}}}(A)$ *and* $K^{\mathcal{L}}_{\Re}(A)$ *is the smallest* $\mathcal{L}\Re_{\mathcal{U}}$*-part containing* A*;*

(ii) We pose $K^{\mathcal{R}}_{1,\Re}(A) = A$ *and*

$$K^{\mathcal{R}}_{n+1,\Re}(A) = \{x \in R \mid \exists (v,u) \in \Re, x \in u \text{ and } v \cap K^{\mathcal{R}}_{n,\Re}(A) \neq \emptyset\},$$

if we consider $K^{\mathcal{R}}_{\Re}(A) = \cup_{n\geq 1} K^{\mathcal{R}}_{n,\Re}(A)$*, then* $K^{\mathcal{R}}_{\Re}(A) = \overline{\mathcal{R}\Re_{\mathcal{U}}}(A)$ *and* $K^{\mathcal{R}}_{\Re}(A)$ *is the smallest* $\mathcal{R}\Re_{\mathcal{U}}$*-part containing* A*;*

(iii) We pose $K_{1,\Re}(A) = A$ *and*

$$K_{n+1,\Re}(A) = \{x \in R \mid \exists (u,v) \in \Re \cup \Re^{-1}, x \in u \text{ and } v \cap K_{n,\Re}(A) \neq \emptyset\},$$

if $K_{\Re}(A) = \cup_{n\geq 1} K_{n,\Re}(A)$*, then* $K_{\Re}(A) = \overline{\Re_{\mathcal{U}}}(A)$ *and* $K_{\Re}(A)$ *is the smallest* $\Re_{\mathcal{U}}$*-part containing* A*.*

Proof. (i) It is necessary to prove:
(1) $K^{\mathcal{L}}_{\Re}(A)$ is $\mathcal{L}\Re_{\mathcal{U}}$-part,
(2) if $A\subseteq B$ and B is $\mathcal{L}\Re_{\mathcal{U}}$-part, then $K^{\mathcal{L}}_{\Re}(A)\subseteq B$.
For the proof (1) suppose that $v\bigcap K^{\mathcal{L}}_{\Re}(A)\neq\emptyset$ and $u1\Re1v$. Therefore there exists $n\in\mathbb{N}$ such that $v\bigcap K^{\mathcal{L}}_{n,\Re}(A)\neq\emptyset$, from which follows $u1\subseteq1K^{\mathcal{L}}_{n+1,\Re}(A)1\subseteq1K^{\mathcal{L}}_{\Re}(A)$. Now we prove (2) by induction on n. We have $K^{\mathcal{L}}_{1,\Re}(A)1=1A1\subseteq1B$. Suppose that $K^{\mathcal{L}}_{n,\Re}(A)1\subseteq1B$. We prove that $K^{\mathcal{L}}_{n+1,\Re}(A)1\subseteq1B$. If $z\in K^{\mathcal{L}}_{n+1,\Re}(A)$, then there exists $(u,v)\in\mathcal{U}\times\mathcal{U}$ such that $z\in u$, $u1\Re1v$ and $v\bigcap K^{\mathcal{L}}_{n,\Re}(A)\neq\emptyset$. Therefore $v\bigcap B\neq\emptyset$ and hence $z\in u1\subseteq1B$. So $K^{\mathcal{L}}_{n+1,\Re}(A)1\subseteq1B$.
(ii) We have

$$\begin{aligned}K^{\mathcal{R}}_{\Re}(A)&=K^{\mathcal{L}}_{\Re^{-1}_{\mathcal{U}}}(A)\\&=\overline{\mathcal{L}\Re^{-1}_{\mathcal{U}}}(A),4\text{by part (i)}\\&=\overline{\mathcal{R}\Re_{\mathcal{U}}}(A),4\text{by Proposition 3.4.}\end{aligned}$$

(iii) Follows from (i) and (ii). □

Proposition 3.8. *Suppose that B is a non-empty subset of R and $\Re$ is a relation on $\mathcal{U}$. Then we have:*
(i) $\overline{\mathcal{L}\Re_{\mathcal{U}}}(B)=\bigcup\limits_{b\in B}\overline{\mathcal{L}\Re_{\mathcal{U}}}(b)$;
(ii) $\overline{\mathcal{R}\Re_{\mathcal{U}}}(B)=\bigcup\limits_{b\in B}\overline{\mathcal{R}\Re_{\mathcal{U}}}(b)$;
(iii) $\overline{\Re_{\mathcal{U}}}(B)=\bigcup\limits_{b\in B}\overline{\Re_{\mathcal{U}}}(b)$.

Proof. (i) It is clear that for all $b\in B$, $\overline{\mathcal{L}\Re_{\mathcal{U}}}(b)1\subseteq1\overline{\mathcal{L}\Re_{\mathcal{U}}}(B)$. By Theorem 3.7(i), $\overline{\mathcal{L}\Re_{\mathcal{U}}}(B)=\bigcup\limits_{n\geq1}K^{\mathcal{L}}_{n,\Re}(B)$. We follow the proposition by induction on n. For $n=1$, $K^{\mathcal{L}}_{1,\Re}(B)=B=\bigcup\limits_{b\in B}\{b\}=\bigcup\limits_{b\in B}K^{\mathcal{L}}_{1,\Re}(b)$. Supposing it is true for n, we show that $K^{\mathcal{L}}_{n+1,\Re}(B)1\subseteq1\bigcup\limits_{b\in B}K^{\mathcal{L}}_{n+1,\Re}(b)$. If $z\in K^{\mathcal{L}}_{n+1,\Re}(B)$, then there exists $(u,v)\in\Re$ such that $z\in u$ and $v\bigcap K^{\mathcal{L}}_{n,\Re}(B)\neq\emptyset$. From this it follows, by the hypothesis of induction, $v\bigcap(\bigcup\limits_{b\in B}K^{\mathcal{L}}_{n,\Re}(b))\neq\emptyset$ and therefore $b'\in B$ exists such that $v\bigcap K^{\mathcal{L}}_{n,\Re}(b')\neq\emptyset$. So $z\in K^{\mathcal{L}}_{n+1,\Re}(b')$ and hence $\overline{\mathcal{L}\Re_{\mathcal{U}}}(B)1\subseteq1\bigcup\limits_{b\in B}\overline{\mathcal{L}\Re_{\mathcal{U}}}(b)$. □

Theorem 3.9. *Suppose that $\Re1\subseteq1\mathcal{U}\times\mathcal{U}$. The relation $K^{\mathcal{L}}_{\Re}$ (resp. $K^{\mathcal{R}}_{\Re}$) on R defined by:*

$$x1K^{\mathcal{L}}_{\Re}1y\Leftrightarrow x\in K^{\mathcal{L}}_{\Re}(y)(x\in K^{\mathcal{R}}_{\Re}(y)),$$

where $K^{\mathcal{L}}_{\Re}(y)=K^{\mathcal{L}}_{\Re}(\{y\})$ (resp. $K^{\mathcal{R}}_{\Re}(y)=K^{\mathcal{R}}_{\Re}(\{y\})$) is a preorder. Furthermore if $\Re$ is symmetric, then $K^{\mathcal{L}}_{\Re}$ (resp. $K^{\mathcal{R}}_{\Re}$) is an equivalence relation.

Proof. It is easy to see that $K_{\Re}^{\mathcal{L}}$ is reflexive. Now suppose that $x1K_{\Re}^{\mathcal{L}}1y$ and $y1K_{\Re}^{\mathcal{L}}1z$. So $x \in K_{\Re}^{\mathcal{L}}(y)$ and $y \in K_{\Re}^{\mathcal{L}}(z)$. By Theorem 3.7(i) we have $K_{\Re}^{\mathcal{L}}(z)$ is $\mathcal{L}\Re_{\mathcal{U}}$-part thus $K_{\Re}^{\mathcal{L}}(y) \subseteq K_{\Re}^{\mathcal{L}}(z)$ and hence $x \in K_{\Re}^{\mathcal{L}}(z)$. Therefore $K_{\Re}^{\mathcal{L}}$ is preorder. Now let $\Re$ be symmetric. We prove that $K_{\Re}^{\mathcal{L}}$ is symmetric as well. To this end the following is premised:

(1) for all $n \geq 2$ and $x \in R$, $K_{n,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) = K_{n+1,\Re}^{\mathcal{L}}(x)$;

(2) $x \in K_{n,\Re}^{\mathcal{L}}(y)$ if and only if $y \in K_{n,\Re}^{\mathcal{L}}(x)$.

We prove (1) by induction on n. Suppose that $z \in K_{2,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x))$ so there exists $(u,v) \in \Re$ such that $z \in u$ and $v \bigcap K_{2,\Re}^{\mathcal{L}}(x) \neq \emptyset$. Thus $z \in K_{3,\Re}^{\mathcal{L}}(x)$. Let $K_{n,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) = K_{n+1,\Re}^{\mathcal{L}}(x)$ so we have:

$$\begin{aligned} z \in K_{n+1,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) &\Leftrightarrow \exists (u,v) \in \Re, z \in u, v \cap K_{n,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) \neq \emptyset \\ &\Leftrightarrow \exists (u,v) \in \Re, z \in u, v \cap K_{n+1,\Re}^{\mathcal{L}}(x) \neq \emptyset \\ &\Leftrightarrow z \in K_{n+2,\Re}^{\mathcal{L}}(x). \end{aligned}$$

We also prove (2) by induction on n. It is clear that $x \in K_{2,\Re}^{\mathcal{L}}(y)$ if and only if $y \in K_{2,\Re}^{\mathcal{L}}(x)$. Suppose $x \in K_{n,\Re}^{\mathcal{L}}(y)$ if and only if $y \in K_{2,\Re}^{\mathcal{L}}(x)$. Let $x \in K_{n+1,\Re}^{\mathcal{L}}(y)$ be given, so there exist $(u,v) \in \Re$ such that $x \in u$ and $v \bigcap K_{n,\Re}^{\mathcal{L}}(y) \neq \emptyset$. Therefore there exists $b \in v \bigcap K_{n,\Re}^{\mathcal{L}}(y)$ and hence $y \in K_{n,\Re}^{\mathcal{L}}(b)$. Since $\Re$ is symmetric and $(u,v) \in \Re$, $b \in v$ and $x \in u \bigcap K_{1,\Re}^{\mathcal{L}}(x)$ implies that $b \in K_{2,\Re}^{\mathcal{L}}(x)$ and hence $y \in K_{n,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) = K_{n+1,\Re}^{\mathcal{L}}(x)$. Similarly we can show if $y \in K_{n+1,\Re}^{\mathcal{L}}(x)$, then $x \in K_{n+1,\Re}^{\mathcal{L}}(x)$. □

Proposition 3.10. *Let $\Re$ be a relation on $\mathcal{U}$ and A be a non-empty subset of the hyperring R. The following conditions are equivalent:*

(i) A is a ($\mathcal{R}\Re_{\mathcal{U}}$-part) $\mathcal{L}\Re_{\mathcal{U}}$-part of R;

(ii) $x \in A, (x1K_{\Re}^{\mathcal{L}}1z)z1K_{\Re}^{\mathcal{L}}1x \Rightarrow z \in A$.

Proof. (i) $\Rightarrow$ (ii) If $x \in A$ and $z \in R$ such that $z1K_{\Re}^{\mathcal{L}}1x$, then there exists $(u,v) \in \Re$ such that $z \in u$ and $v \cap K_{n,\Re}^{\mathcal{L}}(A) \neq \emptyset$ for some $n \in \mathbb{N}$. Since A is a $\mathcal{L}\Re_{\mathcal{U}}$-part by Theorem 3.7, $K_{n,\Re}^{\mathcal{L}}(A)1 \subseteq 1A$ and so $v \cap A \neq \emptyset$. Therefore $u1 \subseteq 1A$ and hence $z \in A$.

(ii) $\Rightarrow$ (i) Let $u \cap A \neq \emptyset$ and $v1\Re1u$. So there exists $x \in A \cap u$ and $x \in u$, $u \cap K_{1,\Re}^{\mathcal{L}}(x) \neq \emptyset$. Now suppose that $z \in v$ is given. So

$$\begin{aligned} v1\Re1u &\Rightarrow z \in K_{2,\Re}^{\mathcal{L}}(x), 4 \text{ because } x \in u \\ &\Rightarrow z1K_{\Re}^{\mathcal{L}}1x \\ &\Rightarrow z \in A, 15 \text{ because } x \in A. \end{aligned}$$

Therefore $v1 \subseteq 1A$ and hence A is $\mathcal{L}\Re_{\mathcal{U}}$-part of R. □

4. Rings derived from hyperrings

In this section we give the notion of *(strongly) normal relation* on $\mathcal{U}$ and then we construct a ring from a hyperring.

Definition 4.1. *Suppose that* $\Re 1\subseteq 1\mathcal{U}\times\mathcal{U}$.

(i) for all $(x,y)\in R^2$ *define the relation* $\rho_{_{\mathcal{L},\Re}}$ *on* R *by:*

$$x1\rho_{_{\mathcal{L},\Re}}y \Leftrightarrow [x=y1 \text{ or } 1\exists(u,v)\in\Re 2 \text{ such that } 1x\in u1 \text{ and } 1y\in v]$$

and $\rho^*_{_{\mathcal{L},\Re}}$ *is the transitive closure of* $\rho_{_{\mathcal{L},\Re}}$*;*

(ii) for all $(x,y)\in R^2$ *define the relation* $\rho_{_{\mathcal{R},\Re}}$ *on* R *by:*

$$x1\rho_{_{\mathcal{R},\Re}}y \Leftrightarrow [x=y1 \text{ or } 1\exists(v,u)\in\Re 2 \text{ such that } 1x\in u1 \text{ and } 1y\in v]$$

and $\rho^*_{_{\mathcal{R},\Re}}$ *is the transitive closure of* $\rho_{_{\mathcal{R},\Re}}$*;*

(iii) for all $(x,y)\in R^2$ *define the relation* $\rho_{_\Re}$ *on* R *by:*

$$x1\rho_{_\Re}y \Leftrightarrow [x=y1 \text{ or } 1\exists(u,v)\in\Re\bigcup\Re^{-1}2 \text{ such that } 1x\in u1 \text{ and } 1y\in v]$$

and $\rho^*_{_\Re}$ *is the transitive closure of* $\rho_{_\Re}$.

Theorem 4.2. *Suppose that* $\Re 1\subseteq 1\mathcal{U}\times\mathcal{U}$*. For all* $(x,y)\in R^2$ *we have:*

(i) $x1K^{\mathcal{L}}_{\Re}1y$ *if and only if* $x1\rho^*_{_{\mathcal{L},\Re}}1y$*;*

(ii) $x1K^{\mathcal{R}}_{\Re}1y$ *if and only if* $x1\rho^*_{_{\mathcal{R},\Re}}1y$.

Proof. (i) It is easy to see that $\rho^*_{_{\mathcal{L},\Re}}1\subseteq 1K^{\mathcal{L}}_{\Re}$. Converesly suppose that $x1K^{\mathcal{L}}_{\Re}1y$ so by Theorem 3.9 we have $x\in K^{\mathcal{L}}_{n+1,\Re}(y)$ for some $n\in\mathbb{N}$. So there exists $(u_1,v_1)\in\Re$ such that $x\in u_1$ and $v_1\bigcap K^{\mathcal{L}}_{n,\Re}(y)\neq\emptyset$ thus there exists $x_1\in v_1\bigcap K^{\mathcal{L}}_{n,\Re}(y)$ and hence $x1\rho_{_{\mathcal{L},\Re}}1x_1$. Since $x_1\in K^{\mathcal{L}}_{n,\Re}(y)$, there exists $(u_2,v_2)\in\Re$ such that $x_1\in u_2$ and $v_2\bigcap K^{\mathcal{L}}_{n-1,\Re}(y)\neq\emptyset$. Therefore $x_11\rho_{_{\mathcal{L},\Re}}1x_2$, where $x_2\in v_2\bigcap K^{\mathcal{L}}_{n-1,\Re}(y)$. As a consequence we conclude that $x_n\in v_n\bigcap K^{\mathcal{L}}_{n-(n-1),\Re}(y)$ exists such that $x_{n-1}1\rho_{_{\mathcal{L},\Re}}1x_n$. Thus we have,

$$x1\rho_{_{\mathcal{L},\Re}}1x_11\rho_{_{\mathcal{L},\Re}}1x_21\ldots 1x_n1\rho_{_{\mathcal{L},\Re}}1y.$$

From this follows $K^{\mathcal{L}}_{\Re}1\subseteq 1\rho^*_{_{\mathcal{L},\Re}}$ and the proof is complete.

Similarly we have (ii). □

Proposition 4.3. *Suppose that* $\Re$ *is a permutation of finite order in* $S_{_\mathcal{U}}$*, then* $\rho^*_{_{\mathcal{L},\Re}}=\rho^*_{_\Re}$.

Proof. Since $K^{\mathcal{L}}_{\Re}(y)$ is $\mathcal{L}\Re_{_\mathcal{U}}$-part by Theorem 3.6, $K^{\mathcal{L}}_{\Re}(y)$ is $\mathcal{R}\Re_{_\mathcal{U}}$-part and hence $K^{\mathcal{R}}_{\Re}\subseteq K^{\mathcal{L}}_{\Re}$. Analogously $K^{\mathcal{L}}_{\Re}\subseteq K^{\mathcal{R}}_{\Re}$ and so $K^{\mathcal{L}}_{\Re}=K^{\mathcal{R}}_{\Re}$. From this it follows that $\rho^*_{_{\mathcal{L},\Re}}=\rho^*_{_\Re}$. □

Definition 4.4. *If* $(R,+,\circ)$ *is a hyperring and* $\rho 1\subseteq 1R\times R$ *is an equivalence, then we set:*

$$A\ \overline{\overline{\rho}}\ B \Leftrightarrow a1\rho 1b, 5\forall a\in A, \forall b\in B,$$

for all pairs (A, B) *of non-empty subsets of* R. *The relation* ρ *is said to be strongly regular to the left (resp. to the right) if (i)* $x1\rho1y \Rightarrow a + x \overset{=}{\rho}\ a + y$ *and (ii)* $x1\rho1y \Rightarrow a \circ x \overset{=}{\rho}\ a \circ y$ *(resp. (i)* $x1\rho1y \Rightarrow x + a \overset{=}{\rho}\ y + a$ *and (ii)* $x1\rho1y \Rightarrow a \circ x \overset{=}{\rho}\ a \circ y$*), for all* $(x, y, a) \in R^3$. ρ *is called strongly regular if it is (i) strongly regular to the right and to the left and moreover (ii) there exists* e *in* R *such that:* $\rho(x) = \rho(t)$, *for all* $t \in x \circ e \bigcap e \circ x$.

Definition 4.5. *Let* R *be a hyperring, then*

(i) a relation $\Re$ *on* $\mathcal{U}$ *is called normal if for all* $x \in R$, *one has* $K^{\mathcal{L}}_{\Re}(x) = K^{\mathcal{R}}_{\Re}(x)$,

(ii) a normal relation $\Re$ *on* $\mathcal{U}$ *is called strongly normal to the left (resp. to the right) if* $\mathsf{\rho}^*_{\mathcal{L},\Re}$ *(resp.* $\mathsf{\rho}^*_{\mathcal{R},\Re}$*) is strongly regular to the left (resp. to the right),*

(iii) a normal relation $\Re$ *on* $\mathcal{U}$ *is called strongly normal if* $\mathsf{\rho}^*_{\Re}$ *is strongly regular.*

Suppose that $\Re1\subseteq1\mathcal{U} \times \mathcal{U}$. For every element x of a hyperring R, set:

$P^n_{\mathcal{L},\Re}(x) = \bigcup\{v1 \mid 1v1\Re1u_n, u_n = \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} x_{ij}, x \in u_n\}$;

$P_{\mathcal{L},\Re}(x) = \bigcup\limits_{n\geqslant1} P^n_{\mathcal{L},\Re}(x) \bigcup\{x\}$;

$\mathsf{\rho}^*_{\mathcal{L},\Re}(x) = \{y \in R1 \mid 1y1\mathsf{\rho}^*_{\mathcal{L},\Re}1x\}$.

Theorem 4.6. *Let* R *be a hyperring and* $\Re$ *be a relation on* $\mathcal{U}$. *The following conditions are equivalent:*

(i) $\mathsf{\rho}_{\mathcal{L},\Re}$ *is transitive;*

(ii) for every $x \in R$, $\mathsf{\rho}^*_{\mathcal{L},\Re}(x) = P_{\mathcal{L},\Re}(x)$;

(iii) for every $x \in R$, $P_{\mathcal{L},\Re}(x)$ *is a* $\mathcal{L}\Re_{\mathcal{U}}$*-part of* R.

Proof. (i) $\Rightarrow$ (ii) For every pair (x, y) of elements of R we have:

$$y \in \mathsf{\rho}^*_{\mathcal{L},\Re}(x) \Leftrightarrow y1\mathsf{\rho}^*_{\mathcal{L},\Re}1x \Leftrightarrow y1\mathsf{\rho}_{\mathcal{L},\Re}1x \Leftrightarrow y \in P_{\mathcal{L},\Re}(x).$$

(ii) $\Rightarrow$ (iii) Let $(v, u) \in \Re$ such that $u \cap P_{\mathcal{L},\Re}(x) \neq \emptyset$ be given. So $u \cap \mathsf{\rho}^*_{\mathcal{L},\Re}(x) \neq \emptyset$ and hence there exists $z \in R$ such that $z \in u$ and $z \in \mathsf{\rho}^*_{\mathcal{L},\Re}(x)$, thus $z \in K^{\mathcal{L}}_{\Re}(x)$, by Theorem 4.2. On the other hand, $z \in K^{\mathcal{L}}_{\Re}(z)$, so $u \cap K^{\mathcal{L}}_{\Re}(z) \neq \emptyset$ and hence $v1\subseteq1K^{\mathcal{L}}_{\Re}(z)$, because $v1\Re1u$ and $K^{\mathcal{L}}_{\Re}(z)$ is a $\mathcal{L}\Re_{\mathcal{U}}$-part of R, by Theorem 3.7. Now suppose that $t \in v$ is an arbitrary element, thus $t \in K^{\mathcal{L}}_{\Re}(x)$ and hence $t1\mathsf{\rho}^*_{\mathcal{L},\Re}1x$. Therefore $t \in \mathsf{\rho}^*_{\mathcal{L},\Re}(x) = P_{\mathcal{L},\Re}(x)$ and so $v1\subseteq1P_{\mathcal{L},\Re}(x)$.

(iii) $\Rightarrow$ (i) Let x, y and z in R be given such that $x1\mathsf{\rho}_{\mathcal{L},\Re}1y$ and $y1\mathsf{\rho}_{\mathcal{L},\Re}1z$. Since $x1\mathsf{\rho}_{\mathcal{L},\Re}1y$, there exists $(u, v) \in \Re$ such that $x \in u$ and $y \in v$. Therefore $v \cap P_{\mathcal{L},\Re}(y) \neq \emptyset$ and since $P_{\mathcal{L},\Re}(y)$ is a $\mathcal{L}\Re_{\mathcal{U}}$-part, $u1\subseteq1P_{\mathcal{L},\Re}(y)$ and hence $x \in P_{\mathcal{L},\Re}(y)$. We can see that $P_{\mathcal{L},\Re}(y)1\subseteq1P_{\mathcal{L},\Re}(z)$, because $y1\mathsf{\rho}_{\mathcal{L},\Re}1z$ and so by above $y \in P_{\mathcal{L},\Re}(z)$. Therefore $x \in P_{\mathcal{L},\Re}(z)$ and hence $x1\mathsf{\rho}_{\mathcal{L},\Re}1z$. □

Proposition 4.7. *If* $\Re$ *is a normal relation on* $\mathcal{U}$, *then:*

(i) $\Re^{-1}$ *is a normal relation;*

(ii) $\rho^*_{\mathcal{L},\Re} = \rho^*_{\Re}$ *and* $\rho^*_{\mathcal{L},\Re}$ *is an equivalence relation.*

Proof. The proof follows from Proposition 3.4 and Theorem 4.2. □

Theorem 4.8. *Suppose that* $(R, +, \circ)$ *is a hyperring and* $\Re$ *is a strongly normal relation on* $\mathcal{U}$. *A ring structure turns out to be define on* $R/\rho^*_{\Re}$ *with respect to the operations:*

$$\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y) = \rho^*_{\Re}(z), 2 where 1 z \in x + y.$$

$$\rho^*_{\Re}(x) \odot \rho^*_{\Re}(y) = \rho^*_{\Re}(z), 2 where 1 z \in x \circ y.$$

Proof. We will prove that the operation $\oplus$ is well defined. Let $\rho^*_{\Re}(x_0) = \rho^*_{\Re}(x_1)$ and $\rho^*_{\Re}(y_0) = \rho^*_{\Re}(y_1)$. It is necessary to verify that $\rho^*_{\Re}(x_0) \oplus \rho^*_{\Re}(y_0) = \rho^*_{\Re}(x_1) \oplus \rho^*_{\Re}(y_1)$. By hypothesis $(m, n) \in \mathbb{N}^2$, $(z_0, z_1, ..., z_m) \in R^{m+1}$ and $(t_0, t_1, ..., t_n) \in R^{n+1}$ exist such that

$$x_0 = z_0 1 \rho_{\Re} 1 z_1 1 \rho_{\Re} 1 z_2 1 \ldots 1 z_{m-1} 1 \rho_{\Re} 1 z_m = x_1$$

and

$$y_0 = t_0 1 \rho_{\Re} 1 t_1 1 \rho_{\Re} 1 t_2 1 \ldots 1 t_{n-1} 1 \rho_{\Re} 1 t_n = y_1$$

Since $\Re$ is normal, for all $u \in z_{s-1} + t_{s-1}$ and $v \in z_s + t_s$, where $1 \leqslant s \leqslant k$ and $k = min\{m, n\}$, we have $u 1 \rho^*_{\Re} v$. Therefore $\rho^*_{\Re}(x_0) \oplus \rho^*_{\Re}(y_0) = \rho^*_{\Re}(z_1) \oplus \rho^*_{\Re}(t_1) = ... = \rho^*_{\Re}(z_k) \oplus \rho * (t_k) = \rho^*_{\Re}(a_{k+i}) \oplus \rho^*_{\Re}(b_{k+i})$, where $k + 1 \leqslant k + i \leqslant max\{m, n\}$ and:

$$(a_{k+i}, b_{k+i}) = \begin{cases} (x_1, t_{k+i}) & \text{if } 1 k = m; \\ (z_{k+i}, y_1) & \text{if} 1 k = n. \end{cases}$$

Hence $\oplus$ is well defined. Similarly the operation $\odot$ is well defined and Theorem 31 of [2] shows that $(R/\rho^*_{\Re}, \oplus)$ is a group. By strongly normality of $\Re$ we conclude that $(R/\rho^*_{\Re}, \odot)$ is a monoid with unit $\rho^*_{\Re}(e)$. The commutativity of $\oplus$ is related with the existence of the unit in multiplication. Since $\Re$ is strong, there exists e in R such that $\rho(x) = \rho(t)$ for all $t \in x \circ e \bigcap e \circ x$ which means $\rho^*_{\Re}(e)$ is the unit of multiplication so we have:

$[\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)] \odot [\rho^*_{\Re}(e) \oplus \rho^*_{\Re}(e)] = (\rho^*_{\Re}(x) \odot [\rho^*_{\Re}(e) \oplus \rho^*_{\Re}(e)]) \oplus (\rho^*_{\Re}(y) \odot [\rho^*_{\Re}(e) \oplus \rho^*_{\Re}(e)]) = (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(x)) \oplus (\rho^*_{\Re}(y) \oplus \rho^*_{\Re}(y))$ and also $[\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)] \odot [\rho^*_{\Re}(e) \oplus \rho^*_{\Re}(e)] = ([\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)] \odot \rho^*_{\Re}(e)) \oplus ([\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)] \odot \rho^*_{\Re}(e)) = (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)) \oplus (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y))$.

So $(\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(x)) \oplus (\rho^*_{\Re}(y) \oplus \rho^*_{\Re}(y)) = (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)) \oplus (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y))$ gives, $\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y) = \rho^*_{\Re}(y) \oplus \rho^*_{\Re}(x)$. □

Let $(R, +, \circ)$ and $(R', +', \circ')$ be two hyperrings. We say that $f : R \to R'$ is a homomorphism if for every $(x, y) \in R^2$ we have $f(x + y) = f(x) +' f(y)$ and $f(x \circ y) = f(x) \circ' f(y)$.

Definition 4.9. *Let R is a hyperring and $\Re$ be a strongly normal relation on $\mathcal{U}$. If $\varphi_{\Re} : R \to R/\rho^*_{\Re}$ be the canonical projection, we set $\omega_{\Re} = \varphi_{\Re}^{-1}(1_{R/\rho^*_{\Re}})$, and called the heart of $\varphi_{\Re}$.*

Theorem 4.10. *Let $(R, +, \circ)$ is a hyperfield (i.e, $(R, +, \circ)$ be a hyperring and $(R, \circ)$ is a hypergroup) and B is a non-empty subset of R, then we have $\omega_{\Re} \circ B = B \circ \omega_{\Re} = \varphi_{\Re}^{-1}(\varphi_{\Re}(B))$.*

Proof. Clearly $\varphi_{\Re}^{-1}(\varphi_{\Re}(B)) = \{x \in R \mid \exists b \in B : \varphi_{\Re}(b) = \varphi_{\Re}(x)\}$. Let $y \in \varphi_{\Re}^{-1}(\varphi_{\Re}(B))$, thus for some $b \in B$, $\varphi_{\Re}(b) = \varphi_{\Re}(y)$. Since $(R, \circ)$ is a hypergroup, $u \in R$ exists such that $y \in b \circ u$, so $\varphi_{\Re}(y) = \varphi_{\Re}(b) \odot \varphi_{\Re}(u)$. Since $(R/\rho^*_{\Re}, \odot)$ is a group and $\varphi_{\Re}(b) = \varphi_{\Re}(y)$, we obtain $\varphi_{\Re}(u) = 1_{R/\rho^*_{\Re}}$ and so $u \in \varphi_{\Re}^{-1}(1_{R/\rho^*_{\Re}}) = \omega_{\Re}$. Therefore, $\varphi_{\Re}^{-1}(\varphi_{\Re}(B)) \subseteq B \circ \omega_{\Re}$.
Converesly if $z \in B \circ \omega_{\Re}$, then $\varphi_{\Re}(z) \in \varphi_{\Re}(B)$ and so $z \in \varphi_{\Re}^{-1}(\varphi_{\Re}(B))$. It is proved that $\omega_{\Re} \circ B = \varphi_{\Re}^{-1}(\varphi_{\Re}(B))$ by a similar way and we obtain $\varphi_{\Re}^{-1}(\varphi_{\Re}(B)) = \omega_{\Re} \circ B = B \circ \omega_{\Re}$. □

Theorem 4.11. *If $(R, +, \circ)$ is a hyperfield and B is a non-empty subset of R, then we have $\omega_{\Re} \circ B = B \circ \omega_{\Re} = \overline{\Re_{\mathcal{U}}}(B)$.*

Proof. If $\varphi_{\Re}(b) = \varphi_{\Re}(x)$ then $x \in \overline{\Re_{\mathcal{U}}}(b)$. Therefore $\varphi_{\Re}^{-1}(\varphi_{\Re}(B)) = \bigcup\limits_{b \in B} \overline{\Re_{\mathcal{U}}}(b) = \overline{\Re_{\mathcal{U}}}(B)$. □

5. $\Re$-parts and A_R-hyperrings

We recall that a K_H hypergroup is a hypergroup constructed from a hypergroup $(H, \circ)$ and a family $\{A(x)\}_{x \in H}$ of non-empty subsets that are mutually disjoint. Put $K_H = \bigcup\limits_{x \in H} A(x)$ and define the hyperoperation $*$ on K_H as following,

$$\forall (a, b) \in K_H^2, 2a \in A(x), b \in A(y), 3a * b :\stackrel{def}{=} \bigcup_{z \in x \circ y} A(z).$$

$(H, \circ)$ is a hypergroup if and only if $(K_H, *)$ is a hypergroup. In this case K_H is said to be a *K_H-hypergroup generated by H*.

Now let $(R, \dagger, \star)$ be a commutative hyperring, S_r , $r \in R$ be a family of non-empty sets indexed in R such that for all $r_1, r_2 \in R$, $r_1 \neq r_2$, $S_{r_1} \bigcap S_{r_2} = \emptyset$. We set $A = \bigcup_{r \in R} S_r$ and we define the hyperoperations $\uplus$ and $\odot$ in A in the following way:

$$\forall (x, y) \in S_{r_1} \times S_{r_2}, \quad x \uplus y = \bigcup_{t \in r_1 \dagger r_2} S_t \quad \text{and} \quad x \odot y = \bigcup_{u \in r_1 \star r_2} S_u.$$

It is easy to see that the structure $(A, \uplus, \odot)$ is a hyperring. The hyperring $(A, \uplus, \odot)$ is called a *A_R-hyperring with suport A* or *A_R-hyperring generated by*

R.
For all $P \in P^*(R)$, let $S(P) = \bigcup\limits_{x \in P} S_x$.

Theorem 5.1. *Let $\Re$ be a relation on $\mathcal{U}$. Then P is $\mathcal{L}\Re_{\mathcal{U}}$-part of R if and only if $S(P)$ is $\mathcal{L}\widehat{\Re}_{\mathcal{U}}$-part of A_R, where the relation $\widehat{\Re}$ is defined as follows:*

$$\sum_{i=1}^{n}\prod_{j=1}^{t_i} x_{ij} \Re \sum_{i=1}^{m}\prod_{j=1}^{k_i} y_{ij} \Leftrightarrow \bigcup_{v \in \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{t_i} x_{ij}} S_v 3\widehat{\Re}1 \bigcup_{u \in \sum\limits_{i=1}^{m}\prod\limits_{j=1}^{k_i} y_{ij}} S_u.$$

Proof. Let $S(P)$ be a $\mathcal{L}\widehat{\Re}_{\mathcal{U}}$-part of A_R and $(\prod\limits_{i=1}^{n} x_i, \prod\limits_{i=1}^{m} y_i) \in \Re$ such that $\prod\limits_{i=1}^{m} y_i \bigcap P \neq \emptyset$ be given. So $\bigcup\limits_{v \in \prod\limits_{i=1}^{n} x_i} S_v 3\widehat{\Re}1 \bigcup\limits_{u \in \prod\limits_{i=1}^{m} y_i} S_u$ and we have,

$$\begin{aligned}
\prod_{i=1}^{m} y_i \cap P \neq \emptyset \Rightarrow & \ \exists p \in P, \text{ such that } p \in \prod_{i=1}^{m} y_i \\
& \Rightarrow \exists p \in P, \text{ such that } S_p 1\subseteq 1 \bigcup_{u \in \prod\limits_{i=1}^{m} y_i} S_u \\
& \Rightarrow \bigcup_{u \in \prod\limits_{i=1}^{m} y_i} S_u \cap S(P) \neq \emptyset \\
& \Rightarrow \bigcup_{v \in \prod\limits_{i=1}^{n} x_i} S_v 1\subseteq 1 S(P), \text{ because } S(P) \text{ is a } \mathcal{L}\widehat{\Re}_{\mathcal{U}} - \text{part.}
\end{aligned}$$

Now suppose that $t \in \prod\limits_{i=1}^{n} x_i$ is given. Then $S_t 1\subseteq 1 S(P)$ and so there exists $q \in P$ such that $S_t \cap S_q \neq \emptyset$. Therefore $t = q$ and hence $t \in P$, thus $\prod\limits_{i=1}^{n} x_i 1\subseteq 1 P$. For the proof of the converse implication let $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{t_i} z_{ij} \bigcap S(P) \neq \emptyset$ and $\sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} t_{ij} 1\widehat{\Re}1 \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{t_i} z_{ij}$ be given. Therefore there exists $x_{ij} \in A$ such that for all $1 \leqslant i \leqslant m'$,$1 \leqslant j \leqslant k'_i$, $z_{ij} \in S_{x_{ij}}$. Suppose that $u \in \bigcup\limits_{y \in \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{t_i} x_{ij}} S_y$, thus $u \in S_{y_0}$ for some $y_0 \in \prod\limits_{i=1}^{n} x_i$. Since $u \in S(P)$, then there exists $y_1 \in P$ such that $u \in S_{y_1}$. Therefore $S_{y_0} \cap S_{y_1} \neq \emptyset$, which implies $y_0 = y_1 \in \prod\limits_{i=1}^{n} x_i \cap P$. Since

P is $\mathcal{L}\Re_{\mathcal{U}}$-part of R and $\sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} x'_{ij} 1\Re 1 \prod\limits_{i=1}^{n} x_i$, where $t_{ij} \in S_{x'_{ij}}$ for all $1 \leqslant i \leqslant s$, then $\sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} x'_{ij} 1{\subseteq}1 P$. So $\sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} t_{ij} = \bigcup\limits_{w \in \sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} x'_{ij}} S_w 1{\subseteq}1 \bigcup\limits_{u \in P} S_u = S(P)$. $\square$

6. Conclusion

In this paper we introduce and analyze a generalization of the notion of a complete part in a hyperring. We call this generalization $\Re$-part of a hyperring. Several properties are investigated, such as the structure of $\Re$-closures of a subset. This research can be continuated, for instance in the study of some particular classes of hyperrings.

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