

## New results on $p$ -Carleson measures and some related measures in the unit disk

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ABSTRACT. We provide some new sharp embeddings for  $p$ -Carleson measures and some related measures in the unit disk of the complex plane.

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### 1. INTRODUCTION

The purpose of this paper is to continue the study of  $p$ -Carleson and related measures in the unit disk. In recent years many papers appeared where  $p$ -Carleson measures were studied intensively from various point of view by many authors (see [9], [10] and references therein). We intend to study again  $p$ -Carleson measures in complex plane in simplest case, namely in the unit disk. Our intention in particular is to provide some direct generalizations of known one dimensional results about  $p$ -Carleson measures (or  $Q_p$  analytic classes). On the other hand, the intention of this note is to provide sharp embeddings for certain new analytic spaces in the unit disk. This paper can be viewed as a continuation of [7] where similar assertions were recently proved.

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Throughout the paper, we write  $C$  (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed. Given two expressions  $A$  and  $B$ , we shall write  $A \asymp B$  if there is a positive constant  $C$  such that  $\frac{B}{C} \leq A \leq CB$ . We will write  $A <_{\prec} B$  if there is a constant  $C$  such that  $A < CB$ .

In the next section we provide some new sharp embedding theorems for so called  $p$ -Carleson measures and related measures in the unit disk which are closely connected with  $Q_p$  spaces in the unit disk (see [9], [10]).

Let  $\mathbb{D}$  be, as usual, the unit disk in the complex plane, and let  $\mathbb{T}$  be the unit circle  $\{\xi : |\xi| = 1\}$ . By  $dm_2$  we mean the standard normalized Lebesgue area measure on  $\mathbb{D}$ . We denote the Lebesgue arc length measure on the unit circle by  $d\xi$  or  $dm(\xi)$ . A positive Borel measure on  $\mathbb{D}$  will be denoted by  $\mu$ . Let further  $H(\mathbb{D})$  denote the space of holomorphic functions in the unit disk. Moreover

$$\square I = \{z = re^{i\varphi} : 1 - |I| < r < 1, \xi = e^{i\varphi} \in I\}$$

is a usual classical Carleson box (see [7]) in  $\mathbb{D}$  where  $I$  is an arc on  $\mathbb{T}$ . Let also  $L^{p,q}(\mathbb{T})$ ,  $0 < p < \infty, 0 < q < \infty$  be the classical Lorentz space on  $\mathbb{T}$  (for definition see [1]) and

$$H^{q,\infty}(\mathbb{T}) = H(\mathbb{D}) \cap L^{q,\infty}(\mathbb{T}).$$

By  $\Gamma_\sigma(\xi)$  we mean the Lusin cone (see [3])

$$\Gamma_\sigma(\xi) = \{z \in \mathbb{D} : |1 - \bar{\xi}z| < \sigma(1 - |z|)\}, \quad \sigma > 1, \xi \in \mathbb{T}.$$

Let

$$\Delta_z = \{z' = r'e^{i\theta'} : |\theta - \theta'| < 1 - r, \frac{1-r}{2} < 1 - r' < 2(1-r)\}, \quad z = re^{i\theta}.$$

For a fixed  $a \in \mathbb{D}$ , we let  $\varphi_a$  be a Möbius transformation on  $\mathbb{D}$  interchanging zero and  $a$  (that is,  $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ ). The Bergman distance in the unit disk which is Möbius invariant is defined by

$$d(z, w) = \frac{1}{2} \log \left( \frac{1 + |\varphi_w(z)|}{1 - |\varphi_w(z)|} \right), \quad z, w \in \mathbb{D}.$$

Given a fixed  $t > 0$  and  $a \in \mathbb{D}$ , the Bergman disk of radius  $t$  and center  $a$  is defined to be (see [11]):

$$D(a, t) = \{w \in \mathbb{D} : d(w, a) < t\}.$$

It is well known that the Bergman disk  $D(a, t)$  is a Euclidean disk with center  $\tilde{a} = \frac{1-r^2}{1-|a|^2r^2}a$  and the radius  $R = \frac{1-|a|^2}{1-|a|^2r^2}r$  where  $r = \frac{e^{2t}-1}{e^{2t}+1}$ ; (see [11]). Note that

$$m_\alpha(D(a, t)) \asymp (1 - |a|)^{2+\alpha}$$

and  $dm_\alpha(z) = (1 - |z|)^\alpha dm_2(z)$  for  $\alpha > -1$  (for details, see [11]). We also mention that for  $a \in \mathbb{D}$  and  $z \in D(a, t)$  we have

$$|1 - \bar{a}z| \asymp (1 - |z|) \asymp (1 - |a|).$$

It is well known that for every  $\delta > 0$ , there exists a distinct sequence  $\{z_j\}$  in  $\mathbb{D}$  called a  $\delta$ -lattice, such that  $d(z_j, z_k) > \delta/5$  if  $j \neq k$ , and

$$\cup_j D(z_j, \delta) = \mathbb{D}; \quad \sum_j \chi_{D(z_j, 5\delta)}(z) \leq L, \quad z \in \mathbb{D},$$

where  $L$  is a constant,  $\chi_E(z)$  is the characteristic function of a set  $E$  (see [11]).

## 2. MAIN RESULTS

We now provide some new embedding theorems for  $p$ -Carleson measures in the unit disk. In all our results,  $\mu$  is a positive Borel measure on  $\mathbb{D}$ , and  $f$  is an analytic function in  $\mathbb{D}$ .

**Theorem 1.** Let  $q > 1$ ,  $\alpha > -1$ , and  $f$  be an analytic function in  $\mathbb{D}$ . Then the following conditions are equivalent.

(a)

$$\left\| \int_{\Gamma_\sigma(\xi)} \frac{|f(z)|}{1 - |z|} d\mu(z) \right\|_{L^{q,\infty}(T)} \leq C \left\| \int_{\Gamma_\sigma(\xi)} |f(z)|(1 - |z|)^{\alpha-1} dm_2(z) \right\|_{L^{q,\infty}(T)}.$$

(b)  $\mu(D(z, t)) \leq C(1 - |z|)^{\alpha+2}$ ,  $t > 0$ ,  $z \in \mathbb{D}$ .

(c)  $\mu(\square I) \leq C|I|^{\alpha+2}$ .

*Proof.* We first show that (a) is equivalent to (b). For this purpose we use the following two important known facts from [2] and [5] (page 234):

$$(2.1) \quad A(\mu) = \sup_{I \subset T} \frac{1}{|I|^s} \int_{\square I} \frac{d\mu(z)}{1 - |z|} \leq C \sup_{I \subset T} \frac{1}{|I|^s} \int_I \int_{\Gamma_\sigma(\xi)} \frac{d\mu(z)}{(1 - |z|)^2} = B(\mu)$$

where  $s \in (0, \infty)$  and

$$(2.2) \quad A(\mu) \asymp B(\mu) \quad \text{for } s \in (0, 1),$$

$$(2.3) \quad \left( \|f\|_{L^{q,\infty}(T)}^r \right) = \sup_{I \subset T} \frac{1}{|I|^{1-r/q}} \int_I |f(\xi)|^r dm(\xi), \quad 0 < r < q.$$

As it was shown in [6]

$$(2.4) \quad \frac{1}{(1 - z\xi)^{1/q}} \in H^{q,\infty}(\mathbb{T}), \quad \xi \in \mathbb{T},$$

and we also have (see [11])

$$(2.5) \quad \int_{\Gamma_\sigma(\xi)} \frac{|f(z)d\mu(z)|}{1 - |z|} \leq \sup_{z \in \mathbb{D}} \left( \frac{\mu(D(z, t))}{(1 - |z|)^{\alpha+2}} \right) \left( \int_{\Gamma_\sigma(\xi)} |f(z)|(1 - |z|)^{\alpha-1} dm_2(z) \right).$$

Combining the relations (2.1) to (2.5) provided above, we obtain the desired result. Indeed, one part follows directly from (2.5). For the other part we note that our estimate in Theorem 1 by (2.2) and (2.3) is equivalent to

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/q}} \int_{\square I} |f(z)| d\mu(z) \leq C \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/q}} \int_I \int_{\Gamma_\sigma(\xi)} |f(z)|(1-|z|)^{\alpha-1} dm_2(z).$$

Let

$$f(z) = \frac{(1-|w|)^\beta}{(1-zw)^{\beta+\alpha+1+1/q}}, \quad z \in \mathbb{D},$$

where  $\beta$  is a big enough positive number. Then the left side is bigger than  $\mu(\square I)/|I|^{\alpha+2}$  if  $w$  is a center of  $\square I$ , here  $\square I$  is fixed and the right side is bounded from above by some constant; here we can use the fact that

$$|1-\bar{w}z| \asymp (1-|w|) \asymp |I|, \quad z \in \square I, w \in \mathbb{D},$$

where  $w$  is a center of Carleson box, and

$$(1-|w|)^{-\beta} \leq C|1-\bar{w}z|^{-\beta}, \quad \beta > 0.$$

Other implications are well known; see [9, 10, 11].  $\square$

We list two assertions concerning the actions of Lusin area integral again, but we do not provide complete proofs for them, since the main idea we have in this proofs coincides with the ideas mentioned in the proof of Theorem 1. We have the following two statements.

**Theorem 2.** Let  $\beta > 0, q > 1, t > \beta + 1$ , and  $f$  be an analytic function in  $\mathbb{D}$ . Then

$$\left\| \int_{\Gamma_\alpha(\xi)} \frac{(1-|z|)^t |f(z)| d\mu(z)}{1-|z|} \right\|_{L^{q,\infty}(\mathbb{T})} \leq C \sup_{|z|<1} |f(z)|(1-|z|^\beta)$$

if and only if

$$\left\| \int_{\Gamma_\alpha(\xi)} (1-|z|)^{t-1} \left( \int_{D(z,\tau)} d\mu(w) \right) \frac{(1-|z|)^{-\beta}}{(1-|z|)^2} \right\|_{L^{q,\infty}(\mathbb{T})} \leq C.$$

**Theorem 3.** Let  $1/p = 1/q + 1/r, q, r, p > 1, \beta > 0, t > -1$ . Then for each  $f \in H(\mathbb{D})$  and each positive Borel measure  $\mu$  on  $\mathbb{D}$  we have

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/p}} \int_{\square I} (1-|z|)^t |f(z)| d\mu(z) \leq C \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/q}} \int_I \sup_{z \in \Gamma_\sigma(\xi)} |f(z)|(1-|z|)^\beta dm(\xi)$$

if and only if

$$\sup_{I \subset \mathbb{T}} \left( \int_{\square I} (1-|z|)^t d\mu(z) \right) \frac{1}{|I|^{\beta+1-1/r}} < \infty.$$

We now provide the complete sketch of proofs for this last statements.

**Proof of Theorem 2.** The idea of the proof is similar to the ideas discussed in the proof of Theorem 1. One part of the theorem follows immediately from [11]:

$$\begin{aligned} \int_{\Gamma_\alpha(\xi)} (1 - |z|)^{t-1} |f(z)| d\mu(z) \\ \leq C \int_{\Gamma_\alpha(\xi)} |f(z)| (1 - |z|)^{t-1} \left( \int_{D(z, \tau)} d\mu(w) \right) \frac{dm_\alpha(z)}{(1 - |z|)^{\alpha+2}}, \end{aligned}$$

for  $\alpha > \beta - 1$ ,  $t > \beta + 1$ .

To show the reverse implication, we need to use the estimates (2.1)-(2.4), the test function  $f(z) = \frac{1}{(1-wz)^\beta}$  for a large enough number  $\beta$ , and a fixed complex number  $w$  in the unit disk  $\mathbb{D}$ . To be more precise, we use (2) to show that the required estimate is equivalent to

$$\frac{1}{|I|^{1-1/q}} \int_{\square_I} |f(z)| (1 - |z|)^t d\mu(z) \leq C \sup_{|z| < 1} |f(z)| (1 - |z|)^\beta,$$

and also the facts that (see [11]):

$$\begin{aligned} \frac{C_1}{(1 - |a|)^{\alpha+2}} \int_{D(a, r+s)} d\mu(z) \\ \leq \frac{1}{(1 - |a|)^{\alpha+2}} \int_{D(a, r)} (1 - w)^{-\alpha-2} \int_{D(w, t)} d\mu(z) dm_\alpha(w) \\ \leq \frac{C}{(1 - |a|)^{\alpha+2}} \int_{D(a, r+t)} d\mu(z), \quad t > s, r > 0. \end{aligned}$$

This implies that the condition in the theorem is equivalent to the fact that our measure is a  $\tau$ -Carleson measure for some  $\tau > 0$ . The rest of proof is routine.  $\square$

We now give a sketch for the proof of Theorem 3. First we verify the necessity of the condition

$$\begin{aligned} \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/p}} \int_{\square_I} (1 - |z|)^t |f(z)| d\mu(z) \leq \\ C \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/q}} \int_I \sup_{z \in \Gamma_\sigma(\xi)} |f(z)| (1 - |z|)^\beta dm(\xi). \end{aligned}$$

Since  $(1 - z)^{-1/q} \in H^{q, \infty}$  (see [5]) we can put  $f(z) = \frac{1}{(1-z)^{\beta+1/q}}$ . The right side is bounded. As for the left hand side of the estimate above, we can estimate it from below. Indeed, we have to verify that the following estimate is true. We

have

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/p}} \int_{\square I} \frac{(1-|z|)^t d\mu(z)}{|1-\bar{w}z|^{\beta+1/q}} \geq C \sup_{I \subset \mathbb{T}} \left( \int_{\square I} (1-|z|)^t d\mu(z) \right) \frac{|I|^{-\beta}}{|I|^{1-1/r}}$$

where  $w$  is a center of  $\square I$ , and  $1/p = 1/q + 1/r$ ,  $q, r, p > 1$ ,  $\beta > 0$ ,  $t > -1$ . This is true, since if  $w$  is a center of  $\square I$ , then

$$|1 - \bar{w}z| \asymp (1 - |w|) \asymp |I|, \quad z \in \square I, \quad w \in \mathbb{D}.$$

Hence we obtain

$$(1 - |w|)^{-\beta} \leq C |1 - \bar{w}z|^{-\beta}, \quad \beta > 0.$$

To show the reverse, we have to modify the proof of Theorems 1,2 and use the estimates (2.1)-(2.5). We have

$$(2.6) \quad \int_{\Gamma_\sigma(\xi)} \frac{(1-|z|)^t |f(z)| d\mu(z)}{1-|z|} \leq C \left( \sup_{z \in \Gamma_\sigma(\xi)} |f(z)| (1-|z|)^\beta \right) \times \\ \times \int_{\Gamma_\sigma(\xi)} \frac{(1-|z|)^t}{(1-|z|)^{\alpha+2}} \left( \int_{D(z,r)} d\mu(w) \right) \frac{dm_{\alpha-\beta}(z)}{1-|z|}, \quad \beta > 0, t > 0.$$

Then we use (see [5])

$$\|f_1 f_2\|_{L^{p,\infty}(\mathbb{T})} \leq C \|f_1\|_{L^{q,\infty}(\mathbb{T})} \|f_2\|_{L^{r,\infty}(\mathbb{T})}, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \quad q, r, p > 1.$$

Using (2.2) in (2.6) we will have the estimate

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/p}} \int_{\square I} (1-|z|)^t |f(z)| d\mu(z) \leq C \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/q}} \int_I \sup_{z \in \Gamma_\sigma(\xi)} |f(z)| (1-|z|)^\beta dm(\xi) \times \\ \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/r}} \int_I \int_{\Gamma_\sigma(\xi)} \frac{(1-|z|)^t}{(1-|z|)^{\alpha+2}} \left( \int_{D(z,r)} d\mu(w) \right) \frac{dm_{\alpha-\beta}(z)}{1-|z|} = G(f)M(\mu).$$

It can be easily shown that

$$M(\mu) \leq C \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-1/r}} \int_{\square I} (1-|z|)^{t-\alpha-2} \left( \int_{D(z,r)} d\mu(w) \right) dm_{\alpha-\beta}(z) \\ \leq C \sup_{\tilde{z} \in \mathbb{D}} \frac{1}{(1-|\tilde{z}|)^{1-1/r}} \int_{D(\tilde{z}, r_1)} (1-|z|)^{t-\alpha-2} \left( \int_{D(z,r)} d\mu(w) \right) dm_{\alpha-\beta}(z).$$

Put  $1 - 1/r = \alpha + 2$ , then it follows that  $\alpha = -1 - 1/r$ . Therefore

$$\begin{aligned} M(\mu) &\leq C \sup_{\tilde{z} \in \mathbb{D}} \frac{1}{(1 - |\tilde{z}|)^{\alpha+2}} \int_{D(\tilde{z}, r_1)} \frac{1}{(1 - |z|)^{\alpha+2}} \\ &\quad \times \left( \int_{D(z, r)} (1 - |\tilde{z}|)^{t-\beta} d\mu(\tilde{z}) \right) dm_\alpha(z) \\ &\leq C \sup \left( \int_{D(\tilde{z}, r_1+r)} (1 - |\tilde{z}|)^t d\mu(\tilde{z}_1) \right) \frac{1}{(1 - |\tilde{z}_1|)^{\alpha+2+\beta}} = M_1(\mu). \end{aligned}$$

**Theorem 4.** Let  $0 < p < q$ ,  $\alpha > 0$ , and  $f$  be an analytic function in  $\mathbb{D}$ . Then

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-p/q}} \int_I \int_{\Gamma_\sigma(\xi)} \frac{|f(z)|^p d\mu(z)}{(1 - |z|)} \leq C \left( \int_{\mathbb{D}} |f(z)|^q (1 - |z|)^{\alpha q - 1} dm_2(z) \right)^{1/q}$$

if and only if

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_{\square_I} \left( \frac{\mu(\Delta_z)}{(1 - |z|)^{\alpha p + 1}} \right)^{\frac{q}{q-p}} \frac{dm_2(z)}{1 - |z|} \leq C.$$

Ideas for the proof of this theorem are the same as those used in the proof of Theorem 2, so that we omit the details.

We remark that the conditions on measures appeared in the formulations of Theorems 2 and 3 can be reformulated in terms of some conditions on  $D(z, t)$ , as we have done in Theorem 1.

**Remark.** Some results of this paper can be generalized to higher dimensions by using the so called diagonal map technique (see [8]). We note in addition that the estimates obtained here are not sharp.

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