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Canonical $(m,n)-$Ary Hypermodules over Krasner $(m,n)-$Ary Hyperrings

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Abstract. The aim of this research work is to define and characterize a new class of n-ary multialgebra that may be called canonical $(m,n)-$ hypermodules. These are a generalization of canonical n-ary hypergroups, that is a generalization of hypermodules in the sense of canonical and a subclasses of $(m,n)-$ary hypermodules. In addition, three isomorphism theorems of module theory and canonical hypermodule theory are derived in the context of canonical (m,n)-hypermodules.

Keywords: Canonical m-ary hypergroup, Krasner (m,n)-hyperring, $(m,n)-$ary hypermodules.

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1. Introduction

Dörnte introduced n-ary groups in 1928 [15], which is a natural generalization of groups. The notion of $n-$hypergroups was first introduced by Davvaz and Vougiouklis as a generalization of $n-$ary groups [11], and studied mainly by Davvaz, Dudek and Vougiouklis [13] and many other authors [13, 21, 22]. Generalization of algebraic hyperstructures (see [14, 18, 24]) especially of $n-$ary

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hyperstructures is a natural way for further development and deeper understanding of their fundamental properties.

Krasner has studied the notion of a hyperring in [19]. Hyperrings are essentially rings, with approximately modified axioms in which addition is a hyperoperation (i.e., $a + b$ is a set). Then this concept has been studied by a number of authors. The principal notions of hyperstructure and hyperring theory can be cited in [6, 7, 10, 12, 25, 26].

(m, n)-rings were studied by Crombez [8], Crombez and Timm [9] and Dudek [16]. Recently, the notation for (m, n)-hyperrings using was defined by Mirvakili and Davvaz and they obtained (m, n)-rings from (m, n)-hyperrings using fundamental relations [23]. Also, they defined a certain class of (m, n)-hyperrings called Krasner (m, n)-hyperrings. Krasner (m, n)-hyperrings are a generalization of (m, n)-rings and a generalization of Krasner hyperrings [23].

Recently, the reseearch of (m, n)-ary hypermodules over (m, n)-ary hyperrings has been initiated by Anvariyeh, Mirvakili and Davvaz who introduced these hyperstructures in [4]. In addition, in [5], Anvariyeh and Davvaz defined a strongly compatible relation on a $(m, n)-$ary hypermodule and determined a sufficient condition such that the strongly compatible relation is transitive.

In this paper, we consider a new class of n-ary multialgebra and we defined a certain class of $(m, n)-$ary hypermodules called canonical $(m, n)-$ary hypermodules. Canonical $(m, n)-$ary hypermodules can be considered as a natural generalization of hypermodules with canonical hypergroups and also a generalization of $(m, n)-$ary modules. In addition, several properties of canonical $(m, n)-$hypermodules are presented.

Finally, we adopt the concept of normal $(m, n)-$ary canonical subhypermodules and we prove the isomorphism theorems for canonical $(m, n)-$ary hypermodules.

2. Preliminaries and basic definition

Let H be a non-empty set and h be a mapping $h : H \times H \longrightarrow \wp^*(H)$, where $\wp^*(H)$ is the set of all non-empty subsets of H. Then h is called a binary hyperoperation on H. We denote by H^n the cartesian product $H \times \ldots \times H$, which appears n times and an element of H^n will be denoted by $(x_1, \ldots, x_n)$, where $x_i \in H$ for any i with $1 \leq i \leq n$. In general, a mapping $h : H^n \longrightarrow \wp^*(H)$ is called an $n-$ary hyperoperation and n is called the arity of hyperoperation.

Let h be an $n-$ary hyperoperation on H and $A_1, \ldots, A_n$ subsets of H. We define

$$h(A_1, \ldots, A_n) = \bigcup \{h(x_1, \ldots, x_n) | x_i \in A_i, i = 1, \ldots, n\}.$$

We shall use the following abbreviated notations: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. Also, for every $a \in H$, we write $h(\underbrace{a, \ldots, a}_{n}) = h(\overset{(n)}{a})$ and for $j<i,\ x_i^j$ is the empty set. In this convention for $j<i,\ x_i^j$ is the empty set and also

$$h(x_1,\dots,x_i,y_{i+1},\dots,y_j,x_{j+1},\dots,x_n)$$

will be written as $h(x_1^i,y_{i+1}^j,x_{j+1}^n)$.

A non-empty set H with an $n-$ary hyperoperation $h: H^n \longrightarrow P^*(H)$ will be called an $n-$ary hypergroupoid and will be denoted by (H,h). An $n-$ary hypergroupoid (H,h) is commutative if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$, we have $h(a_1^n)=h(a_{\sigma(1)}^{\sigma(n)})$.

An element $e \in H$ is called *scalar neutral element*, if $x=h(\overset{(i-1)}{e},x,\overset{(n-i)}{e})$ for every $1 \leq i \leq n$ and for every $x \in H$.

An $n-$ary hypergroupoid (H,h) will be an $n-$ary semihypergroup if and only if the following associative axiom holds:

$$h(x_1^{i-1},h(x_i^{n+i-1}),x_{n+i}^{2n-1}))=h(x_1^{j-1},h(x_j^{n+j-1}),x_{n+j}^{2n-1})),$$

for every $i,j \in \{1,2,\dots,n\}$ and $x_1,x_2,\dots,x_{2n-1} \in H$.

An $n-$ary semihypergroup (H,h), in which the equation $b \in h(a_1^{i-1},x_i,a_{i+1}^n)$ has the solution $x_i \in H$ for every $a_1,\dots,a_{i-1},a_{i+1},\dots,a_n,b \in H$ and $1 \leq i \leq n$, is called $n-$ ary hypergroup.

If H is an $n-$ary groupoid and $t=l(n-1)+1$, then the $t-$ary hyperoperation given by

$$h_{(l)}(x_1^{l(n-1)+1})=h(h(\dots,h(h(x_1^n),x_{n+1}^{2n-1}),\dots),x_{(l-1)(n-1)+2}^{l(n-1)+1}),$$

will be denoted by $h_{(l)}$.

According to [17], an *n-ary polygroup* is an n-ary hypergroup (P,f) such that the following axioms hold for all $1 \leq i,j \leq n$ and $x,x_1^n \in P$:

1. There exists a unique element $0 \in P$ such that $x=f(\overset{(i-1)}{0},x,\overset{(n-i)}{0})$,
2. There exists a unitary operation $-$ on P such that $x \in f(x_1^n)$ implies that $x_i \in f(-x_{i-1},\dots,-x_1,x,-x_n,\dots,-x_{i+1})$.

It is clear that every 2-ary polygroup is a polygroup. Every n-ary group with a scalar neutral element is an n-ary polygroup. Also, Leoreanu-Fotea in [20] defined a canonical n-ary hypergroup. A *canonical n-ary hypergroup* is a commutative n-ary polygroup.

An element 0 of an n-ary semihypergroup (H,g) is called *zero element* if for every $x_2^n \in H$ we have

$$g(0,x_2^n)=g(x_2,0,x_3^n)=\ldots=g(x_2^n,0)=0.$$

If 0 and $0'$ are two zero elements, then $0=g(0',\overset{(n-1)}{0})=0'$ and so zero element is unique.

A *Krasner hyperring* [19] is an algebraic structure $(R,+,\cdot)$ which satisfies the following axioms:

(1) $(R,+)$ is a *canonical hypergroup*, i.e.,

i) for every $x, y, z \in R,\ x + (y + z) = (x + y) + z$,
ii) for every $x, y \in R,\ x + y = y + x$,
iii) there exists $0 \in R$ such that $0 + x = x$ for all $x \in R$,
iv) for every $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$;
(We shall write $-x$ for x' and we call it the opposite of x.)
v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$;

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element.
(3) The multiplication is distributive with respect to the hyperoperation $+$.

Definition 2.1. [23]. A Krasner (m, n)-hyperring is an algebraic hyperstructure (R, f, g) which satisfies the following axioms:

(1) (R, f) is a canonical m-ary hypergroups,
(2) (R, g) is a n-ary semigroup,
(3) the $n-$ary operation g is distributive with respect to the $m-$ary hyperoperation f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R,\ 1 \le i \le n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)),$$

(4) 0 be a zero element (absorbing element) of $n-$ary operation g, i.e., for every $x_2^{n-1} \in R$, we have

$$g(0, x_2^n) = g(x_2, 0, x_3^n) = \ldots = g(x_2^n, 0) = 0.$$

Example 1. Let $(R, +, \cdot)$ be a ring and G be a normal subgroup of $(R, \cdot)$, i.e., for every $x \in R$, $xG = Gx$. Set $\bar{R} = \{\bar{x} | x \in R\}$, where $\bar{x} = xG$ and define m-ary hyperoperation f and n-ary multiplication g as follows:

$$\begin{cases} f(\bar{x}_1, \ldots, \bar{x}_m) & = \{\bar{z} | \bar{z} \subseteq \bar{x}_1 + \ldots + \bar{x}_m\}, \\ g(\bar{x}_1, \ldots, \bar{x}_n) & = \overline{x_1 x_2 \ldots x_n}. \end{cases}$$

It can be verified obviously that $(\bar{R}, f, g)$ is a Krasner (m, n)-hyperring.

Example 2. If $(L, \wedge, \vee)$ is a relatively complemented distributive lattice and if f and g are defined as:

$$\begin{cases} f(a_1, a_2) = \{c \in L |\ a_1 \wedge c = a_2 \wedge c = a_1 \wedge a_2,\ a_1, a_2 \in L\}, \\ \\ g(a_1, \ldots, a_n) = \vee_{i=1}^n a_i,\ \forall a_1^n \in L. \end{cases}$$

Then it follows that (L, f, g) is a Krasner $(2, n)$-hyperring.

Definition 2.2. A non-empty set $M = (M, h, k)$ is an $(m, n)-$ary hypermodule over an $(m, n)-$ary hyperring (R, f, g), if (M, h) is an $m-$ary hypergroup and there exists the map

$$k : \underbrace{R \times \ldots \times R}_{n-1} \times M \longrightarrow \wp^*(M)$$

such that

(1) $k(r_1^{n-1}, h(x_1^m)) = h(k(r_1^{n-1}, x_1), \ldots, k(r_1^{n-1}, x_m))$,
(2) $k(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x) = h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))$,
(3) $k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x) = k(r_1^{n-1}, k(r_m^{n+m-2}, x))$,
(4) $k(r_1^{i-1}, 0, r_{i+1}^{n-1}, x) = 0$,

where $r_i, s_i \in R$ and $x, x_i \in M$.

3. Canonical $(m,n)-$ary hypermodules

A canonical $(m,n)-$ary hypermodule (namely canonical $(m,n)-$hypermodule) is an $(m,n)-$ary hypermodule with a canonical $m-$ary hypergroup (M,h) over a Krasner (m,n)-hyperring (R,f,g).

In the following in this paper, an $(m,n)-$ary hypermodule is a canonical $(m,n)-$ary hypermodule.

Example 3. Let M be a module over ring $(R,+,\cdot)$ and G be a normal subgroup of $(R,\cdot)$, then by Example 1, $(\bar{R},f,g)$ is a Krasner (m,n)-hyperring. Now, we define on M an equivalence relation $\sim$ defined as follows:

$$x \sim y \iff x = ty,\ t \in G.$$

Let $\bar{M} = \{\bar{x} | x \in M\}$ be the set of the equivalence classes of M modulo $\sim$. We define hyperoperation h and k as follows:

$$h(\bar{x}_1, \ldots, \bar{x}_m) = \{\bar{w} | \bar{w} \subseteq \bar{x}_1 + \ldots + \bar{x}_m\}, \text{ where } x_1^m \in M$$
$$k(\bar{r}_1, \ldots, \bar{r}_{n-1}, \bar{x}) = \overline{r_1 r_2 \ldots r_{n-1} x}, \text{ where } r_1^{n-1} \in R \text{ and } x \in M.$$

It is not difficult to verify that $(\bar{M},h,k)$ is a canonical $(m,n)-$hypermodule over a Krasner (m,n)-hyperring $(\bar{R},f,g)$.

Example 4. Let (H,f,g) be a Krasner (m,n)-hyperring in Example 1, and set $M = H$, $h = f$ and $k = g$, then (M,h,k) is a canonical (m,n)-hypermodule over the Krasner (m,n)-hyperring (H,f,g). In general, If R is a Krasner (m,n)-hyperring, then (R,f,g) is a canonical (m,n)-hypermodule over the Krasner (m,n)-hyperring R.

Lemma 3.1. *Let (M,h,k) be a canonical (m,n)-hypermodule over an $(m,n)-$ary hyperring (R,f,g), then*

(1) *For every $x \in M$, we have $-(-x) = x$ and $-0 = 0$.*
(2) *For every $x \in M$, $0 \in h(x, -x, \overset{(m-2)}{0})$.*
(3) *For every x_1^m, $-h(x_1, \ldots, x_m) = h(-x_1, \ldots, -x_m)$, where $-A = \{-a \mid a \in A\}$.*
(4) *For every $r_1^{n-1} \in R$, we have $k(r_1^{n-1}, 0) = 0$.*

Proof. (1) $x = h(x, \overset{(m-1)}{0})$, hence we have $0 \in h(-x, x, \overset{(m-2)}{0})$ and this means $x \in h(-(-x), \overset{(m-1)}{0}) = -(-x)$.

(2) $x = h(x, \overset{(m-1)}{0})$ implies that $0 \in h(x, -x, \overset{(m-1)}{0})$.

(3) We have

$$\begin{aligned} 0 &\in h(x_1, -x_1, \overset{(m-1)}{0}) \\ &\subseteq h_{(2)}(x_1^2, -(x_1^2), \overset{(2m-5)}{0}) \\ &\cdots \\ &\subseteq h(h(x_1^m), h(-(x_1^m)), \overset{(m-2)}{0}). \end{aligned}$$

Thus, we obtain

$$h(-(x_1^m)) \subseteq h(-h(x_1^m), \overset{(m-1)}{0}) = -h(x_1^m)$$

and

$$h(x_1^m) \subseteq h(-h(-(x_1^m), \overset{(m-1)}{0}) = -h(-(x_1^m)).$$

So $-h(x_1^m) \subseteq -(-h(-(x_1^m))) = h(-(x_1^m))$. Hence $-h(x_1, \ldots, x_m) = h(-x_1, \ldots, -x_m)$.

(4) We have

$$\begin{aligned} k(r_1^{n-1}, 0) &= k(r_1^{n-1}, k(\overset{(n-1)}{0}, 0)) \\ &= k(r_1^{n-2}, g(r_{n-1}, \overset{(n-1)}{0}), 0) \\ &= k(r_1^{n-2}, 0, 0) \\ &= 0. \end{aligned}$$

□

Let N be a non-empty subset of canonical (m, n)-hypermodule (M, h, k). If (N, h, k) is a canonical (m, n)-hypermodule, then N called a *subhypermodule* of M. It is easily to see that N is a subhypermodule of M if and only if

1. N is a subhypergroup of the canonical m-ary hypergroup (M, h), i.e., (N, h) is a canonical m-ary hypergroup.
2. For every $r_1^{n-1} \in R$ and $x \in M$, $k(r_1^{n-1}, x) \subseteq N$.

Lemma 3.2. *A non-empty subset N of a canonical (m,n)-hypermodule is a subhypermodule if*

1. $0 \in N$.
2. *For every* $x \in N$, $-x \in N$.
3. *For every* $a_1^m \in N$, $h(a_1^m) \subseteq N$.
4. *For every* $r_1^{n-1} \in R$, *and* $x \in N$, $k(r_1^{n-1}, x) \subseteq N$.

Proof. It is straightforward. □

Lemma 3.3. *Let M be a canonical $(m, n)-$hypermodule. Then*

1. *If $N_1, \ldots, N_m$ are subhypermodules of M, then $h(N_1^m)$ is a subhypermodule of M.*

(2) *If* $\{N_i\}_{i\in I}$ *are subhypermodules of* M*, then* $\bigcap_{i\in I} N_i$ *is a subhypermodule of* M*.*

(3) *If* N *is a subhypermodule of* M *and* $a_2^m \in N$*, then* $h(N, a_2^m) = N$*.*

Proof. (1) Let $N = h(N_1^m)$. Then for every $a_1^m \in N$ we have $a_i = h(x_{i1}^{im})$, where $x_{ij} \in N_j$ and $1 \le i, j \le m$. Hence

$$\begin{aligned} h(a_1^m) &= h(h(x_{11}^{1m}), \ldots, h(x_{m1}^{mm})),\ h \text{ is commutative and associative,} \\ &= h(h(x_{11}^{m1}), \ldots, h(x_{1m}^{mm})),\ N_i \text{ is a subhypermodule,} \\ &\subseteq h(N_1, \ldots, N_m). \end{aligned}$$

Let $a \in N$, then there exists $x_i \in N_i$, $1 \le i \le m$ such that $a = h(x_1^m)$. Hence we obtain $-a = -h(x_1^m) = h(-(x_1^m)) \in h(N_1^m) = N$. Also, $0 = h(\overset{(m)}{0}) \in h(N_1^m) = N$. Therefore (N, h) is a canonical m-ary hypergroup.

Now, let $r_1^{n-1} \in R$, then

$$k(r_1^{n-1}, h(x_1^m)) = h(k(r_1^{n-1}, x_1), \ldots, k(r_1^{n-1}, x_m)) \quad \subseteq h(N_1^m)$$

Therefore (N, h, k) is a subhypermodule of M.

(2) It is clear.

(3) Since N is a subhypermodule, then for every $a_2^m \in N$, we have $h(N, a_2^m) \subseteq N$. Also, we obtain

$$\begin{aligned} N = h(N, \overset{(m-1)}{0}) &\in h(N, h(a_2^m, 0), -h(a_2^m, 0), \overset{(m-3)}{0}) \\ &= h(N, h(a_2^m, 0), h(-(a_2^m), 0), \overset{(m-3)}{0}) \\ &= h(h(N, -(a_2^m)), a_2^{m-1}, h(a_m, \overset{(m-1)}{0})) \\ &\subseteq h(N, a_2^m). \end{aligned}$$

Therefore $N = h(N, a_2^m)$. □

Definition 3.4. A subhypermodule N of M is called *normal* if and only if for every $x \in M$,

$$h(-x, N, x, \overset{(m-3)}{0}) \subseteq N.$$

If N is a normal subhypermodule of a canonical (m, n)-hypermodule M, then

$$N = h(N, \overset{(m-1)}{0}) \subseteq h(N, h(-x, x, \overset{(m-2)}{0}), \overset{(m-2)}{0}) = h(-x, N, x, \overset{(m-3)}{0}) \subseteq N.$$

Thus for every $x \in M$, $h(-x, N, x, \overset{(m-3)}{0}) = N$.

If $s \in h(N, x, \overset{(m-2)}{0})$, then $h(N, s, \overset{(m-3)}{0}) \subseteq h(N, h(N, x, \overset{(m-3)}{0})$
$= h(N, x, \overset{(m-3)}{0})$.

Also, $s \in h(N, x, \overset{(m-2)}{0})$ implies that $r \in h(-N, s, \overset{(m-2)}{0}) = h(N, s, \overset{(m-2)}{0})$ and so we obtain $h(N, x, \overset{(m-2)}{0}) \subseteq h(N, s, \overset{(m-2)}{0})$. Therefore we have

$$s \in h(N, x, \overset{(m-2)}{0}) \Longrightarrow h(N, x, \overset{(m-2)}{0}) = h(N, s, \overset{(m-2)}{0}).$$

Lemma 3.5. *Let N be a normal subhypermodule of a canonical (m, n)-hypermodule M. Then for every $s_i \in h(N, x_i, \overset{(m-2)}{0})$, $i = 2, \ldots, m$, we have $h(N, x_2^m) = h(N, s_2^m)$.*

Proof. We have

$$\begin{aligned} h(N, s_2^m) &\subseteq h(N, h(N, x_2, \overset{(m-2)}{0}), \ldots, h(N, x_m, \overset{(m-2)}{0})) \\ &\subseteq h(h(\overset{(m)}{N}), x_2^m, h_{(m-2)}(\overset{((m-2)(m-1)+1)}{0})) \\ &= h(N, x_2^m). \end{aligned}$$

Also, we have $h(N, x_i, \overset{(m-2)}{0}) = h(N, s_i, \overset{(m-2)}{0})$ and so $x_i \in h(N, s_i, \overset{(m-2)}{0})$. The similar way implies $h(N, x_2^m) \subseteq h(N, s_2^m)$. □

Example 5. (Construction). Let $(M, +, \cdot)$ be a canonical $R-$hypermodule over a Krasner hyperring R. Let f be an m-ary hyperoperation and g be an n-ary operation on R as follows:

$$f(x_1^m) = \sum_{i=1}^{m} x_i, \quad \forall x_1^m \in R,$$

$$g(x_1^n) = \prod_{i=1}^{n} x_i, \quad \forall x_1^n \in R.$$

Then it follows that (R, f, g) is a Krasner (m, n)-hyperring. Let h be an m-ary hyperoperation and k be an n-ary scalar hyperoperation on M as follows:

$$h(x_1^m) = \sum_{i=1}^{m} x_i, \quad \forall x_1^m \in M,$$

$$k(r_1, \ldots, r_{n-1}, x) = (\prod_{i=1}^{n-1} r_i) \cdot x.$$

Since $+$ and $\cdot$ are well-defined and associative so h and k are well-defined and associative. If 0 is a zero element of $(M, +, \cdot)$, then 0 is a zero element of (M, h, k). Now, let $1 \leq j \leq m$ and $x, x_1^m \in M$. Then

$$\begin{aligned} x &\in h(x_1^m) \\ &= \textstyle\sum_{i=1}^{m} x_i, \quad + \text{ is commutative} \\ &= x_1 + \ldots + x_{j-1} + x_{j+1} + \ldots + x_m + x_j \\ &= X + x_j, \quad X = x_1 + \ldots + x_{j-1} + x_{j+1} + \ldots + x_m. \end{aligned}$$

Thus $x \in z + x_j$ such that $z \in X$ and hence $x_j \in -z + x$, But $-z \in -X = -(x_1 + \ldots + x_{j-1} + x_{j+1} + \ldots + x_m)$. Therefore

$$x_j \in (-x_{j-1}) + \ldots + (-x_1) + x + (-x_m) + \ldots + (-x_{j+1}) =$$

$$h(-x_{j-1}, \ldots, -x_1, x, -x_m, \ldots, -x_{j+1}).$$

This implies that (M, h) is a canonical $m-$ary hypergroup.

Since M is an R-hypermodule, it is not difficult to see that the properties of M as an R-hypermodule, guarantee that the canonical $m-$hypergroup (M, h, k) is a canonical (m, n)-hypermodule.

Definition 3.6. The canonical (m, n)-hypermodule (M, h, k) derived from canonical hypermodule $(M, +, \circ)$ in Example 5, denote by
$(M, h, k) = der_{(m,n)}(M, +, \cdot)$.

Theorem 3.7. *Every canonical (m, n)-hypermodule M extended by a canonical $(2, n)$-hypermodule.*

Proof. We define the hyperoperation $+$ as follows:

$$x + y = h(x, y, \overset{(m-2)}{0}),\ \forall x, y \in R.$$

It is clear that $+$ is commutative and associative. Also, 0 is a scalar neutral and a zero element of $(M, +, k)$. Now, let $x \in y + z$ then $x \in h(x, y, \overset{(m-2)}{0})$. This implies that $y \in h(-x, y, \overset{(m-2)}{0}) = -x + y$ and so $(M, +)$ is a canonical hypergroup. It is easy to see that $n-$ary operation k is distributive with respect to the hyperoperation $+$. Therefore $(M, +, k)$ is a canonical $(2, n)$-hypermodule. □

4. Relations on a canonical (m, n)-hypermodules

In this section, we introduce two relations on a canonical (m, n)-hypermodule M. In addition, three isomorphism theorems of module theory and canonical hypermodule theory are derived in the context of canonical (m, n)-hypermodules by these relations. In order to see the relations on the hypermodules, one can see [1, 2, 3]. Also, the concepts of normal (m, n)-ary canonical subhypermodules are defined.

Suppose that N is a normal subhypermodule of M.

(1) The relation N^* on M is defined as follows:

$$x\ N^*\ y \text{ if and only if } h(x, -y, \overset{(m-2)}{0}) \cap N \neq \emptyset,\ \forall\ x, y \in M.$$

(2) Also, the relation N_* on M may be defined as follows:

$x\ N_*\ y$ if and only if there exist $x_2^m \in M$, such that $x, y \in h(N, x_2^m)$, $\forall\ x, y \in M$.

Lemma 4.1. *The relation N^* is an equivalence relation on a canonical (m,n)-hypermodule M.*

Proof. Since $0 \in h(x,-x,\overset{(m-2)}{0}) \cap N$, then the relation N^* is reflexive. If xN^*y, then there exists an element $a \in N$ such that $a \in h(x,-y,\overset{(m-2)}{0})$. Therefore, we have $-a \in -h(x,-y,\overset{(m-2)}{0}) = h(-x,y,\overset{(m-2)}{0})$ and commutativity of (M,h) implies that $-a \in h(y,-x,\overset{(m-2)}{0}) \cap N$. So yN^*x and the relation N^* is symmetric. Now, suppose that xN^*y and yN^*z. Then there exist $a,b \in N$ such that $a \in h(x,-y,\overset{(m-2)}{0})$ and $b \in h(y,-z,\overset{(m-2)}{0})$. Thus $x \in h(a,y,\overset{(m-2)}{0})$ and $-z \in h(-y,b,\overset{(m-2)}{0})$. But, N is a normal subhypermodule of N and we obtain:

$$\begin{aligned} h(x,-z,\overset{(m-2)}{0}) &\subseteq h(h(a,y,\overset{(m-2)}{0}),h(-y,b,\overset{(m-2)}{0}),\overset{(m-2)}{0}) \\ &= h(y,h(a,b,\overset{(m-2)}{0}),-y,\overset{(m-3)}{0}) \\ &\subseteq h(y,N,-y,\overset{(m-3)}{0}) \\ &\subseteq N. \end{aligned}$$

Therefore xN^*z and the relation N^* is transitive. □

Let $N^*[x]$ be the equivalence class of the element $x \in M$, then

Lemma 4.2. *If N is a normal subhypermodule of a canonical (m,n)-hypermodule M, then*

$$N^*[x] = h(N,x,\overset{(m-2)}{0}).$$

Proof. we have

$$\begin{aligned} N^*[x] &= \{y \in M \mid yN^*x\} \\ &= \{y \in M \mid \exists a \in N \text{ such that } a \in h(y,-x,\overset{(m-2)}{0})\} \\ &= \{y \in M \mid \exists a \in N \text{ such that } y \in h(a,x,\overset{(m-2)}{0})\} \\ &= h(N,x,\overset{(m-2)}{0}). \end{aligned}$$

□

Lemma 4.3. *Let N be a normal subhypermodule of a canonical (m,n)-hypermodule M. Then for all $a_2^m \in M$, we have $h(N,a_2^m) = N^*[x]$ for all $x \in h(N,a_2^m)$.*

Proof. By Lemma 4.2, we prove that $h(N,a_2^m) = h(N,x,\overset{(m-2)}{0})$, for all $x \in h(N,a_2^m)$.

Let $x \in h(N, a_2^m)$, so

$$\begin{aligned} h(N, x, \overset{(m-2)}{0}) &\subseteq h(N, h(N, a_2^m), \overset{(m-2)}{0}) \\ &= h(h(N, N, \overset{(m-2)}{0}), a_2^m) \\ &= h(N, a_2^m). \end{aligned}$$

Also, $x \in h(N, x, \overset{(m-2)}{0}) \subseteq h(N, h(N, a_2^m), \overset{(m-2)}{0})$ implies that $h(N, a_2^m) \in h(-N, x, \overset{(m-2)}{0}) = h(N, x, \overset{(m-2)}{0})$. Therefore, we obtain $h(N, a_2^m) = h(N, x, \overset{(m-2)}{0})$. □

Corollary 4.4. Let N be a normal subhypermodule of a canonical (m,n)-hypermodule M and $h(N, a_2^m) \cap h(N, b_2^m) \neq \emptyset$, then $h(N, a_2^m) = h(N, b_2^m)$.

Proof. Let $x \in h(N, a_2^m) \cap h(N, b_2^m)$, then Lemma 4.3, implies $h(N, a_2^m) = N^*[x] = h(N, b_2^m)$ □

Corollary 4.5. Let N be a normal subhypermodule. Then $N^* = N_*$ and the relation N_* is an equivalence relation.

Proof. Let $N_*[x]$ be the equivalence class of the element $x \in M$. Then

$$\begin{aligned} N_*[x] &= \{y \in M \mid xN_*y\} \\ &= \{y \in M \mid \exists a_2^m \in M, x, y \in h(N, a_2^m)\}. \end{aligned}$$

Since $x \in h(N, a_2^m)$, thus by Lemma 4.3, $N^*[x] = h(N, x, \overset{(m-2)}{0}) = h(N, a_2^m)$ and we obtain $N_*[x] = \{y \in M \mid y \in N^*[x]\} = N^*[x]$. Therefore $N^* = N_*$. □

Lemma 4.6. *Let N be a normal subhypermodule of a canonical (m,n)-hypermodule (M,h,k), then for all $a_1^m \in M$, we have $N^*[h(a_1^m)] = N^*[a]$ for all $a \in h(a_1^m)$.*

Proof. Suppose that $a \in h(a_1^m)$, then $N^*[a] \subseteq N^*[h(a_1^m)]$.

On the other hand, let $a \in N^*[h(a_1^m)] = h(N, h(a_1^m), \overset{(m-2)}{0}) = h(h(N, a_1^{m-1}), \overset{(m-2)}{0}$

$, a_m)$. Thus $a_m \in h(-h(N, a_1^{m-1}), \overset{(m-2)}{0}, a)$ and so

$$\begin{aligned}
h(a_1^m) &\subseteq h(a_1^{m-1}, h(h(-N, -(a_1^{m-1})), \overset{(m-2)}{0}, a)) \\
&= h_{(2)}(h(a_1, N, -a_1, \overset{(m-3)}{0}), a_2^{m-1}, -(a_2^{m-1}), 0, a), \ N \text{ is normal,} \\
&\subseteq h_{(2)}(N, a_2^{m-1}, -(a_2^{m-1}), 0, a) \\
&= h_{(2)}(h(a_2, N, -a_2, \overset{(m-3)}{0}), a_3^{m-1}, -(a_3^{m-1}), \overset{(3)}{0}, a), \ N \text{ is normal,} \\
&\subseteq h_{(2)}(N, a_3^{m-1}, -(a_3^{m-1}), 0, a) \\
&\ldots \\
&= h_{(2}(h(a_m, N, -a_m, \overset{(m-3)}{0}), \overset{(2m-2)}{0}, a) \\
&\subseteq h(N, \overset{(m-2)}{0}, a) \\
&= h(N, a, \overset{(m-2)}{0}) \\
&= N^*[a].
\end{aligned}$$

Therefore $h(a_1^m) \subseteq N^*[a]$ and so $N^*[h(a_1^m)] \subseteq N^*[a]$ and this completes the proof. □

Theorem 4.7. *Let N be a normal subhypermodule of a canonical (m, n)-hypermodule (M, h, k). Then*

(1) *For all $x_1^m \in M$, we have $N^*[h(N^*[x_1], \ldots, N^*[x_m])] = h(N^*[x_1], \ldots, N^*[x_m])$.*
(2) *For all $r_1^{n-1} \in R$ and $x \in M$, we have $N^*[N^*[k(r_1^{n-1}, x)]] = N^*[k(r_1^{n-1}, x)]$.*

Proof. (1) The proof easily follows from Lemma 4.6.

(2) We have $N^*[k(r_1^{n-1}, x)] \subseteq N^*[N^*[k(r_1^{n-1}, x)]]$. Now, let $a \in N^*[N^*k(r_1^{n-1}, x)]]$. Then, there exists $b \in N^*[k(r_1^{n-1}, x)]$ such that $a \in N^*[b]$. So aN^*b and $bN^*k(r_1^{n-1}, x)$ which implies that $aN^*k(r_1^{n-1}, x)$. Hence $a \in N^*[k(r_1^{n-1}, x)]$ and $N^*[N^*[k(r_1^{n-1}, x)]] \subseteq N^*[k(r_1^{n-1}, x)]$ □

By definition of a canonical (m, n)-hypermodule and Theorem 4.7, we have:

Theorem 4.8. *(Construction). Let N be a normal subhypermodule of a canonical (m, n)-hypermodule (M, h, k). Then the set of all equivalence classes $[M : N] = \{N^*[x] \mid x \in M\}$ is a canonical (m, n)-hypermodule with the m-ary hyperoperation h/N and the scalar n-ary operation k/N, defined as follows:*

$$h/N(N^*[x_1], \ldots, N^*[x_m]) = \{N^*[z] \mid z \in h(N^*[x_1], \ldots, N^*[x_m])\}, \ \forall x_1^m \in M,$$

$$k/N(r_1^{n-1}, N^*[x]) = N^*[k(r_1^{n-1}, N^*[x])], \ \forall \ r_1^{n-1} \in R, \ x \in M.$$

EXAMPLE 6. Suppose $R := \{0, 1, 2, 3\}$ and define a 2-ary hyperoperation $+$ on R as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	$\{0,1\}$	3	$\{2,3\}$
2	2	3	0	1
3	3	$\{2,3\}$	1	$\{0,1\}$.

It follows that $(R, +)$ is a canonical 2-ary hypergroup. If g is an n-ary operation on R such that

$$g(x_1^n) = \begin{cases} 2 & if \;\; x_1^n \in \{2,3\}, \\ 0 & else. \end{cases}$$

Then, we have $(R, +, g)$ is a Krasner $(2, n)-$hyperring.

Now, set $M = R$, $\oplus = +$ and $k = g$, then it can be verified $(M, \oplus, k)$ is a canonical $(2, n)-$hypermodule over Krasner $(2, n)$-hyperring $(R, +, g)$.

Let $N := \{0, 1\}$, then N is a normal subhypermodule of M. Also, it is not difficult to see that $N^*[0] = \{0, 1\}$ and $N^*[2] = \{2, 3\}$ and so

$\oplus/N$	$N^*[0]$	$N^*[2]$
$N^*[0]$	$N^*[0]$	$N^*[2]$
$N^*[2]$	$N^*[2]$	$N^*[0]$

and

$$N^*[k/N(r_1^{n-1}, N^*[x])] = \begin{cases} N^*[2], & if \; r_1^{n-1}, x \in \{2,3\}, \\ \\ N^*[0], & else. \end{cases}$$

Then it is easily to see that $([M : N], \oplus/N) \cong (\mathbb{Z}_2, +)$.

Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two canonical (m, n)-hypermodules, a mapping $\varphi : M_1 \to M_2$ is called an $R-$*homomorphism (or homomorphism)*, if for all $r_1^{n-1} \in R$ and $x_1^m, x \in M$ we have:

$$\varphi(h_1(x_1, \ldots, x_m)) = h_2(\varphi(x_1), \ldots, \varphi(x_m))$$

$$\varphi(k_1(r_1^{n-1}, x)) = k_2(r_1^{n-1}, \varphi(x))$$

A homomorphism φ is an isomorphism if φ is injective and onto and we write $M_1 \cong M_2$ if M_1 is isomorphic to M_2.

Lemma 4.9. *Let $\varphi : M_1 \to M_2$ be a homomorphism, then*

(1) $\varphi(0_{M_1}) = 0_{M_2}$.
(2) *For all $x \in M$, $\varphi(-x) = -\varphi(x)$.*

(3) *Let* $\ker\varphi = \{x \in M_1 \mid \varphi(x) = 0_{M_2}\}$, *then* φ *is injective if and only if* $\ker\varphi = \{0_{M_1}\}$.

Proof. It is straightforward. □

Lemma 4.10. *Let* N_1^m *be subhypermodules of a canonical (m,n)-hypermodule* M *and there exists* $1 \le j \le m$ *such that* N_j *be a normal subhypermodule. Then*

(1) $\bigcap\limits_{i=1}^{m} N_i$ *is a normal subhypermodule of* N_k, *where* $1 \le k \le m$.

(2) N_j *is a normal subhypermodule of* $h(N_1^m)$.

Proof. It is straightforward. □

The First Isomorphism Theorem comes next.

Theorem 4.11. *(First Isomorphism Theorem). Let* φ *be a homomorphism from the canonical* (m,n)*-hypermodule* (M_1, h_1, k_1) *into the canonical* (m,n)*-hypermodule* (M_2, h_2, k_2) *such that* $K = \ker\varphi$ *is a normal subhypermodule of* M_1, *then* $[M_1 : K^*] \cong Im\varphi$.

Proof. We define $\rho : [M_1 : K^*] \to Im\varphi$ by $\rho(K^*[x]) = \varphi(x)$. First, we prove that ρ is well-define. Suppose that $K^*[x] = K^*[y]$. Then

$$\begin{aligned} K^*[x] = K^*[y] \quad &\Leftrightarrow h_1(K, x, \overset{(m-2)}{0_{M_1}}) = h_1(K, y, \overset{(m-2)}{0_{M_1}}) \\ &\Leftrightarrow \varphi(h_1(K, x, \overset{(m-2)}{0_{M_1}})) = \varphi(h_1(K, y, \overset{(m-2)}{0_{M_1}})) \\ &\Leftrightarrow h_2(\varphi(K), \varphi(x), \overset{(m-2)}{\varphi(0_{M_1})}) = h_2(\varphi(K), \varphi(y), \overset{(m-2)}{\varphi(0_{M_1})}) \\ &\Leftrightarrow h_2(0_{M_2}, \varphi(x), \overset{(m-2)}{0_{M_2}}) = h_2(0_{M_2}, \varphi(y), \overset{(m-2)}{0_{M_2}}) \\ &\Leftrightarrow \varphi(x) = \varphi(y). \end{aligned}$$

Therefore ρ is well-define.

Let $K^*[x_1], \ldots, K^*[x_m] \in [M_1 : K^*]$. Then

$$\begin{aligned} \rho(h_1/K(K^*[x_1], \ldots, K^*[x_m])) &= \rho(\{K^*[z] \mid z \in h_1(K^*[x_1], \ldots, K^*[x_m])\}) \\ &= \rho(\{K^*[z] \mid z \in h_1(h_1(K, x_1, \overset{(m-2)}{0_{M_1}}), \ldots, h_1(K, x_m, \overset{(m-2)}{0_{M_1}}))\}) \\ &= \rho(\{K^*[z] \mid z \in h_1(K, h_1(x_1^m), \overset{(m-2)}{0_{M_1}})\}) \\ &= \{\varphi(z) \mid z \in K^*[h_1(x_1^m)]\} \\ &= \varphi(K^*[h_1(x_1^m)]) \\ &= \varphi(h_1(K, h_1(x_1^m), \overset{(m-2)}{0_{M_1}})) \\ &= h_2(\varphi(K), \varphi(h_1(x_1^m)), \overset{(m-2)}{\varphi(0_{M_1})}) \\ &= h_2(0_{M_2}, h_2(\varphi(x_1), \ldots, \varphi(x_m)), \overset{(m-2)}{0_{M_2}}) \\ &= h_2(\varphi(x_1), \ldots, \varphi(x_m)) \\ &= h_2(\rho(x_1), \ldots, \rho(x_m)). \end{aligned}$$

Also, let $r_1^{n-1} \in R$ and $K^*[x] \in [M_1 : K^*]$. Then

$$\begin{aligned}\rho(k_1/K(r_1^{n-1}, K^*[x])) &= \rho(K^*(k_1(r_1^{n-1}, K^*[x]))) \\ &= \{\varphi(k_1(r_1^{n-1}, x)|x \in K^*[x])\} \\ &= k_2(r_1^{n-1}, x)|x \in \varphi(K^*[x])) \\ &= k_2(r_1^{n-1}, \rho(K^*[x])).\end{aligned}$$

Therefore ρ is an $R-$homomorphism.

Also, we have $\rho(0_{[M_1:K^*]}) = \rho(K^*[0_{M_1}]) = \varphi(0_{M_1}) = 0_{M_2}$.

Let $y \in Im\varphi$, so there exists $x \in M_1$ such that $y = \varphi(x) = \rho(K^*[x])$. Thus ρ is onto.

Now, we show that ρ is an injective homomorphism. We have

$$\begin{aligned}\ker\rho &= \{K^*[x] \in [M_1 : K^*] \mid \rho(K^*[x]) = 0_{M_2}\} \\ &= \{K^*[x] \in [M_1 : K^*] \mid \varphi(x) = 0_{M_2}\} \\ &= K^*(\ker\varphi), \quad \text{Since } K = \ker\varphi, \\ &= h_1(K, K, \overset{(m-2)}{0_{M_1}}) \\ &= K = 0_{[M_1:K^*]}.\end{aligned}$$

Therefore ρ is an isomorphism and so $[M_1 : K^*] \cong Im\varphi$. □

Theorem 4.12. *(Second Isomorphism Theorem). If N_1^n are subhypermodules of a canonical $(m,n)-$hypermodule (M,h,k) and there exists $1 \le j \le m$ such that N_j be a normal subhypermodule of M. Let for every $r_1^{n-1} \in R$ and $y \in M$, we have $N_j^*[k(r_1^{n-1}, y)] = k(r_1^{n-1}, N_j^*(y)]$. Then*

$$[h(N_1^j, 0, N_{j+1}^m) : (h(N_1^j, 0, N_{j+1}^m) \cap N_j)^*] \cong [h(N_1^m) : N_j^*],$$

where N_{j+1}^m are subhypermodules of M.

Proof. By Lemma 4.10, N_j is a normal subhypermodule of $h(N_1^m)$ and so $[h(N_1^m) : N_j^*]$ is defined. Define $\rho : h(N_1^j, 0, N_{j+1}^m) \to [h(N_1^m) : N_j^*]$ by $\rho(x) = N_j^*[x]$. Since N^* is an equivalence relation then ρ is well-defined. It is not difficult to see that ρ is an $R-$homomorphism. Consider $N_j^*[y] \in [h(N_1^m) : N_j^*]$, $y \in h(N_1^m)$. Thus, there exists $a_k \in N_k$, $1 \le k \le m$ such that $y \in h(a_1^m)$. By

Lemma 4.6, we have

$$\begin{aligned} N_j^*[y] &= N_j^*[h(a_1^m)] \\ &= h(N_j, h(a_1^m), \overset{(m-2)}{0}) \\ &= h(a_1^{j-1}, h(N_j, a_j, \overset{(m-2)}{0}), a_{j+1}^m) \\ &= h(a_1^{j-1}, N_j, a_{j+1}^m) \\ &= h(N_j, h(a_1^{j-1}, 0, a_{j+1}^m), \overset{(m-2)}{0}) \\ &= N_j^*[h(a_1^{j-1}, 0, a_{j+1}^m)] \\ &= h_j^*[x], \quad x \in h(a_1^{j-1}, 0, a_{j+1}^m) \subseteq h(N_1^{j-1}, 0, N_{j+1}^m), \\ &= \rho(x), \quad x \in h(N_1^{j-1}, 0, N_{j+1}^m). \end{aligned}$$

Therefore ρ is onto. Now, we prove that $\ker\rho = h(N_1^j, 0, N_{j+1}^m) \cap N_j$.

$$\begin{aligned} x \in \ker\rho &\Leftrightarrow \rho(x) = N_j \\ &\Leftrightarrow N_j^*[x] = N_j \\ &\Leftrightarrow h(N_j, x, \overset{(m-2)}{0}) = N_j \\ &\Leftrightarrow x \in N_j \cap h(N_1^j, 0, N_{j+1}^m). \end{aligned}$$

Now, we have $[M : (\ker\rho)^*] \cong Im\rho$ and so

$$[h(N_1^j, 0, N_{j+1}^m) : (h(N_1^j, 0, K_{j+1}^m) \cap N_j)^*] \cong [h(N_1^m) : N_j^*].$$

□

Theorem 4.13. *(Third Isomorphism Theorem). If A and B are normal subhypermodules of a canonical (m,n)-hypermodule M such that $A \subseteq B$, then $[B : A^*]$ is a normal subhypermodule of canonical (m,n)-hypermodule $[M : A^*]$ and $[[M : A^*] : [B : A^*]] \cong [M : B^*]$.*

Proof. First, we show that $[B : A^*]$ is a normal subhypermodule of canonical (m,n)-hypermodule $[M : A^*]$. Since $0 \in B$ then $0_{[M:A^*]} = A^*[0] \in [B : A^*]$. If $A^*[x_1], \ldots, A^*[x_m] \in [B : A^*]$, then $A^*[x_1], \ldots, A^*[x_m] \subseteq B$ and since B is a subhypermodule of M, we obtain $h(A^*[x_1], \ldots, A^*[x_m]) \subseteq B$. Thus $h/N(A^*[x_1], \ldots, A^*[x_m]) \in [B : A^*]$. If $A^*[x] \in [B : A^*]$ then $A^*[x] \subseteq B$ and so $-A^*[x] \subseteq -B = B$. We leave it to reader to verify that for every $r_1^{n-1} \in R$ and $A^*[x] \in [B : A^*]$, $k/N(r_1^{n-1}, A^*[x]) \in [B : A^*]$. Now, Lemma 3.2 implies that $[B : A^*]$ is a subhypermodule of M.

Also, let $A^*[y] \in [M : A^*]$ and $A^*[x] \in [B : A^*]$, so $A^*[y] \subseteq M$ and $A^*[x] \subseteq B$. Since B is a normal subhypermodule, then $h(-y, x, y, \overset{(m-3)}{0}) \subseteq B$. This implies that

$$h(-A^*[y], A^*[x], A^*[y], \overset{(m-3)}{A^*[0]}) = A^*[h(-y, x, y, \overset{(m-3)}{0})] \in [B : A^*].$$

Therefore $[B : A^*]$ is a normal subhypermodule of canonical (m,n)-hypermodule $[M : A^*]$.

Now, $\rho : [M : A^*] \to [M : B^*]$ defined by $\rho(A^*[x]) = B^*[x]$ is an $R-$ homomorphism and onto with kernel $\ker\rho = [B : A^*]$. □

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