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Uniform Boundedness Principle for Operators on Hypervector Spaces

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Abstract. The aim of this paper is to prove the Uniform Boundedness Principle and Banach-Steinhaus Theorem for anti linear operators and hence strong linear operators on Banach hypervector spaces. Also we prove the continuity of the product operation in such spaces.

Keywords: Hypervector space, Normed hypervector space, Operator.

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1. Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions has been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. A wealth of applications of this concepts are given in $[1, 2, 4, 12 - 14]$.

In 1988 the concept of hypervector space was first introduced by Tallini. She studied more properties of this new structure in [6]. We considered the

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generalization of a vector space in the viewpoint of analysis and proved some important results in this field. See $[7-11]$. This paper is arranged as follows. In section 2 we define the hypervector spaces, norm and different types of operators in such spaces and give some examples. In section 3 we prove the Uniform Boundedness Principle and Banach-Steinhaus Theorem for anti linear operators and hence strong linear operators on Banach hypervector spaces. Also we show the continuity of the product operation in these spaces.

We denote the set of all complex numbers by $\mathbb{C}$ and real numbers by $\mathbb{R}$. Also in this note the field F is either $\mathbb{C}$ or $\mathbb{R}$.

2. Preliminaries

Definition 2.1. ([6]) Let $(X, +)$ be an abelian group and F be a field. Then a hypervector space is a quadruple $(X,+,o,F)$ where o is a mapping:

$$o : F \times X \longrightarrow P_*(X)$$

such that the following conditions are satisfied:

(1) $\forall a \in F, \forall x, y \in X,\ ao(x + y) \subseteq aox + aoy$ (right distributivity),

(2) $\forall a, b \in F, \forall x \in X,\ (a + b)ox \subseteq aox + box$ (left distributivity),

(3) $\forall a, b \in F, \forall x \in X,\ ao(box) = (ab)ox$ (associativity),

(4) $\forall a \in F, \forall x \in X,\ ao(-x) = (-a)ox$,

(5) $\forall x \in X,\ x \in 1ox$.

Note that the set $ao(box)$ in (3) is of the form $\cup_{y\in box} aoy$.

Example 2.2. Suppose z and a are two nonzeros arbitrary elements of $\mathbb{C}$ and $\mathbb{R}$, respectively. $\mathbb{C}$ with the usual sum and the following product is a weak hypervector space on $\mathbb{R}$:

$$aoz = \{re^{i\theta}; 0 < r \leq| a || z |, \theta = arg(z)\}.$$

If $a = 0$ or $z = 0$, then we define $aoz = 0$.

Example 2.3. Suppose z and a are arbitrary elements of $\mathbb{C}$ and $\mathbb{R}$, respectively. $\mathbb{C}$ with the usual sum and the following product is a weak hypervector space on $\mathbb{R}$:

$$a.z = \{re^{i\theta}; 0 \leq r \leq| a || z |, 0 \leq \theta \leq 2\pi\}.$$

Definition 2.4. ([6]) Let $(X, +, o, F)$ be a hypervector space over a field F. We define a pseudonorm in X as being a mapping $\| \,.\, \|: X \longrightarrow R$, of X into the real numbers such that:

(i) $\| 0 \|= 0$,
(ii) $\forall x, y \in X,\ \| x + y \|\leq\| x \| + \| y \|$,
(iii) $\forall a \in F,\ \forall x \in X,\ \sup \| aox \|=| a ||| x \|$.

A pseudonorm in X is called norm if:
(iv) $\| x \| = 0 \ \Leftrightarrow \ x = 0$.

Definition 2.5. Let X and Y be hypervector spaces over F. A map $T : X \longrightarrow Y$ is called
(i) linear if and only if
$T(x+y) = T(x) + T(y),\ T(aox) \subseteq aoT(x),\ \forall x, y \in X, a \in F$
(ii) anti linear if and only if
$T(x+y) = T(x) + T(y),\ T(aox) \supseteq aoT(x),\ \forall x, y \in X, a \in F,$
(iii) strong linear if and only if
$T(x+y) = T(x) + T(y),\ T(aox) = aoT(x),\ \forall x, y \in X, a \in F.$

Example 2.6. Let T be a map on hypervector space $\mathbb{C}$ (that was defined in example 2.3) into $\mathbb{C}$ (that was defined in example 2.2) over $\mathbb{R}$ and defined by $x \mapsto x$. We see that T is an anti linear operator, because in this space for any $a \in \mathbb{R}$ and $x \in \mathbb{C}$ we have

$$\begin{aligned} T(aox) &= \{Ty; y \in a.x\} \\ &= \{y; y \in a.x\} \\ &= \{re^{i\theta}; 0 < r \leq | a || z |, 0 \leq \theta \leq 2\pi\} \end{aligned}$$

and

$$aoTx = a.x = \{re^{i\theta}; 0 \leq r \leq | a || z |, \theta = arg(z)\}.$$

So $T(a.x) \supseteq aoTx$ and hence T is anti linear.

3. Main results

Lemma 3.1. ([7]) *If X is a weak hypervector space over F, $0 \neq a \in F$ and $x \in X$, then there exists a z in aox such that we have $x \in a^{-1}oz$.*

Note that if X is a normed weak hypervector space, then it is easy to check that $\| z \| = | a | \| x \|$.

Definition 3.2. A Banach hypervector space is a complete normed hypervector space in the metric defined by its norm.

Theorem 3.3. *Let A be a set of bounded anti linear operators on a Banach hypervector space X into a normed hypervector space Y, such that $\{\| Tx \| ; T \in A\}$ is bounded for every $x \in X$, say,*

$$\| Tx \| \leq c_x, \quad \forall T \in A,$$

where c_x is a real number. Then the set of the norms $\{\| T \|; T \in A\}$ is bounded, that is, there is a c such that

$$\| T \| \leq c, \quad \forall T \in A.$$

Proof. For every $k \in \mathbb{N}$, let $A_k \subseteq X$ be defined by the following form

$$A_k = \{x \in X; \| Tx \| \leq k, \ \forall T \in A\}.$$

A_k is closed. Indeed, for any $x \in \overline{A}_k$ there is a sequence $\{x_j\}$ in A_k converging to x. This means that for every fixed T we have $\| Tx_j \| \leq k$ and obtain $\| Tx \| \leq k$, because T is continuous and so is the norm. Hence $x \in A_k$, and A_k is closed.

By assumption, each $x \in X$ belongs to some A_k. Hence

$$X = \bigcup_{k=1}^{\infty} A_k.$$

Since X is complete, Baire's Theorem implies that some A_k contains an open ball, say,

$$B_0 = B(x_0, r) \subset A_{k_0}. \qquad (1)$$

Let $x \in X$ be arbitrary, not zero. We set

$$Z = x_0 + \gamma o x, \qquad (2)$$

where $\gamma = \frac{r}{2\|x\|}$. Then $sup \| Z - x_0 \| = sup \| \gamma o x \| = \frac{r}{2} < r$, so that $Z \subseteq B_0$. By (1) and the definition of A_{k_0} we thus have

$$\| Tz \| \leq k_0, \quad \forall T \in A, \forall z \in Z. \qquad (3)$$

Also since $x_0 \in B_0$

$$\| Tx_0 \| \leq k_0. \qquad (4)$$

On the other hand, by $T(\gamma o x) \supseteq \gamma o Tx$ and (2) we obtain $T(Z - x_0) \supseteq \gamma o Tx$. So $\| T(\gamma o x) \| \supseteq \| \gamma o Tx \|$ and Lemma 3.1 imply that $\gamma \| Tx \| \in \| T(Z - x_0) \|$. Thus there exists a $z_0 \in Z$ such that $\| T(z_0 - x_0) \| = \gamma \| Tx \|$. (3) and (4) yield for all $T \in A$

$$\gamma \| Tx \| = \| T(z_0 - x_0) \| \leq \| Tz_0 \| + \| Tx_0 \| \leq 2k_0,$$

this implies

$$\| Tx \| \leq \frac{4}{r} \| x \| k_0.$$

Hence by Proposition 3.7 in [6] for all $T \in A$,

$$\| T \| = \sup_{\|x\|=1} \| Tx \| \leq \frac{4}{r} k_0$$

which is the assertion with $c = 4k_0/r$. □

By Theorem 3.3 we easily have the following corollary.

Corollary 3.4. *Let A be a set of bounded strong linear operators on a Banach hypervector space X into a normed hypervector space Y such that $\{\| Tx \|; T \in A\}$ is bounded for every $x \in X$. Then the set of the norms $\{\| T \|; T \in A\}$ is bounded.*

We want to prove the Banach-Steinhaus Theorem for hypervector spaces. To this end, we need the following Lemmas.

The proof of the following Lemma is not difficult. Hence it is omitted.

Lemma 3.5. *Let X be a normed hypervector space and A and B be subsets of $P_*(X)$. A map $D : P_*(X) \times P_*(X) \to \mathbb{R}$ that is defined as following, is a meter on this space:*

$$D(A, B) = max\{sup_{x\in A} dist\{x, B\}, sup_{y\in B} dist\{A, y\}\}.$$

Definition 3.6. Let X be a normed hypervector space, A_n be a sequence of subsets of X and A be a subset of X. We say that A_n converges to A and write $lim_{n\to\infty} A_n = A$ or $A_n \to A$, when for any $\varepsilon > 0$ there exists a $N > 0$ such that $D(A_n, A) < \varepsilon$, for all $n > N$.

Lemma 3.7. *Let X be a normed hypervector space and A and B be subsets of X. Let also A_n and B_n be sequences of $P_*(X)$ that converges to A and B, respectively. If there exists a N such that for any $n > N$ we have $A_n \subseteq B_n$, then $A \subseteq B$.*

Proof. It is clear that $A_n - B_n = \emptyset$ for any $n > N$. So $lim_{n\to\infty}(A_n - B_n) = \emptyset$ or $lim_{n\to\infty} A_n - lim_{n\to\infty} B_n = \emptyset$. This implies $A - B = \emptyset$ and hence $A \subseteq B$. □

Lemma 3.8. *Let X be a normed hypervector space over F with the following property:*

$$\forall a \in \lambda o x,\ \exists b \in \mu o x \Rightarrow a + b \in (\lambda + \mu) o x, \quad \forall \lambda, \mu \in F,\ \forall x \in X.$$

Then o is a continuous map with respect to x and by the meter defined in Lemma 3.5.

Proof. Let $x \in X$, a be a fixed element of F, $\{x_n\}$ be a sequence in X such that $x_n \to x$ and $\varepsilon > 0$ be arbitrary. So there exists $N > 0$ such that for any $n > N$ we have $\| x_n - x \| \leq \varepsilon \mid a \mid^{-1}$. Now if $y \in aox$, then by assumption for every fixed n there exists $y_n \in aox_n$ such that

$$y_n - y \in ao(x_n - x),$$

and hence

$$\| y_n - y \| \leq \mid a \mid \| x_n - x \| < \varepsilon, \quad \forall n > N.$$

This implies

$$dist\{aox_n, y\} < \varepsilon, \quad \forall n > N,$$

and so for $n > N$ we obtain

$$sup_{y\in aox} dist\{aox_n, y\} < \varepsilon.$$

On the other hand, for $n > N$ if $y_n \in aox_n$, there exists $y \in aox$ such that $y_n - y \in ao(x_n - x)$ and hence $\parallel y_n - y \parallel \leq | a || \parallel x_n - x \parallel < \varepsilon$. This implies

$$sup_{y_n \in aox_n} dist\{y_n, aox\} < \varepsilon, \quad \forall n > N.$$

Thus for any $n > N$ we obtain

$$D(aox_n, aox) = max\{sup_{y_n \in aox_n} dist\{y_n, aox\}, sup_{y \in aox} dist\{aox_n, y\}\} < \varepsilon,$$

and this completes the proof. $\square$

Theorem 3.9. *Let $\{T_n\}$ be a sequence of bounded anti linear operators on a Banach hypervector space X into a normed hypervector space Y with the following property:*

$$\forall a \in \lambda ox, \ \exists b \in \mu ox \Rightarrow a + b \in (\lambda + \mu)ox, \quad \forall \lambda, \mu \in F, \ \forall x \in X.$$

Also if for any $x \in X$ the limit of $\{T_n x\}$ exists and it is equal to Tx, then T is a bounded anti linear operator.

Proof. We first show that T is an anti linear operator. It is clear that T is additive. So it is enough to show $T(aox) \supseteq aoT(x)$ for all $a \in F$ and $x \in X$. Since $T_n x \to Tx$, so by Lemma 3.8 we have

$$aoT_n x \to aoTx, \qquad (5)$$

by the defined meter in Lemma 3.5. Set

$$A = \{Ty; Ty = lim_{n \to \infty} T_n y, \ y \in aox\}.$$

If $z \in A$, then for a y in aox we have $z = Ty$ and $T_n y \to Ty$. If $\varepsilon > 0$ be arbitrary so there exists $N > 0$ such that $\parallel T_n y - z \parallel < \varepsilon$. Thus we obtain

$$dist\{T_n(aox), z\} < \varepsilon, \quad \forall n > N.$$

Now let n_0 be an arbitrary number and $y_{n_0} \in T_{n_0}(aox)$. So $y_{n_0} = T_{n_0} z$, for a z in aox. Set $y_n = T_n z$. It is clear that $y_n \to Tz$. So there exists $M > 0$ such that for all $n > M$ we have $\parallel y_n - Tz \parallel < \varepsilon$. Thus we obtain

$$dist\{y_n, A\} < \varepsilon, \quad \forall n > N.$$

Finally, for $n > max\{N, M\}$ we obtain

$$D(T_n(aox), A) = max\{sup_{y_n \in T_n(aox)} dist\{y_n, A\}, sup_{z \in A} dist\{T_n(aox), z\}\} < \varepsilon,$$

and hence

$$T_n(aox) \to A. \qquad (6)$$

So by (5) and (6) and Lemma 3.7 we obtain

$$aoTx \subseteq A.$$

On the other hand, $A \subseteq T(aox)$. So $aoTx \subseteq T(aox)$ and hence T is an anti linear operator. Now we must show that T is bounded. Since $\{T_n x\}$ is convergent for all $x \in X$, so $\{T_n x\}$ is bounded for all $x \in X$. Thus, by

Theorem 3.3 there exists a constant $c > 0$ such that $\| T_n \| \leq c$ for all n. If x be an arbitrary element of closed unit ball, then for every n we have

$$\| Tx \| \leq \| Tx - T_n x \| + \| T_n x \| \leq \| Tx - T_n x \| + c,$$

where for enough large n this implies

$$\| Tx \| \leq c,$$

and since x is belong to closed unit ball, by proposition 3.7 in [5] we obtain

$$\| T \| \leq c,$$

and this completes the proof. □

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