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z-Weak Ideals and Prime Weak Ideals

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Abstract. In this paper, we study a generalization of z-ideals in the ring $C(X)$ of continuous real valued functions on a completely regular Hausdorff space X. The notion of a weak ideal and naturally a weak z-ideal and a prime weak ideal are introduced and it turns out that they behave such as z-ideals in $C(X)$.

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1. Introduction

Throughout this paper, $C(X)$ will denote the ring of real continuous functions defined on a completely regular Hausdorff space. As usual, if $f \in C(X)$, its zero set $f^{\leftarrow}(0)$ and its cozero set $X \setminus f^{\leftarrow}(0)$ are denoted by $Z(f)$ and $Coz(f)$, respectively. Also if $S \subseteq C(X)$, $Z[S] = \{Z(f) : f \in S\}$ and $Coz[S] = \{Coz(f) : f \in S\}$. Whenever I is an ideal in $C(X)$, we call I a *z-ideal* in $C(X)$ if $g \in C(X)$ and $Z(g) \in Z[I]$ imply that $g \in I$. The partial ordering on $C(X)$ is defined by:

$$f \leq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in X.$$

A proper ideal I of $C(X)$ is called a *convex ideal* if whenever $0 \leq f \leq g$, and $g \in I$, then $f \in I$ and it is called an *absolutely convex ideal* if whenever $|f| \leq |g|$,

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and $g \in I$, then $f \in I$. Recall that βX is the Stone-Čech compactification of X. For undefined terms and notations, the readers are referred to [5, 7, 8, 9].

Let R always denote a commutative ring with identity. A proper ideal I of R is called a *prime ideal* of R if for every $a, b \in R$, $ab \in I$ implies $a \in I$ or $b \in I$. A prime ideal P in R is called a *minimal prime ideal* of the ideal I if $I \subseteq P$ and there is no prime ideal P' such that $I \subseteq P' \subset P$. Let $Min(I)$ denotes the set of minimal prime ideals of I in R. An ideal I of R is called an *unit ideal* of R if $I = R$.

We need the following well known facts in the sequel, see [5] and [14].

(1) If P is a prime ideal of $C(X)$, then $|\bigcap Z[P]| \leq 1$.
(2) Every z-ideal in $C(X)$ is an intersection of prime z-ideals.
(3) Every prime ideal of $C(X)$ is absolutely convex.
(4) If I is a z-ideal in $C(X)$ and $P \in Min(I)$, then P is a z-ideal in $C(X)$.
(5) The sum of two z-ideals in $C(X)$ is either a z-ideal or is the unit ideal.
(6) The sum of two prime ideals in $C(X)$ is either a prime ideal or is the unit ideal.

L. Gilman and C. W. Kohls have remarked [[6], p. 401] that the proofs of items (5) and (6) seem to depend strongly on properties of βX and David Rudd has proved both items by an elementary methods, see [14].

It is well known that $C(X)$ with pointwise multiplication operation is a semigroup. In this paper we study the ideals in semigroup $(C(X), .)$ by similar tools which are used in the ring $C(X)$.

2. z-weak ideal

The structure of the prime ideals and the z-ideals of $C(X)$ has been the subject of much investigation (see [1, 2, 10, 11, 12]). In this section we introduce prime weak ideal and z-weak ideal in $C(X)$.

Definition 2.1. A nonempty subset I of a ring R is called a *weak ideal* of R if $\{ri : r \in R \& i \in I\} \subseteq I$.

It is easy to see that a nonempty subset I of R is a weak ideal if and only if $I = \bigcup_{a \in I} aR$.

Definition 2.2. A proper weak ideal I of $C(X)$ is called a *z-weak ideal* if $Z(f) \in Z[I]$ implies that $f \in I$.

It is obvious that the intersection (or union) of an arbitrary (non empty) family of z-weak ideals of $C(X)$ is a z-weak ideal of $C(X)$.

Definition 2.3. A proper weak ideal I of $C(X)$ is called a *C-weak ideal* if for every $Z_1, Z_2 \in Z[I]$, we have $Z_1 \cap Z_2 \in Z[I]$, i.e., $Z[I]$ is closed under finite intersection.

Example 2.4. Let $f, g \in C(X)$ such that $Z(f) = Z(g) \neq \emptyset$, $f \notin gC(X)$, and $g \notin fC(X)$. We have $I = fC(X) \cup gC(X)$ is a C-weak ideal of $C(X)$, but it is not an ideal of $C(X)$ (see [[4], Example 1]).

It is clear that for a z-weak ideal I of $C(X)$, I is an ideal of $C(X)$ if and only if I is a C-weak ideal of $C(X)$.

For every $f \in C(X)$, we put $\mathcal{M}_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ and this notation is first used in [2].

Proposition 2.5. *Every z-weak ideal of $C(X)$ is a union of z-ideals of $C(X)$.*

Proof. Let I be a z-weak ideal of $C(X)$. Clearly, for every $f \in I$, $\mathcal{M}_f$ is a z-ideal of $C(X)$ and $I = \bigcup_{f \in I} \mathcal{M}_f$. □

Definition 2.6. A proper weak ideal I of $C(X)$ is called a *convex weak ideal* if whenever $0 \leq f \leq g$, and $g \in I$, then $f \in I$ and it is called an *absolutely convex weak ideal* if whenever $|f| \leq |g|$, and $g \in I$, then $f \in I$.

Trivially, an absolutely convex weak ideal of $C(X)$ is convex weak ideal, but the converse is not true. Furthermore, it is clear that every z-weak ideal of $C(X)$ is an absolutely convex weak ideal.

A space X is called *F-space* if each finitely generated ideal of $C(X)$ is a principal ideal. It is well known (see [[5], Theorem 14.25]) that X is an F-space if and only if every ideal of $C(X)$ is a convex ideal.

Proposition 2.7. *The following statements are equivalent:*

(1) *X is an F-space.*
(2) *Every weak ideal of $C(X)$ is a convex ideal.*
(3) *Every C-weak ideal of $C(X)$ is a convex ideal.*

Proof. It is clear. □

Definition 2.8. A proper weak ideal I of R is called a *prime weak ideal* if for every $a, b \in R$, $ab \in I$ implies $a \in I$ or $b \in I$.

Remark 2.9. We recall that a nonempty subset S of a ring R is *multiplicative* provided that precisely $s_1, s_2 \in S$ implies $s_1 s_2 \in S$. If S is a multiplicative subset of R which is disjoint from a weak ideal I of R, then

$$\mathcal{S} = \{Q \subseteq R : Q \cap S = \emptyset \,\&\, I \subseteq Q \,\&\; Q \text{ is a proper weak ideal of } R\ \}$$

is partially ordered by inclusion. By Zorn's Lemma, there is a weak ideal P of R which is maximal in $\mathcal{S}$. Furthermore any such weak ideal P is prime weak ideal of R.

Proposition 2.10. *Every prime weak ideal of R is a union of prime ideals of R.*

Proof. Let Q be a prime weak ideal of R. If $f \in Q$, then $fR \cap (R \setminus Q) = \emptyset$ and $R \setminus Q$ is a multiplicative subset of R. By Theorem 2.2, in [8], there is a prime ideal P_f (in ring of R) disjoint from $R \setminus Q$ that contains fR and hence $fR \subseteq P_f \subseteq Q$. Thus $Q = \bigcup_{f \in Q} P_f$, whence Q is a union of prime ideals of R. □

Corollary 2.11. *If P is a prime weak ideal of $C(X)$, then $|\bigcap Z[P]| \leq 1$.*

Proof. By Proposition 2.10, there exists a prime ideal P' of $C(X)$ such that $P' \subseteq P$ and hence $|\bigcap Z[P]| \leq |\bigcap Z[P']| \leq 1$ (see [5]). □

By Theorem 5.5 in [5], every prime ideal P of $C(X)$ is absolutely convex ideal. Therefore the union of prime ideals of $C(X)$ is an absolutely convex weak ideal. So it is evident that:

Corollary 2.12. *Every prime weak ideal P of $C(X)$ is absolutely convex weak ideal.*

Example 2.13. It is well known that, the prime ideals in $C(X)$ containing a given prime ideal form a chain (see [5] and [14]). Let $X = \mathbb{R}$, $I = M_2 \cup M_3$, $P = M_2 \cup M_3 \cup M_4$, and $Q = M_2 \cup M_3 \cup M_5$. Clearly, I, P and Q are prime weak ideals of $C(X)$ and $I \subseteq P$, $I \subseteq Q$, but P, Q are primes which are not in a chain.

Corollary 2.14. *Let I be a prime weak ideal of $C(X)$ and let P and Q be prime ideals of $C(X)$. If $I \subseteq P$ and $I \subseteq Q$, then either $P \subseteq Q$ or $Q \subseteq P$.*

Proof. By Proposition 2.10, there exists a prime ideal P' of $C(X)$ such that $P' \subseteq I$ and hence either $P \subseteq Q$ or $Q \subseteq P$ (see [[5], 14.3(c)]). □

Remark 2.15. Let I be a weak ideal of R. The *radical* (or *nilradical*) of I, denoted by $RadI$, is the weak ideal $\bigcap P$, where the intersection is taken over all prime weak ideals P of R containing I. If the set of prime weak ideals of R containing I is empty, then $RadI$ is defined to be R. Also $RadI = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$.

Proposition 2.16. *Every z-weak ideal of $C(X)$ is an intersection of prime weak ideals of $C(X)$.*

Proof. For every $n \in \mathbb{N}$ and $f \in C(X)$, $Z(f^n) = Z(f)$. Hence if I is any z-weak ideal of $C(X)$, then $f^n \in I$ implies $f \in I$. Hence by Remark 2.15, $I = RadI$ is the intersection of all prime weak ideals of $C(X)$ containing I. □

3. Sum of two z-ideals and sum of two prime ideals

This section is devoted to the study of the smallest z-weak ideal of $C(X)$ containing a given weak ideal of $C(X)$ and the greatest z-weak ideal of $C(X)$ contained in a given weak ideal of $C(X)$. We show that the sum of two z-weak ideals (prime weak ideals) of $C(X)$ is either a z-weak ideal (a prime weak ideal) or is the unit ideal.

It is evident that if I is z-weak ideal (or prime weak ideal) of $C(X)$ then for every $f, g \in C(X)$, $f^2 + g^2 \in I$ implies that $f, g \in I$.

If A and B are subsets of $C(X)$, we put $A + B = \{f + g : f \in A \,\&\, g \in B\}$.

Theorem 3.1. *The sum of two z-weak ideals of $C(X)$ is either a z-weak ideal or is the unit ideal.*

Proof. Let I and J be z-weak ideals of $C(X)$. By Proposition 2.5, $I = \bigcup_{\lambda\in\Lambda} I_\lambda$ and $J = \bigcup_{\gamma\in\Gamma} J_\gamma$, where for every $\lambda \in \Lambda$ and $\gamma \in \Gamma$, I_λ and J_γ are z-ideal of $C(X)$. Since the sum of two z-ideals in $C(X)$ is either a z-ideal or is the unit ideal and $I + J = \bigcup_{\lambda\in\Lambda \& \gamma\in\Gamma}(I_\lambda + J_\gamma)$, $I + J$ is either a z-weak ideal or is the unit ideal. $\square$

For every ideal I in $C(X)$, it is well known that the smallest ideal containing I is $Z^{\leftarrow}[Z[I]] = \{f \in C(X) : Z(f) \in Z[I]\}$ which is in fact the intersection of all z-ideals containing I and it is also denoted by I_z in [10]. In the notation of Mason in the same reference, for a given ideal I in $C(X)$, the largest z-ideal contained in I is also represented by I^z which is in fact the sum of all z-ideals contained in I. Topological and algebraic characterizations of I_z and I^z are given in [2] by $I_z = \{g \in C(X) : Z(f) \subseteq Z(g) \text{ for some } f \in I\}$ and $I^z = \{f \in C(X) : \mathcal{M}_f \subseteq I\}$ respectively. Using these notations and characterizations, for a given proper weak ideal I in $C(X)$, we let:

$$I_{zw} = \{g \in C(X) : Z(f) \subseteq Z(g) \text{ for some } f \in I\},$$

and

$$I^{zw} = \{f \in C(X) : \mathcal{M}_f \subseteq I\}.$$

Thus I_{zw} is the smallest z-weak ideal of $C(X)$ containing I and also I^{zw} is the greatest z-weak ideal of $C(X)$ contained in I.

We can now give some characterizations and some properties of the smallest (greatest) z-weak ideal in $C(X)$ containing (contained in) I, for a weak ideal I of $C(X)$.

Remark 3.2. Clearly, if I and J are proper weak ideals of $C(X)$, then

(1) For every $f, g \in C(X)$, $\mathcal{M}_g \subseteq \mathcal{M}_f$ if and only if $Z(f) \subseteq Z(g)$.
(2) $I_{zw} = \bigcup_{f\in I} \mathcal{M}_f$ and $I^{zw} = \bigcup_{\mathcal{M}_f \subseteq I} \mathcal{M}_f$.
(3) I is a z-weak ideal if and only if $I = I_{zw}$ if and only if $I = I^{zw}$.
(4) I is a z-weak ideal if and only if for every $f \in I$ and $g \in C(X)$, $\mathcal{M}_g \subseteq \mathcal{M}_f$ implies $g \in I$.
(5) If $n \in \mathbb{N}$ and I^n is a z-ideal of $C(X)$, then I is a z-ideal of $C(X)$ and $I^n = I$.
(6) For every $n \in \mathbb{N}$, $(I^n)_{zw} = I_{zw}$ and $(I^n)^{zw} = I^{zw}$.
(7) If $I \subseteq J$, then $I_{zw} \subseteq J_{zw}$ and $I^{zw} \subseteq J^{zw}$.

(8) $(I \cap J)_{zw} = I_{zw} \cap J_{zw}$ and $(I \cap J)^{zw} = I^{zw} \cap J^{zw}$.
(9) $(I \cup J)_{zw} = I_{zw} \cup J_{zw}$.
(10) If $I + J$ is proper weak ideal of $C(X)$, then $I_{zw} + J_{zw} \subseteq (I + J)_{zw}$ and $I^{zw} + J^{zw} = (I + J)^{zw}$.
(11) If $I_{zw} + J_{zw} \neq C(X)$, then $(I + J)_{zw} = (I_{zw} + J_{zw})_{zw}$.

Remark 3.3. Let I be a z-weak ideal of $C(X)$ and $\emptyset \neq A \subseteq C(X)$. We put $(I : A) = \{g \in C(X) : gA \subseteq I\}$. If we suppose that $f \in (I : A)$, $g \in \mathcal{M}_f$, and $h \in A$, then $gh \in \mathcal{M}_{fh} \subseteq I$, which follows that $gh \in I$. Hence $\mathcal{M}_f \subseteq (I : A)$. Now by Remark 3.2, $(I : A)$ is a z-ideal.

Proposition 3.4. *The following statements are equivalent:*

(1) *If I and J are z-weak ideals of $C(X)$, then $I + J$ is a z-weak ideal.*
(2) *If I and J are proper weak ideals of $C(X)$, then $(I + J)_{zw} = I_{zw} + J_{zw}$.*

Proof. It is clear. □

Proposition 3.5. *Let I be a weak ideal of $C(X)$ and $f \in C(X)$.*

(1) *If $\mathcal{M}_f \subseteq RadI$, then $\mathcal{M}_f \subseteq I$.*
(2) *If J is a z-weak ideal of $C(X)$ and $J \subseteq RadI$, then $J \subseteq I$.*
(3) *$\{\mathcal{M}_f : f \in I\} = \{\mathcal{M}_f : f \in RadI\}$.*
(4) *$(RadI)_{zw} = I_{zw}$ and $(RadI)^{zw} = I^{zw}$.*
(5) *I is a z-weak ideal if and only if $RadI$ is a z-weak ideal.*

Proof. (1) See [[2], Proposition 2.1].
(2) By Remark 3.2, $J = \bigcup_{f \in J} \mathcal{M}_f$, and in view of part (1), $J \subseteq I$.
(3) It is clear, Since for every $n \in \mathbb{N}$, $\mathcal{M}_f = \mathcal{M}_{f^n}$.
(4) It is obvious, by part (3).
(5) It is trivial, by Remark 3.2, and in view of part (4). □

If I is a proper weak ideal of R, then by Zorn's Lemma, there is a prime weak ideal P of R which is minimal member with respect to inclusion in

$$\{Q : Q \text{ is prime weak ideal of } R \text{ and } I \subseteq Q\}.$$

Such a minimal member is called a minimal prime weak ideal of I. Let $MPW(I)$ denotes the set of minimal prime weak ideals of I in R. If I is a proper weak ideal of R, then $RadI = \bigcap_{P \in MPW(I)} P$.

It is well known that if I is a z-ideal of $C(X)$ and $P \in Min(I)$, then P is a z-ideal of $C(X)$ (See [[5], p. 197] and [[10], Theorem 1.1]). The converse is also true, see in [[2], Corollary 2.5] and [[13], Corollary 2.5]. Similarly, we have:

Corollary 3.6. *Let I be a weak ideal of $C(X)$. I is a z-weak ideal if and only if every $P \in MPW(I)$ is a z-weak ideal.*

Proof. Let P be a prime weak ideal of $C(X)$ and be a z-weak ideal $I \subseteq P$. Suppose that P is not z-weak ideal, then there exist $f \in P$ and $g \in C(X) \setminus P$, such that $Z(f) = Z(g)$. Put

$$S = (C(X) \setminus P) \cup \{hf^n : h \in C(X) \setminus P \,\&\, n \in \mathbb{N}\}.$$

It is clear that S is multiplicative set and $S \cap I = \emptyset$. By Remark 2.9, there exists a prime weak ideal Q of $C(X)$ such that $S \cap Q = \emptyset$ and $I \subseteq Q$. It is manifest that $I \subseteq Q \subseteq P$ and $f \in P \setminus Q$, it follows that $P \notin MPW(I)$.

Conversly, let every prime weak ideal minimal over I be a z-ideal. Since $RadI = \bigcap_{P \in MPW(I)} P$, we conclude that $RadI$ is a z-weak ideal and hence by Proposition 3.5, I is also a z-weak ideal. □

By Corollary 3.6, it is clear that every z-weak ideal of $C(X)$ is an intersection of prime z-weak ideals of $C(X)$. Also since (0) is a z-ideal of $C(X)$, every prime minimal weak ideal of $C(X)$ is z-weak ideal.

The following proposition is a counterpart of Proposition 2.8 in [2].

Proposition 3.7. *Let I be a weak ideal in $C(X)$ and let P and Q be prime weak ideals of $C(X)$.*

(1) *If $I \cap P$ is a z-weak ideal, then either I is a z-weak ideal or P is a z-weak ideal.*
(2) *If $\{P, Q\}$ is not chain with respect to inclusion and $P \cap Q$ is a z-weak ideal, then P and Q are z-weak ideals.*

Proof. (1) If $I \subseteq P$, then $I = I \cap P$ is a z-weak ideal. Now we may assume that $I \not\subseteq P$ and $g \in I \setminus P$. Let $f \in P$. We show that $\mathcal{M}_f \subseteq P$. If $h \in \mathcal{M}_f$, then $hg \in \mathcal{M}_{fg}$. Since $fg \in I \cap P$and $I \cap P$ is a z-ideal, then $hg \in \mathcal{M}_{fg} \subseteq I \cap P$, thus $h \in P$. Hence $P = \bigcup_{f \in P} \mathcal{M}_f$, i.e.; P is a z-weak ideal of $C(X)$.

(2) It is clear. □

Proposition 3.8. *If P is a prime weak ideal of $C(X)$ which is not a z-weak ideal, then*

$$\mathcal{A} = \{I \subseteq P : I \textit{is a } z\textit{-weak ideal of } C(X)\}$$

has maximal element with respect to inclusion and every maximal element of $\mathcal{A}$ is a prime weak ideal of $C(X)$. In particular, if P is a prime weak ideal of $C(X)$, then P^{zw} is a prime weak ideal.

Proof. Clearly, $(0) \in \mathcal{A}$, so by Zorn's Lemma, $\mathcal{A}$ have maximal element. Let $I \in \mathcal{A}$ be a maximal element. By hypotheses $I \subset P$, hence there exists $Q \in MPW(I)$ such that $I \subseteq Q \subseteq P$. By Corollary 3.6, Q is a z-weak ideal and $Q \subset P$, thus $Q = I$ and the proof is complete. □

We need the following lemma which is proved in [13].

Lemma 3.9. *For any $f_1, ..., f_n \in C(X)$, there exists $g \in C(X)$ such that any natural power of g divides every f_i and $Z(g) = Z(f_1) \cap \cdots \cap Z(f_n)$.*

Proposition 3.10. *If I is a proper ideal of $C(X)$, then I^{zw} is a z-ideal of $C(X)$. In particular, if P is a prime ideal of $C(X)$, then P^{zw} is prime ideal.*

Proof. Let $g, h \in I^{zw}$. By Remark 3.2, $I^{zw} = \bigcup_{\mathcal{M}_f \subseteq I} \mathcal{M}_f$, it follows that there exist $g_1, h_1 \in C(X)$ such that $g \in \mathcal{M}_{g_1} \subseteq I$ and $h \in \mathcal{M}_{h_1} \subseteq I$. Since $Z(g_1^2 + h_1^2) \subseteq Z(g^2 + h^2)$, we can then conclude from the Lemma 1.1 in [2] that $g^2 + h^2 \in \mathcal{M}_{g_1^2+h_1^2} = \mathcal{M}_{g_1} + \mathcal{M}_{h_1} \subseteq I$, hence $g^2 + h^2 \in I^{zw}$. Also, by Lemma 3.9, there exists $f, g_2, h_2 \in C(X)$ such that $g = fg_2$, $h = fh_2$ and $Z(f) = Z(g) \cap Z(h) = Z(g^2 + h^2)$. Since I^{zw} is z-weak ideal of $C(X)$, we conclude that $f \in I^{zw}$, and this follows that $g + h = f(g_1 + h_1) \in I^{zw}$. Therefore I^{zw} is a z- ideal of $C(X)$.

whenever P is a prime ideal of $C(X)$, then by Proposition 3.8, P^{zw} is a prime ideal. □

Proposition 3.11. *If I is a proper ideal of $C(X)$, then I_{zw} is a z-ideal of $C(X)$.*

Proof. Let $g, h \in I_{zw}$. By Remark 3.2, $I_{zw} = \bigcup_{f \in I} \mathcal{M}_f$, so there exist $f_1, f_2 \in I$ such that $Z(f_1) \subseteq Z(g)$ and $Z(f_2) \subseteq Z(h)$. Hence $Z(f_1^2 + f_2^2) \subseteq Z(g^2 + h^2)$. Since I is a proper ideal of $C(X)$, we conclude that $f_1^2 + f_2^2 \in I$. By Remark 3.2, $g^2 + h^2 \in \mathcal{M}_{f_1^2+f_2^2} \subseteq I_{zw}$, which implies that $(g + h)^2 \in I_{zw}$. Therefore $Z(g) \cap Z(h) = Z(g^2 + h^2) \in Z[I_{zw}]$. On the other hand, by Lemma 3.9, there exists $f, g_1, h_1 \in C(X)$ such that $g = fg_1$, $h = fh_1$ and $Z(f) = Z(g) \cap Z(h)$. Since I_{zw} is z-weak ideal of $C(X)$, we conclude that $f \in I_{zw}$, so $g + h = f(g_1 + h_1) \in I_{zw}$. Therefore I_{zw} is a z- ideal of $C(X)$. □

Proposition 3.12. *If P is a prime ideal of $C(X)$, then P_{zw} is a prime ideal of $C(X)$.*

Proof. By Propositions 3.11, P_{zw} is an ideal of $C(X)$. Let for some $f, g \in C(X)$, $fg \in P_{zw}$. Put $h = |g| - |f|$. It is clear that $(h \wedge 0)(h \vee 0) = 0 \in P$. This follows that $(h \wedge 0) \in P$ or $(h \vee 0) \in P$. If $(h \wedge 0) \in P$, then $Z(h \wedge 0) \cap Z(fg) \subseteq Z(f)$ and $Z(h \wedge 0) \cap Z(fg) \in Z(P_{zw})$. Since P_{zw} is a z-ideal of $C(X)$, we conclude that $f \in P_{zw}$. Similarly, if $(h \vee 0) \in P$, then $g \in P_{zw}$ which completes the proof. □

Proposition 3.13. *If P is a prime weak ideal of $C(X)$, then P_{zw} is a prime weak ideal of $C(X)$.*

Proof. By Proposition 2.10, there exists $\{P_\lambda\}_{\lambda \in \Lambda} \subseteq Spec(C(X))$ such that $P = \bigcup_{\lambda \in \Lambda} P_\lambda$. Now, by Remark 3.2, we have $P_{zw} = \bigcup_{f \in P} \mathcal{M}_f = \bigcup_{f \in \bigcup_{\lambda \in \Lambda} P_\lambda} \mathcal{M}_f = \bigcup_{\lambda \in \Lambda} \bigcup_{f \in P_\lambda} \mathcal{M}_f = \bigcup_{\lambda \in \Lambda} (P_\lambda)_{zw}$. By Proposition 3.12, P_{zw} is a prime weak ideal of $C(X)$. □

Example 3.14. If $P = \{f \in C(\mathbb{R}) : f(2)f(3) = 0\}$, then P is prime weak ideal of $C(\mathbb{R})$ and $P = P_{zw}$ is not C-weak ideal.

We say that a proper weak ideal Q of $C(X)$ is a *primary weak ideal*, if $RadQ$ is a prime weak ideal of $C(X)$ and if $RadQ = P$, then Q is said to be P-primary weak ideal. The following proposition is a counterpart of Proposition 2.8 in [2].

Proposition 3.15. *Let I be a weak ideal in $C(X)$ and let Q and Q' be respectively P-primary and $P'-$primary weak ideals of $C(X)$.*

1. *If $I \cap Q$ is a z-weak ideal, then either $RadI = I$ is a z-weak ideal or $RadQ = P$ is a z-weak ideal.*
2. *Q^{zw} is a prime weak ideal.*
3. *If $\{Q, Q'\}$ is not chain with respect to inclusion and $Q \cap Q'$ is a z-weak ideal, then Q and Q' are prime z-weak ideals.*

Proof. (1) Since by Proposition 3.5, $Rad(I \cap Q) = Rad(I) \cap Rad(Q) = Rad(I) \cap P$ is a z-weak ideal, then by Proposition 3.7, $RadI = I$ is a z-weak ideal or $RadQ = P$ is a z-weak ideal.

(2) Since $Q^{zw} = (RadQ)^{zw} = P^{zw}$, we conclude from Proposition 3.8 that Q^{zw} is a prime weak ideal of $C(X)$.

(3) Since $Q \cap Q' = Rad(Q \cap Q') = Rad(Q) \cap Rad(Q') = P \cap P'$ is z-weak ideal and $\{P, P'\}$ is not chain with respect to the inclusion, then by Proposition 3.7, and Proposition 3.5, Q and Q' are prime z-weak ideals.

□

The following proposition is a counterpart of $14B(1)$ in [5].

Proposition 3.16. *The sum of two prime weak ideals of $C(X)$ is either a prime weak ideal or is the unit ideal.*

Proof. Let P and Q be prime weak ideals of $C(X)$. By Proposition 2.10, $P = \bigcup_{\lambda \in \Lambda} P_\lambda$ and $Q = \bigcup_{\gamma \in \Gamma} Q_\gamma$, where for every $\lambda \in \Lambda$ and $\gamma \in \Gamma$, P_λ and Q_γ are prime ideal of $C(X)$. Since the sum of two prime ideals in $C(X)$ is either prime ideal or is the unit ideal and $P + Q = \bigcup_{\lambda \in \Lambda \& \gamma \in \Gamma} (P_\lambda + Q_\gamma)$, we conclude that $P + Q$ is either prime weak ideal or is the unit ideal by Problem $14B(1)$ in [5]. □

A space X is called *P-space* if each finitely generated ideal of $C(X)$ is a direct summand. Clearly, X is a P-space if and only if $C(X)$ is a regular ring or equivalently if each G_δ set is open, see [5], 4J.

Proposition 3.17. *The following statements are equivalent:*

1. *X is a P-space.*
2. *Every prime weak ideal of $C(X)$ is a union of maximal ideals of $C(X)$.*
3. *Every weak ideal of $C(X)$ is a z-weak ideal.*
4. *Every C-weak ideal of $C(X)$ is a z-weak ideal.*

Proof. By Theorem 14.29 in [5], and Proposition 2.10, the proof is clear. □

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