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Weakly $g(x)$-Clean Rings

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Abstract. A ring R with identity is called "clean" if for every element $a \in R$, there exist an idempotent e and a unit u in R such that $a = u + e$. Let $C(R)$ denote the center of a ring R and $g(x)$ be a polynomial in $C(R)[x]$. An element $r \in R$ is called "g(x)-clean" if $r = u + s$ where $g(s) = 0$ and u is a unit of R and R is $g(x)$-clean if every element is $g(x)$-clean. In this paper we define a ring to be weakly $g(x)$-clean if each element of R can be written as either the sum or difference of a unit and a root of $g(x)$.

Keywords: Clean ring, $g(x)$-clean ring, Weakly $g(x)$-clean ring.

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1. Introduction

Throughout this note, R is an associative ring with identity. A ring R is called clean if for every element $a \in R$, there exist an idempotent e and a unit u in R such that $a = e + u$ [9] and R is called strongly clean if, in addition, $eu = ue$ [10].
Let $C(R)$ denote the center of a ring R and $g(x)$ be a polynomial in $C(R)[x]$. Following Camillo and Simon [2], an element $r \in R$ is called $g(x)$-clean if $r = u + s$ where $g(s) = 0$ and u is a unit of R, and R is $g(x)$-clean if every element in R is $g(x)$-clean. It is clear that the $(x^2 - x)$- clean rings are precisely

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the clean rings.
Camillo and Simon [2] proved that if V is a countable dimensional vector space over a division ring D and $g(x)$ is any polynomial with coefficients in $K = C(D)$ and two distinct roots in K, then $End(V_D)$ is $g(x)$-clean. Nicholson and Zhou [11] generalized Camillo and Simon's result by proving that $End({}_RM)$ is $g(x)$-clean where ${}_RM$ is a semisimple left R-module and $g(x) \in (x-a)(x-b)C(R)[x]$ with $a, b \in C(R)$ and $b, b-a \in U(R)$. $g(x)$-clean rings have also been studied in [3], [7] and [6].
It is easy to see that a ring R is $g(x)$-clean if and only if each $x \in R$ can be written in the form $x = u - s$ where $u \in U(R)$ and $g(s) = 0$. This raises the question of whether a ring with the property that, for each $x \in R$, either $x = u + s$ or $x = u - s$ for some $u \in U(R)$ and $g(s) = 0$ must be cleaned. Let us call rings with this property weakly $g(x)$-clean. Here we study weakly $g(x)$-clean rings and also investigate the general properties of weakly $g(x)$-clean rings which are similar to those of $g(x)$-clean rings. For example we prove the following results:

Proposition 1.1. *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_i$ is weakly $g(x)$-clean if and only if for all $i \in I$, R_i is weakly $g(x)$-clean.*

Theorem 1.2. *Let R be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then R is weakly $g(x)$-clean if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is weakly $g(x)$-clean.*

Theorem 1.3. *Let R be a commutative ring and M an R-module. Let $g(x) \in C(R)[x]$. If R is weakly $g(x)$-clean, then the idealization $R(M)$ of R and M is also weakly $g(x)$-clean.*

In section 3 we consider the weakly $(x^n - x)$-clean rings and weakly 2-clean rings.

An usual, $\mathbb{T}_n(R)$ denotes the upper triangular matrix ring of order n over R; $GL_n(R)$ denotes the general linear group over R; and $gcd(m, n)$ means the greatest common divisor of the integers m and n. All polynomials are in the polynomial ring $C(R)[x]$ and $U(R)$ denotes the multiplicative unit group of R.

2. Weakly $g(x)$-clean rings

In this section first we define the weakly $g(x)$-clean rings, then we explain the relation between weakly $g(x)$-clean and $g(x)$-clean rings.

Definition 2.1. Let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called weakly $g(x)$-clean if $r = u + s$ or $r = u - s$ where $g(s) = 0$ and

$u \in U(R)$. We say that R is weakly $g(x)$-clean if every element is weakly $g(x)$-clean.

Obviously, $g(x)$-clean rings are weakly $g(x)$-clean and also if $g(x)$ is an odd or an even polynomial (i.e $g(-x) = -g(x)$ or $g(-x) = g(x)$), then the concepts $g(x)$-clean and weakly $g(x)$-clean coincide, that is, if R is a weakly $g(x)$-clean ring then R is also $g(x)$-clean. So the interesting case is when $g(x)$ is neither an even nor an odd polynomial. In [1, Proposition 16] it was shown that if R has exactly two maximal ideals and $2 \in U(R)$, then each $x \in R$ has the form $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in \{0, 1\}$. Thus $\mathbb{Z}_{(3)} \bigcap \mathbb{Z}_{(5)}$ is weakly clean but is not clean since an indecomposable clean ring is quasilocal [1, Thedrem 3]. But since weakly $(x^2 - x)$-clean rings are precisely the weakly clean rings, we can say that for $g(x) = x^2 - x$, the ring $\mathbb{Z}_{(3)} \bigcap \mathbb{Z}_{(5)}$ is weakly $g(x)$-clean, but it is not $g(x)$-clean.

The following two examples explain the relations between weakly $g(x)$-clean rings and weakly clean rings.

Example 2.2. Let $R = \mathbb{Z}_{(p)} = \{\frac{m}{n}\ ;\ gcd(p, n) = 1\ \textit{and}\ \text{p}\ \textit{prime}\}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ and $g(x) = (x - a)(x^2 + 1) \in C(R)[x]$. Then R is a weakly clean ring, because local rings are strongly clean, thus R is clean (it is of course weakly clean). But as a is the single root of $g(x)$, R is not a weakly $g(x)$-clean ring.

Example 2.3. Let R be a Boolean ring with the number of elements $|R| > 2$ and $c \in R$ with $0 \neq c \neq 1$. Define $g(x) = (x + 1)(x + c)$. Then R is not weakly $g(x)$-clean.
Because if $c = u \pm s$ where $u \in U(R)$ and g(s)=0, then it must be that $u = 1$ and $s = \pm(c \pm u)$. But, clearly, $g(c + 1) \neq 0$. However, R is certainly weakly clean.

Let R and S be rings and $\theta : C(R) \longrightarrow C(S)$ be a ring homomorphism with $\theta(1) = 1$. Then θ induces a map θ' from C(R)[x] to $C(S)[x]$ such that For $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, $\theta'(g(x)) := \sum_{i=0}^{n} \theta(a_i)x^i \in C(S)[x]$. Clearly, if $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then $\theta'(g(x)) = g(x)$. We give some properties of weakly $g(x)$-clean rings which are similar to those of weakly clean rings.

Proposition 2.4. *Let $\theta : R \longrightarrow S$ be a ring epimorphism. If R is weakly $g(x)$-clean, then S is weakly $\theta'(g(x))$-clean.*

Proof. Let $g(x) = a_0 + a_1 x + ... + a_n x^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + ... + \theta(a_n)x^n \in C(S)[x]$. As θ is a ring epimorphism so for any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since R is weakly $g(x)$-clean, there exist $u \in U(R)$ and $s_0 \in R$ such that $r = u \pm s_0$ and $g(s_0) = 0$. Then $s = \theta(r) = \theta(u \pm s_0) = \theta(u) \pm \theta(s_0)$ with $\theta(u) \in U(S)$. But $\theta'(g(\theta(s_0))) = \theta(a_0) + \theta(a_1)\theta(s_0) + ... + \theta(a_n)\theta(s_0^n) = \theta(a_0 + a_1 s_0 + ... + a_n s_0^n) = \theta(g(s_0)) = \theta(0) = 0$, we have s is weakly $\theta'(g(x))$-clean. Therefore S is weakly $\theta'(g(x))$-clean.

□

Corollary 2.5. *If R is weakly $g(x)$-clean, then for any ideal I of R, R/I is weakly $\bar{g}(x)$-clean where $\bar{g}(x) \in C(R/I)[x]$.*

Proposition 2.6. *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i\in I}$ be a family of rings. Then $\prod_{i\in I} R_i$ is weakly $g(x)$-clean if and only if for all $i \in I$, R_i is weakly $g(x)$-clean.*

Proof. Let $\prod_{i\in I} R_i$ be a weakly $g(x)$-clean. Define $\pi_j : \prod_{i\in I} R_i \longrightarrow R_j$ by $\pi_j(\{a_i\}_{i\in I}) = a_j$. Since for all $j \in I$, π_j is a ring epimorphism, so by Proposition 2, for every $i \in I$, each R_i is a weakly $g(x)$-clean ring.
For the converse, let $x = \{x_i\}_{i\in I} \in R = \prod_{i\in I} R_i$. In R_{i_0}, we can write $x_i = u_{i_0} + s_{i_0}$ or $x_i = u_{i_0} - s_{i_0}$ where $u_{i_0} \in U(R_{i_0})$ and $g(s_{i_0}) = 0$. If $x_{i_0} = u_{i_0} + s_{i_0}$, for $i \neq i_0$, let $x_i = u_i + s_i$ where $u_i \in U(R_i)$, $g(s_i) = 0$; while if $x_{i_0} = u_{i_0} - s_{i_0}$, for $i \neq i_0$, let $x_i = u_i - s_i$ where $u_i \in U(R_i)$, $g(s_i) = 0$. Then $u = \{u_i\}_{i\in I} \in U(R)$ and

$$\begin{aligned} g(s = \{s_i\}_{i\in I}) &= a_0\{1_{R_i}\}_{i\in I} + a_1\{s_i\}_{i\in I} + ... + a_n\{s_i^n\}_{i\in I} \\ &= \{a_0\}_{i\in I} + \{a_1 s_i\}_{i\in I} + ... + \{a_n s_i^n\}_{i\in I} \\ &= \{a_0 + a_1 s_i + ... + a_n s_i^n\}_{i\in I} \\ &= \{g(s_i)\}_{i\in I} = 0 \end{aligned}$$

That is, $\prod_{i\in I} R_i$ is weakly g(x)-clean. □

Define $\pi_n : C(R) \longrightarrow M_n(R)$ by $a \longmapsto aI_n$ with I_n being the identity matrix of $M_n(R)$ and $a \in C(R)$. Then $M_n(R)$ is a $C(R)$-algebra.

Theorem 2.7. *Let R be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then R is weakly $g(x)$-clean if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is weakly $g(x)$-clean.*

Proof. Let R be weakly $g(x)$-clean and $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ for $1 \leq j < i \leq n$. Since R is weakly $g(x)$-clean, for any $1 \leq i \leq n$, there exist $s_{ii} \in R$ and $u_{ii} \in U(R)$ such that $a_{ii} = u_{ii} \pm s_{ii}$ with $g(s_{ii}) = 0$. So we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} u_{11} \pm s_{11} & a_{12} & \dots & a_{1n} \\ 0 & u_{22} \pm s_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \pm s_{nn} \end{bmatrix}$$

In R for any $0 \leq i \leq n$, we can write $a_{ii} = u_{ii} + s_{ii}$ or $a_{ii} = u_{ii} - s_{ii}$ where $u_{ii} \in U(R)$ and $g(s_{ii}) = 0$. If $a_{ii} = u_{ii} + s_{ii}$ for $j \neq i$, let $a_{jj} = u_{jj} + s_{jj}$ where $(u_{jj} \in U(R),\ g(s_{jj}) = 0)$; while if $a_{ii} = u_{ii} - s_{ii}$, for $j \neq i$, let $a_{jj} = u_{jj} - s_{jj}$ such that $(u_{jj} \in U(R),\ g(s_{jj}) = 0)$. Then by elementary row and column operations we can see that,

$$U = \begin{bmatrix} u_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & u_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix} \in GL_n(R).$$

Suppose $g(x) = \sum_{i=0}^{m} a_i x^i \in C(R)[x]$, then

$$g(S = \begin{bmatrix} s_{11} & 0 & \dots & 0 \\ 0 & s_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_{nn} \end{bmatrix}) = a_0 I_n + a_1 S + ... + a_n S^n$$

$$= \begin{bmatrix} a_0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 \end{bmatrix} + \begin{bmatrix} a_1 s_{11} & 0 & \dots & 0 \\ 0 & a_1 s_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 s_{nn} \end{bmatrix} + ...$$

$$+ \begin{bmatrix} a_m s_{11}^m & 0 & \dots & 0 \\ 0 & a_m s_{22}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m s_{nn}^m \end{bmatrix}$$

$$= \begin{bmatrix} g(s_{11}) & 0 & \dots & 0 \\ 0 & g(s_{22}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g(s_{nn}) \end{bmatrix} = 0.$$

So $\mathbb{T}_n(R)$ is weakly $g(x)$-clean.

Now let $\mathbb{T}_n(R)$ be weakly $g(x)$-clean. Define $\theta : \mathbb{T}_n(R) \longrightarrow R$ by $\theta(A) = a_{11}$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$. Then θ is a ring epimorphism. For any $a \in R$, let B be the diagonal matrix diag$(a, ...a)$. Then $a = \theta(B) = \theta(U \pm S) = \theta(U) \pm \theta(S)$ where $U \in GL_n(R)$ and

$$\begin{aligned} g(\theta(S)) &= a_0 + a_1\theta(S) + ... + a_n\theta(S^n) \\ &= \theta(B_0) + \theta(B_1)\theta(S) + ... + \theta(B_n)\theta(S^n) \\ &= \theta(B_0 + B_1 S + ... + B_n S^n) \\ &= \theta(a_0 I_n + (a_1 I_n)S + ... + (a_n I_n)S^n) \\ &= \theta(g(S)) = 0. \end{aligned}$$

Thus a is weakly $g(x)$-clean, i.e., R is a weakly $g(x)$-clean ring.

□

Remark 2.8. Let R be a ring with identity, then the following hold:

(1) $f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a unit if and only if a_0 is a unit of R.

(2) $U(R[t]) = \{r_0 + r_1 t + ... + r_n t^n \mid r_0 \in U(R),\ r_i \in \sqrt{(0)}\ for\ i = 0, 1, ..., n\}$

Proposition 2.9. *Let R be a ring and $g(x) \in C(R)[x]$. Then the formal power series ring $R[[t]]$ is weakly $g(x)$-clean if and only if R is weakly $g(x)$-clean.*

Proof. Let R be weakly $g(x)$-clean and $f = \sum_{i \geq 0} a_i t^i \in R[[t]]$. Since R is weakly $g(x)$-clean, $a_0 = u \pm s$ for some $s \in R$ and $u \in U(R)$ and $g(s) = 0$. Then $f = (u + \sum_{i \geq 1} a_i t^i) \pm s$. By Remark 6, $u + \sum_{i \geq 1} a_i t^i \in U(R[[t]])$. So f is weakly $g(x)$-clean, i.e., $R[[t]]$ is weakly $g(x)$-clean.
For the converse, let $R[[t]]$ be weakly $g(x)$-clean. Since $\theta : R[[t]] \longrightarrow R$ with $\theta(f) = a_0$ is a ring epimorphism where $f = \sum_{i \geq 0} a_i t^i \in R[[t]]$, by Proposition 2, R is weakly $g(x)$-clean.

□

Remark 2.10. Generally, the polynomial ring $R[t]$ is not weakly $g(x)$-clean for non-zero polynomial $g(x) \in C(R)[x]$. For example let R be a commutative ring and also let $g(x) = x$, we show that t is not weakly $g(x)$-clean. If $t = u \pm s$ then it must be that $s = 0$ and so $t = u$. But, by Remark 6, clearly $t \notin U(R[t])$, i.e., $R[t]$ is not weakly $g(x)$-clean.

For more examples of weakly $g(x)$-clean rings, we consider the method of idealization. Let R be a commutative ring and M an R-module. The idealization of R and M is the ring $R(M) = R \bigoplus M$ with product $(r, m)(r', m') = (rr', rm' + r'm)$ and addition $(r, m)(r', m') = (r + r', m + m')$.

Theorem 2.11. *Let R be a commutative ring, M an R-module and $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. If R is a weakly $g(x)$-clean ring, then the idealization $R(M)$ of R and M is weakly $g(x)$-clean.*

Proof. Let $(r, m) \in R(M)$. Since R is a weakly $g(x)$-clean ring, we have $r = u \pm s$ where $u \in U(R)$ and $g(s) = 0$. So $(r, m) = (u \pm s, m) = (u, m) \pm (s, 0)$. We have $(u, m)(u^{-1}, -u^{-1}mu^{-1}) = (uu^{-1}, u(-u^{-1}mu^{-1}) + mu^{-1}) = (1, 0)$. Therefore $(u, m) \in U(R(M))$. Also we have

$$\begin{aligned} g((s,0)) &= a_0(1,0) + a_1(s,0) + ... + a_n(s,0)^n \\ &= a_0(1,0) + a_1(s,0) + ... + a_n(s^n,0) \\ &= (a_0,0) + (a_1 s,0) + ... + (a_n s^n,0) \\ &= (a_0 + a_1 s + ... + a_n s^n, 0) = (g(s),0) = (0,0). \end{aligned}$$

Thus (r, m) is weakly $g(x)$-clean and so $R(M)$ is a weakly $g(x)$-clean ring. □

3. WEAKLY $(x^n - x)$-CLEAN RINGS

In this section we consider the weakly $(x^n - x)$-clean rings and weakly 2-clean rings.

Theorem 3.1. *Let R be a ring, $n \in \mathbb{N}$ and $a, b \in R$. Then R is weakly $(ax^{2n} - bx)$-clean if and only if R is weakly $(ax^{2n} + bx)$-clean.*

Proof. Suppose R is weakly $(ax^{2n} - bx)$-clean. Then for any $r \in R$, $-r = u \pm s$ where $(as^{2n} - bs) = 0$ and $u \in U(R)$. So $r = (-u) \pm (-s)$ where $(-u) \in U(R)$ and $a(-s)^{2n} + b(-s) = 0$. Hence, r is weakly $(ax^{2n} + bx)$-clean. Therefore, R is weakly c$(ax^{2n} + bx)$-clean. Now suppose R is weakly $(ax^{2n} + bx)$-clean. Let $r \in R$. Then there exist s and u such that $-r = u \pm s$, $as^{2n} + bs = 0$ and $u \in U(R)$. So $r = (-u) \pm (-s)$ satisfies $(as^{2n} - bs) = 0$. Hence, R is weakly $(ax^{2n} - bx)$-clean.

□

Proposition 3.2. *Let $2 \leq n \in \mathbb{N}$. If for every $a \in R$, $a = u \pm v$ where $u \in U(R)$ and $v^{n-1} = 1$, then R is weakly $(x^n - x)$-clean.*

The following Lemma is well-known.

Lemma 3.3. *Let $a \in R$. The following statements are equivalent for $n \geq 1$:*
(1) $a = a(ua)^n$ *for some* $u \in U(R)$*;*
(2) $a = ve$ *for some* $e^{n+1} = e$ *and some* $v \in U(R)$*;*
(3) $a = fw$ *for some* $f^{n+1} = f$ *and some* $w \in U(R)$.

Proof. See Lemma 4.3 of [3]. □

Proposition 3.4. *Let R be a weakly $(x^n - x)$-clean ring where $n \geq 2$ and $a \in R$. Then either (i) $a = u \pm v$ where $u \in U(R)$ and $v^{n-1} = 1$ or (ii) both aR and Ra contain nontrivial idempotents.*

Proof. Since R is weakly $(x^n - x)$-clean, write $a = u \pm e$ where $u \in U(R)$ and $e^n = e$. Then $ae^{n-1} = ue^{n-1} \pm e$. So $a(1 - e^{n-1}) = u(1 - e^{n-1})$. Since $1 - e^{n-1}$ is an idempotent, by Lemma 12, $u(1 - e^{n-1}) = fw$ where $w \in U(R)$ and $f^2 = f \in R$. So $f = a(1 - e^{n-1})w^{-1} \in aR$. Suppose (i) does not hold. Then $1 - e^{n-1} \neq 0$, hence $f \neq 0$. Thus, aR contains a nontrivial idempotent. Similarly, Ra contains a nontrivial idempotent. □

Definition 3.5. An element $r \in R$ is called weakly n-clean if $r = u_1 + u_2 + ... + u_n \pm e$ with $e^2 = e \in R$ and $u_i \in U(R)$ for $1 \leq i \leq n$ and R is called weakly n-clean if every element of R is weakly n-clean.

Definition 3.6. An element $a \in R$ is called right π-regular if it satisfies the following equivalent conditions,
(1) $a^n \in a^{n+1}R$ for some integer $n \geq 1$;
(2) $a^nR = a^{n+1}R$ for some integer $n \geq 1$;
(3) The chain $aR \supseteq a^2R \supseteq$ ...terminates.

The left π-regular elements are defined analogously. An element $a \in R$ is called strongly π-regular if it is both left and right π-regular, and R is called strongly π-regular if every element is strongly π-regular [10].

Proposition 3.7. *Let $n \in \mathbb{N}$, if the ring R is weakly $(x^n - x)$-clean, then R is weakly 2-clean.*

Proof. Let $r \in R$. Then $r = u \pm t$ for some $t^n = t \in R$ and $u \in U(R)$. Since t is a strongly π-regular element and strongly π-regular elements are strongly clean [10] (it is of course clean and weakly clean), $t = v \pm e$ for some $e^2 = e$ and

$v \in U(R)$. Then $r = u \pm v \pm e$ is weakly 2-clean. Hence, R is weakly 2-clean. □

In fact, all weakly $(x^2 - x)$-clean rings and weakly $(x^2 + cx + d)$-clean rings with $d \in U(R)$ discussed above, are weakly 2-clean rings.

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