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# A Note on Tensor Product of Graphs

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Abstract. Let $G$ and $H$ be graphs. The tensor product $G \otimes H$ of $G$ and $H$ has vertex set $V(G \otimes H) = V(G) \times V(H)$ and edge set $E(G \otimes H) = \{(a,b)(c,d)|ac \in E(G) \ \ and \ \ bd \in E(H)\}$. In this paper, some results on this product are obtained by which it is possible to compute the Wiener and Hyper Wiener indices of $K_n \otimes G$.

**Keywords:** Tensor product, connected graph, bipartite graph, Wiener index, hyper Wiener index.

**2000 Mathematics subject classification:** 05C35, 05C12, 05A20, 05C05.

## 1. Introduction

A graph $G$ consists of a set of vertices $V(G)$ and a set of edges $E(G)$. For every vertices $a, b \in V(G)$, the edge connecting $a$ and $b$ is denoted by $ab$. The distance between two vertices in a connected graph $G$ is the number of edges in a shortest path between them. For vertices $a$ and $b$ of $G$, their distance is shown by $d_G(a,b)$. This concept has been known for a very long time and has received considerable attention as a subject of research in metric graph theory. Graph operations play an important role in the study of graph decompositions into isomorphic subgraphs. For more details about graph operations see [1, 4, 6, 7, 8, 9, 10, 11, 15]. For any two simple graphs $G$ and $H$, the tensor product of $G$ and $H$ has vertex set $V(G \otimes H) = V(G) \times V(H)$ and edge set $E(G \otimes H) = \{(a,b)(c,d)|ac \in E(G) \ \ and \ \ bd \in E(H)\}$. We refer the reader to

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[2] for the proof of this fact that $|E(G \otimes H)| = 2|E(G)||E(H)|$. The distance between two vertices under tensor product of graphs is studied in [13]. Here we obtain useful equality for distance of vertices in tensor product of graphs by a new method and using simple concepts. Then we use these results for obtaining simple proofs for some classical theorems [4]. One of the concepts related to distance in graphs is the Wiener index [14]. It is not only an early index which correlates well with many physico-chemical properties of organic compounds but also a subject that has been studied by many mathematicians and chemists. The Wiener index is the sum of distances between all vertex pairs in a connected graph, $W(G) = \sum_{\{a,b\} \subseteq V(G)} d_G(a,b)$. The hyper-Wiener index of acyclic graphs was introduced by Milan Randic in 1993. Then Klein et al. [12], generalized Randics definition for all connected graphs, as a generalization of the Wiener index. It is defined as $WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\sum_{\{a,b\} \subseteq V(G)} d^2(a,b)$. In last section of this paper, formula for the Wiener and hyper-Wiener indices of the tensor product of complete graph $K_n$ by a graph $G$ are obtained. For more details about Wiener and hyper-Wiener index see for example [3, 5].

## 2. Main Result and Discussion

In this section some new concepts are presented by which it is possible to find new proof for some classical results.

**Definition 1.** Let $G$ be a graph and $x, y \in V(G)$ . Define $d'_G(x,y)$ as follows:
(i) If $d_G(x,y)$ is odd then $d'_G(x,y)$ is defined as the length of a shortest even walk joining $x$ and $y$ in $G$, and if there is no shortest even walk then $d'_G(x,y) = +\infty$ .
(ii) If $d_G(x,y)$ is even then $d'_G(x,y)$ is defined as the length of a shortest odd walk joining $x$ and $y$ in $G$, and if there is no shortest odd walk then $d'_G(x,y) = +\infty$.
(iii) If $d_G(x,y) = +\infty$ , then $d'_G(x,y) = +\infty$.

*Example* 1. Let $K_n$ be a complete graph with $n \geq 3$. For each $a, b \in V(K_n)$, if $a \neq b$ then $d_{K_n}(a,b) = 1$ and $d'_{K_n}(a,b) = 2$ and if $a = b$ then $d_{K_n}(a,b) = 0$ and $d'_{K_n}(a,b) = 3$. Also let $C_{2n+1}$ be a cycle of order $2n+1$, then for each two vertices $a, b \in V(C_{2n+1})$, it is easy to see that, $d'_{C_{2n+1}}(a,b) = 2n+1 - d_{C_{2n+1}}(a,b)$. In every even cycle $C_{2n}$, $d'_{C_{2n}}(a,b) = +\infty$, where $a, b \in V(C_{2n})$ .

**Proposition 1.** *Let $G$ be a graph. Then $G$ is not bipartite graph if and only if there exists $x, y \in V(G)$ such that $d'_G(x,y) < +\infty$.*

*Proof.* Let $G$ is not bipartite, then there is an odd cycle $C$ in $G$. Suppose $x$ and $y$ are two vertices on $C$. If $d_G(x,y)$ is even (odd) then there exists an odd (even) path between $x$ and $y$ in $C$, so $d'_G(x,y) < +\infty$. Conversely, let $x, y \in V(G)$ and $d'_G(x,y) < +\infty$. . Therefore is odd and so there exists an odd closed walk contains $x$. Thus there is an odd cycle in $G$ and this proves $G$ is non-bipartite. □

**Note.** If $G$ is a connected and non-bipartite graph, then $d'_G(x,y) < +\infty$, for all $x, y \in V(G)$.

**Lemma 1.** *Let $G$ and $H$ be graphs and $(a,b)(c,d) \in V(G \otimes H)$. Then*

$$Max\{d_G(a,c), d_H(b,d)\} \leq d_{G\otimes H}\big((a,b),(c,d)\big). \tag{2.1}$$

*Proof.* If $d_{G\otimes H}\big((a,b),(c,d)\big) = +\infty$, then inequality (2.1) holds. Suppose that $d_{G\otimes H}\big((a,b),(c,d)\big) = n$. So the following path between $(a,b)$ and $(c,d)$ in $G \otimes H$ exists:

$$P: \ (a,b) = (x_0,y_0),(x_1,y_1),...,(x_n,y_n) = (c,d).$$

Obviously $a = x_0, x_1, ..., x_n = c$ is a walk with length $n$ between $a$ and $c$ in $G$ and $b = y_1, y_2, ...y_n = d$ is a walk with length $n$ between $b$ and $d$ in $H$. Hence $d_G(a,c), d_H(b,d) \leq n$ and then,

$$Max\{d_G(a,c), d_H(b,d)\} \leq n = d_{G\otimes H}\big((a,b),(c,d)\big).$$

□

**Definition 2.** Let $G$ and $H$ be two graphs and $(a,b),(c,d) \in V(G \otimes H)$. The relation $\sim$ on the vertices of $G \otimes H$ is defined as follows:
$(a,b) \sim (c,d)$ if and only if $d_G(a,c), d_H(b,d) < +\infty$ and $d_G(a,c) + d_H(b,d)$ is even.

**Proposition 2.** *Let $G$ and $H$ be graphs and $(a,b),(c,d) \in V(G \otimes H)$ . If $(a,b) \sim (c,d)$, then $(a,b)$ and $(c,d)$ are in the same component of $G \otimes H$. Moreover*

$$Max\{d_G(a,c), d_H(b,d)\} = d_{G\otimes H}\big((a,b),(c,d)\big).$$

*Proof.* Suppose $d_G(a,c) = m$, $d_H(b,d) = n$ and $m \leq n$. The following paths between $a$ and $c$ in $G$ and $b$ and $d$ in $H$ exist:

$$P_1: \ a = x_0, x_1, ..., x_m = c \ \ and \ \ P_2: \ b = y_1, y_2, ..., y_n = d.$$

Since $(a,b) \sim (c,d)$, then $m+n$ and $n-m$ are even. Therefore there exists the following path between $(a,b)$ and $(c,d)$ in $G \otimes H$, with length $n$:

$$\begin{aligned} P:(a,b) \ &= \ (x_0,y_0),(x_1,y_1),...,(x_m,y_m) = (c,y_m), \\ &\quad\ (x_{m-1},y_{m+1}),(x_m,y_{m+2}),...,(x_m,y_n) = (c,d). \end{aligned}$$

So $d_{G\otimes H}\big((a,b),(c,d)\big) \leq n = d_H(b,d) = Max\{d_G(a,c), d_H(b,d)\}$ . Similarly, if $m > n$, then $d_{G\otimes H}\big((a,b),(c,d)\big) \leq n = d_G(a,c) = Max\{d_G(a,c), d_H(b,d)\}$. By using Lemma 1,$Max\{d_G(a,c), d_H(b,d)\} \leq d_{G\otimes H}\big((a,b),(c,d)\big)$, and this completes the proof. □

**Proposition 3.** *Let $G$ and $H$ be graphs and $(a,b),(c,d) \in V(G \otimes H)$ . If $(a,b) \nsim (c,d)$ then,*

$$d_{G\otimes H}\big((a,b),(c,d)\big) = Min\Big\{Max\{d_G(a,c), d'_H(b,d)\}, Max\{d'_G(a,c), d_H(b,d)\}\Big\}.$$

*Proof.* At first we show that the following inequality is holds:

$$d_{G\otimes H}\big((a,b),(c,d)\big) \le Min\Big\{ \quad Max\{d_G(a,c), d'_H(b,d)\}, \quad Max\{d'_G(a,c), d_H(b,d)\} \quad \Big\}. \tag{2.2}$$

If $d_G(a,c) = +\infty$ ($d_H(b,d) = +\infty$), then $d'_G(a,c) = +\infty$ ($d'_G(a,c) = +\infty$), then the above inequality is satisfied.
We now assume that $d_G(a,c), d_H(b,d) < +\infty$. It is enough to show the following inequalities are hold:

$$d_{G\otimes H}\big((a,b),(c,d)\big) \le Max\{d_G(a,c), d'_H(b,d)\}, \tag{2.3}$$

$$d_{G\otimes H}\big((a,b),(c,d)\big) \le Max\{d'_G(a,c), d_H(b,d)\}. \tag{2.4}$$

If $d'_H(b,d) = +\infty$ then inequality (2.3) holds. Suppose that $d'_H(b,d) = n$ and $d_G(a,c) = m$ and $m \le n$. There exist the shortest path $P: a = x_0, x_1, ..., x_m = c$ in $G$ and the walk $W: \ b = y_0, y_1, ..., y_n = d$ in $H$. Since $(a,b) \nsim (c,d)$ then $n+m$ and $n-m$ are even. Therefore there exist the path $P'$ between $(a,b)$ and $(c,d)$ in as follows:

$$\begin{aligned} P' : (a,b) \quad &= \quad (x_0,y_0),(x_1,y_1),...,(x_m,y_m) = (c,y_m), \\ &\quad\ (x_{m-1},y_{m+1}),(x_m,y_{m+2}),...,(x_m,y_n) = (c,d). \end{aligned}$$

Hence, $d_{G\otimes H} \le n = d'_H(b,d) = Max\{d_G(a,c), d'_H(b,d)\}$. Similarly, if $n < m$ then the inequality (2.3) is satisfied. By a similar argument, one can prove the inequality (2.4). By above argument we have shown inequality (2.2) is satisfied. We now prove the following:

$$d_{G\otimes H}\big((a,b),(c,d)\big) \ge Min\Big\{ \quad Max\{d_G(a,c), d'_H(b,d)\}, \quad Max\{d'_G(a,c), d_H(b,d)\} \quad \Big\}. \tag{2.5}$$

If $(a,b)$ and $(c,d)$ are not in the same component of tensor product of $G$ and $H$ then, $d_{G\otimes H}\big((a,b),(c,d)\big) = +\infty$ and (2.5) is satisfied. Suppose $(a,b)$ and $(c,d)$ are in the same component. Then $a$ and $c$ ( $b$ and $d$) are in the same component of $G$ ($H$). If $d_{G\otimes H}\big((a,b),(c,d)\big) = n$ then the path $P''$ exists, $P'': \ (a,b) = (x_0,y_0),(x_1,y_1),...,(x_n,y_n) = (c,d)$ . Since $(a,b) \nsim (c,d)$ then $d_G(a,c) + d_H(b,d)$ is odd. We can suppose that $d_G(a,c)$ is odd and $d_H(b,d)$ is even. Our main proof consider the following cases:

*Case 1*: If $n$ is odd, then the walk $W_1: \ b = y_0, y_1, ..., y_n = d$ is an odd walk with length $n$ between $b$ and $d$ in $H$. So by definition of $d'_H(b,d)$, it is clear that $d'_H(b,d) \leq n$. It is easy to see $d_G(a,c) \leq n$. Therefore $Max\{d_G(a,c), d'_H(b,d)\} \leq n$.

*Case 2*: If $n$ is even, then by a similar method we can see

$$Max\{d'_G(a,c), d_H(b,d)\} \leq n.$$

Hence

$$Min\big\{Max\{d_G(a,c), d'_H(b,d)\}, Max\{d'_G(a,c), d_H(b,d)\}\big\} \leq d_{G\otimes H}\big((a,b),(c,d)\big).$$

and the proof is complete. □

**Corollary 1.** *Let $G$ and $H$ be graphs and $(a,b),(c,d) \in V(G\otimes H)$. Then*

$$d_{G\otimes H}\big((a,b),(c,d)\big) = \begin{cases} d_1\big((a,b),(c,d)\big) & (a,b) \sim (c,d) \\ d_1\big((a,b),(c,d)\big) & (a,b) \nsim (c,d) \end{cases}$$

*where,*

$$d_1\big((a,b),(c,d)\big) = Max\{d_G(a,c), d_H(b,d)\}$$

*and*

$$d_2\big((a,b),(c,d)\big) = Min\big\{Max\{d_G(a,c), d'_H(b,d)\}, Max\{d'_G(a,c), d_H(b,d)\}\big\}.$$

## 3. Application

In this section at first we present new simple proofs for some classical theorems related to the tensor product of graphs and then as an application of previous results, we compute the Wiener and Hyper Wiener indices of the tensor product of $K_n$ and $G$, where $K_n$ is the complete graph of order $n$ and $G$ is a connected graph.

**Proposition 4.** *([4]) Let $G$ and $H$ be connected graphs with at least two vertices. Then $G\otimes H$ is connected if and only if $G$ or $H$ is not bipartite.*

If $G$ or $H$ are not bipartite then by Proposition 4, $G\otimes H$ is connected and has just one component. Now suppose $G$ and $H$ are bipartite. If $|V(G)| = |V(H)| = 2$, then $G\otimes H$ also is connected and has just one component. Let $|V(G)| > 2$ or $|V(H)| > 2$. Since $G$ is bipartite, there exists the path $P : a_1, a_2, a_3$ in $G$. Obviously $d_G(a_1,a_3) = 2$. Let $b_1$ and $b_2$ are adjacent vertices in $H$. Since $(a_1,b_1) \nsim (a_2,b_1)$ , then by using Proposition 3 or Corollary 1, $d_{G\otimes H}\big((a_1,b_1),(a_2,b_1)\big) = +\infty$ . Then $(a_1,b_1)$ and $(a_2,b_1)$ are in different components. For each $(a,b) \in V(G\otimes H)$, two cases are considered:

*Case 1*: If $d_G(a,a_1)+d_H(b,b_1)$ is even, then $(a,b) \sim (a_1,b_1)$ and by Corollary 1, $(a,b)$ and $(a_1,b_1)$ are in the same component.

*Case 2*: If $d_G(a,a_2)+d_H(b,b_1)$ is even, then $(a,b) \sim (a_2,b_1)$ and by Corollary 1, $(a,b)$ and $(a_2,b_1)$ are in the same component. Notice that if $d_G(a,a_1) + d_H(b,b_1)$ and $d_G(a,a_2)+d_H(b,b_1)$ are odd, then both of $d_G(a,a_1)$ and $d_G(a,a_2)$ are even or odd. So $d_G(a,a_1)+d_G(a,a_2)$ is even, and hence there exists an even walk between $a_1$ and $a_2$. So there exists an odd closed walk in $G$ containing

which is a contradiction.
Therefore we can conclude the following well known result.

**Proposition 5.** *([4]) Suppose $G$ and $H$ are connected graphs with at least two vertices, then $G \otimes H$ has at most two components.*

In the following propositions the Wiener and Hyper Wiener indices of the tensor product of $K_n$ and a connected graph $G$ are computed.

**Proposition 6.** *Let $G$ be connected graph of order $m$. Then the Wiener index of the tensor product of $K_n$ and $G$ is given by:*

$$W(K_n \otimes G) = n^2 W(G) + 2\binom{n}{2}|V(G)| + 2n|E(G)| - n|T_G|,$$

*where $n \geq 3$ and $T_G = \{ab \in E(G) | ab \text{ is contained in a triangle}\}$ .*

*Proof.* By definition of the Wiener index,

$$W(K_n \otimes G) = \sum_{\{(a,b),(c,d)\} \subseteq V(K_n \otimes G)} d_{K_n \otimes G}\big((a,b),(c,d)\big)$$

We consider a partition of $V(K_n \otimes G)$ into the following parts:

$$A_1 = \big\{\{(a,b),(c,d)\} | a \neq c, b \neq d, (a,b) \sim (c,d)\big\},$$
$$A_2 = \big\{\{(a,b),(c,d)\} | a \neq c, b \neq d, (a,b) \nsim (c,d)\big\},$$
$$A_3 = \big\{\{(a,b),(c,d)\} | a \neq c, b = d\big\},$$
$$A_4 = \big\{\{(a,b),(c,d)\} | a = c, b \neq d, (a,b) \sim (c,d)\big\},$$
$$A_5 = \big\{\{(a,b),(c,d)\} | a = c, b \neq d, (a,b) \nsim (c,d)\big\}.$$

Therefore,

$$W(K_n \otimes G) = \sum_{\{(a,b),(c,d)\} \in A_1} d_{K_n \otimes G}\big((a,b),(c,d)\big) + \sum_{\{(a,b),(c,d)\} \in A_2} d_{K_n \otimes G}\big((a,b),(c,d)\big) + \sum_{\{(a,b),(c,d)\} \in A_3} d_{K_n \otimes G}\big((a,b),(c,d)\big) + \sum_{\{(a,b),(c,d)\} \in A_4} d_{K_n \otimes G}\big((a,b),(c,d)\big) + \sum_{\{(a,b),(c,d)\} \in A_5} d_{K_n \otimes G}\big((a,b),(c,d)\big).$$

We evaluate each summation separately. It is obvious that if $\{a,c\} \subseteq V(K_n)$, then $d_{K_n}(a,c) = 1$ and $d'_{K_n}(a,c) = 2$. Apply Proposition 3, if $(a,b) \sim (c,d)$ and $a \neq c, b \neq d$ then,

$$Min\big\{Max\{d_{K_n}(a,c), d'_G(b,d)\}, Max\{d'_{K_n}(a,c), d_G(b,d)\}\big\} = d_G(b,d)$$

and

$$Max\{d_{K_n}(a,c), d_G(b,d)\} = d_G(b,d).$$

Therefore,

$$\begin{aligned}\sum_{\{(a,b),(c,d)\}\in A_1\cup A_2} d_{K_n\otimes G}\big((a,b),(c,d)\big) &= \sum_{\{(a,b),(c,d)\}\in A_1} d_{K_n\otimes G}\big((a,b),(c,d)\big)\\ &+ \sum_{\{(a,b),(c,d)\}\in A_2} d_{K_n\otimes G}\big((a,b),(c,d)\big)\\ &= 2\sum_{\{a,c\}\subseteq V(K_n)}\ \sum_{\{b,d\}\subseteq V(G)} d_G(b,d)\\ &= 2\binom{n}{2}W(G).\end{aligned}$$

By considering the set $A_3$, we have:

$$\sum_{\{(a,b),(c,d)\}\in A_3} d_{K_n\otimes G}\big((a,b),(c,d)\big) = \sum_{\substack{\{a,c\}\subseteq V(K_n)\\ b\in V(G)}} 2 = 2|V(G)|\binom{n}{2}.$$

For computing our summation over the set $A_4$, we notice that,

$$A_4 = \Big\{\{(a,b)(a,d)\}|a\in V(K_n),\{b,d\}\subseteq V(G)\ and\ \ 2|d_G(b,d)\ \Big\}.$$

Hence,

$$\begin{aligned}\sum_{\{(a,b),(c,d)\}\in A_4} d_{K_n\otimes G}\big((a,b),(c,d)\big) &= \sum_{\substack{\{b,d\}\subseteq V(G)\\ a\in V(K_n)\\ 2|d_G(b,d)}} d_{K_n\otimes G}\big((a,b),(c,d)\big)\\ &= \sum_{\substack{\{b,d\}\subseteq V(G)\\ a\in V(K_n)\\ 2|d_G(b,d)}} d_G(b,d) = n\sum_{\substack{\{b,d\}\subseteq V(G)\\ 2|d_G(b,d)}} d_G(b,d).\end{aligned}$$

Compute our summation over $A_5$, we have:

$$\sum_{\{(a,b),(c,d)\}\in A_5} d_{K_n\otimes G}\big((a,b),(c,d)\big) = \sum_{\substack{\{b,d\}\subseteq V(G)\\ a\in V(K_n)\\ 2\nmid d_G(b,d)}} d_{K_n\otimes G}\big((a,b),(c,d)\big).$$

If $d_G(b,d)$ is odd then by Proposition 3,

$$\begin{aligned}d_{K_n\otimes G}\big((a,b),(c,d)\big) &= Min\Big\{Max\{d_{K_n}(a,a),d'_G(b,d)\},\\ &\quad Max\{d'_{K_n}(a,a),d_G(b,d)\}\Big\}\\ &= Min\Big\{d'_G(b,d),Max\{3,d_G(b,d)\}\Big\}.\end{aligned}$$

By a case by case calculations, one can see that

$$Min\Big\{d'_G(b,d),Max\{3,d_G(b,d)\}\Big\} = \begin{cases} d_G(b,d) & d_G(b,d)\geq 3\\ 2 & d_G(b,d)=1\ and\ d'_G(b,d)=2\\ 3 & d_G(b,d)=1\ and\ d'_G(b,d)\geq 4\end{cases}$$

Define:

$$A_5' = \Big\{\{(a,b),(c,d)\}|a \in V(K_n), d_G(b,d) \geq 3 \; and \; d_G(b,d) \; is \; odd\Big\},$$

$$A_5'' = \Big\{\{(a,b),(c,d)\}|a \in V(K_n), d_G(b,d) = 1 \; and \; d_G'(b,d) = 2\Big\},$$

$$A_5''' = \Big\{\{(a,b),(c,d)\}|a \in V(K_n), d_G(b,d) = 1 \; and \; d_G'(b,d) \geq \Big\}.$$

Such that $A_5 = A_5' \cup A_5'' \cup A_5'''$. Hence,

$$\sum_{\{(a,b),(c,d)\}\in A_5} d_{K_n\otimes G}\big((a,b),(c,d)\big) = \sum_{\{(a,b),(c,d)\}\in A_5'} d_{K_n\otimes G}\big((a,b),(c,d)\big) +$$

$$\sum_{\{(a,b),(c,d)\}\in A_5''} d_{K_n\otimes G}\big((a,b),(c,d)\big) + \sum_{\{(a,b),(c,d)\}\in A_5'''} d_{K_n\otimes G}\big((a,b),(c,d)\big) =$$

$$\sum_{\{(a,b),(c,d)\}\in A_5'} d_G(b,d) + \sum_{\{(a,b),(c,d)\}\in A_5''} 2 + \sum_{\{(a,b),(c,d)\}\in A_5'''} 3 =$$

$$n\Big(\sum_{\substack{\{b,d\}\subseteq V(G)\\ 2\nmid d_G(b,d)}} d_G(b,d) - |E(G)|\Big) + 2n|T_G| + 3n(|E(G)| - |T_G|).$$

By the above calculations, one can see that,

$$W(K_n \otimes G) = n^2 W(G) + 2\binom{n}{2}|V(G)| + 2n|E(G)| - n|T_G|.$$

□

**Corollary 2.** *The following statements are hold:*

*a) Let $G$ be a triangle-free graph then,*

$$W(K_n \otimes G) = n^2 W(G) + 2\binom{n}{2}|V(G)| + 2n|E(G)|.$$

*b) The Wiener index of two complete graphs of orders $m$ and $n$ is computed as follows:*

$$W(K_n \otimes K_m) = n(n+1)\binom{m}{2} + n(n-1)m.$$

*Proof.* a) It is immediate by Proposition 6.

b) In complete graph $K_n$ , we can see that $T_{K_m} = E(K_m)$ and so by Proposition 6,

$$\begin{aligned} W(K_n \otimes K_m) &= n^2 W(K_m) + 2\binom{n}{2} m + 2n\binom{m}{2} - n\binom{m}{2} \\ &= n(n+1)\binom{m}{2} + n(n-1)m. \end{aligned}$$

□

**Proposition 7.** *Let $G$ be connected graph of order $m$. Then the hyper Wiener index of the tensor product of $K_n$ and $G$ is given by:*

$$WW(K_n \otimes G) = \binom{n+1}{2} WW(G) + 3\binom{n}{2}|V(G)| + 5n|E(G)| - 3n|T_G|.$$

*Proof.* The proof is similar to the proof of Proposition 6. □

**Corollary 3.** *The hyper Wiener index of two complete graphs of orders m and n is computed as follows:*

$$WW(K_n \otimes K_m) = \binom{n+1}{2} m(m-1) + 3\binom{n}{2} m + 5n\frac{m(m-1)}{2} - 3nm(m-1).$$

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