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# Weakly $g(x)$-Clean Rings

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Abstract. A ring $R$ with identity is called "clean" if for every element $a \in R$, there exist an idempotent $e$ and a unit $u$ in $R$ such that $a = u + e$. Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial in $C(R)[x]$. An element $r \in R$ is called "g(x)-clean" if $r = u + s$ where $g(s) = 0$ and $u$ is a unit of $R$ and $R$ is $g(x)$-clean if every element is $g(x)$-clean. In this paper we define a ring to be weakly $g(x)$-clean if each element of $R$ can be written as either the sum or difference of a unit and a root of $g(x)$.

**Keywords:** Clean ring, $g(x)$-clean ring, Weakly $g(x)$-clean ring.

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## 1. Introduction

Throughout this note, $R$ is an associative ring with identity. A ring $R$ is called clean if for every element $a \in R$, there exist an idempotent $e$ and a unit $u$ in $R$ such that $a = e + u$ [9] and $R$ is called strongly clean if, in addition, $eu = ue$ [10].
Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial in $C(R)[x]$. Following Camillo and Simon [2], an element $r \in R$ is called $g(x)$-clean if $r = u + s$ where $g(s) = 0$ and $u$ is a unit of $R$, and $R$ is $g(x)$-clean if every element in $R$ is $g(x)$-clean. It is clear that the $(x^2 - x)$- clean rings are precisely

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the clean rings.
Camillo and Simon [2] proved that if $V$ is a countable dimensional vector space over a division ring $D$ and $g(x)$ is any polynomial with coefficients in $K = C(D)$ and two distinct roots in $K$, then $End(V_D)$ is $g(x)$-clean. Nicholson and Zhou [11] generalized Camillo and Simon's result by proving that $End(_RM)$ is $g(x)$-clean where $_RM$ is a semisimple left $R$-module and $g(x) \in (x-a)(x-b)C(R)[x]$ with $a, b \in C(R)$ and $b, b-a \in U(R)$. $g(x)$-clean rings have also been studied in [3], [7] and [6].
It is easy to see that a ring $R$ is $g(x)$-clean if and only if each $x \in R$ can be written in the form $x = u - s$ where $u \in U(R)$ and $g(s) = 0$. This raises the question of whether a ring with the property that, for each $x \in R$, either $x = u + s$ or $x = u - s$ for some $u \in U(R)$ and $g(s) = 0$ must be cleaned. Let us call rings with this property weakly $g(x)$-clean. Here we study weakly $g(x)$-clean rings and also investigate the general properties of weakly $g(x)$-clean rings which are similar to those of $g(x)$-clean rings. For example we prove the following results:

**Proposition 1.1.** *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i\in I} R_i$ is weakly $g(x)$-clean if and only if for all $i \in I$, $R_i$ is weakly $g(x)$-clean.*

**Theorem 1.2.** *Let $R$ be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then $R$ is weakly $g(x)$-clean if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is weakly $g(x)$-clean.*

**Theorem 1.3.** *Let $R$ be a commutative ring and $M$ an $R$-module. Let $g(x) \in C(R)[x]$. If $R$ is weakly $g(x)$-clean, then the idealization $R(M)$ of $R$ and $M$ is also weakly $g(x)$-clean.*

In section 3 we consider the weakly $(x^n - x)$-clean rings and weakly 2-clean rings.

An usual, $\mathbb{T}_n(R)$ denotes the upper triangular matrix ring of order $n$ over $R$; $GL_n(R)$ denotes the general linear group over $R$; and $gcd(m, n)$ means the greatest common divisor of the integers $m$ and $n$. All polynomials are in the polynomial ring $C(R)[x]$ and $U(R)$ denotes the multiplicative unit group of $R$.

## 2. Weakly $g(x)$-clean rings

In this section first we define the weakly $g(x)$-clean rings, then we explain the relation between weakly $g(x)$-clean and $g(x)$-clean rings.

**Definition 2.1.** Let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called weakly $g(x)$-clean if $r = u + s$ or $r = u - s$ where $g(s) = 0$ and

$u \in U(R)$. We say that $R$ is weakly $g(x)$-clean if every element is weakly $g(x)$-clean.

Obviously, $g(x)$-clean rings are weakly $g(x)$-clean and also if $g(x)$ is an odd or an even polynomial (i.e $g(-x) = -g(x)$ or $g(-x) = g(x)$ ), then the concepts $g(x)$-clean and weakly $g(x)$-clean coincide, that is, if $R$ is a weakly $g(x)$-clean ring then $R$ is also $g(x)$-clean. So the interesting case is when $g(x)$ is neither an even nor an odd polynomial. In [1, Proposition 16] it was shown that if $R$ has exactly two maximal ideals and $2 \in U(R)$, then each $x \in R$ has the form $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in \{0, 1\}$. Thus $\mathbb{Z}_{(3)} \bigcap \mathbb{Z}_{(5)}$ is weakly clean but is not clean since an indecomposable clean ring is quasilocal [1, Thedrem 3]. But since weakly $(x^2 - x)$-clean rings are precisely the weakly clean rings, we can say that for $g(x) = x^2 - x$, the ring $\mathbb{Z}_{(3)} \bigcap \mathbb{Z}_{(5)}$ is weakly $g(x)$-clean, but it is not $g(x)$-clean.

The following two examples explain the relations between weakly $g(x)$-clean rings and weakly clean rings.

**Example 2.2.** Let $R = \mathbb{Z}_{(p)} = \{\frac{m}{n}\ ;\ gcd(p, n) = 1 \text{ and p } prime\}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ and $g(x) = (x - a)(x^2 + 1) \in C(R)[x]$. Then $R$ is a weakly clean ring, because local rings are strongly clean, thus $R$ is clean (it is of course weakly clean). But as $a$ is the single root of $g(x)$, $R$ is not a weakly $g(x)$-clean ring.

**Example 2.3.** Let $R$ be a Boolean ring with the number of elements $|R| > 2$ and $c \in R$ with $0 \neq c \neq 1$. Define $g(x) = (x + 1)(x + c)$. Then $R$ is not weakly $g(x)$-clean.
Because if $c = u \pm s$ where $u \in U(R)$ and g(s)=0, then it must be that $u = 1$ and $s = \pm(c \pm u)$. But, clearly, $g(c + 1) \neq 0$. However, $R$ is certainly weakly clean.

Let $R$ and $S$ be rings and $\theta : C(R) \longrightarrow C(S)$ be a ring homomorphism with $\theta(1) = 1$. Then $\theta$ induces a map $\theta'$ from C(R)[x] to $C(S)[x]$ such that For $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, $\theta'(g(x)) := \sum_{i=0}^{n} \theta(a_i)x^i \in C(S)[x]$. Clearly, if $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then $\theta'(g(x)) = g(x)$. We give some properties of weakly $g(x)$-clean rings which are similar to those of weakly clean rings.

**Proposition 2.4.** *Let $\theta : R \longrightarrow S$ be a ring epimorphism. If $R$ is weakly $g(x)$-clean, then $S$ is weakly $\theta'(g(x))$-clean.*

*Proof.* Let $g(x) = a_0 + a_1x + ... + a_nx^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + ... + \theta(a_n)x^n \in C(S)[x]$. As $\theta$ is a ring epimorphism so for any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since $R$ is weakly $g(x)$-clean, there exist $u \in U(R)$ and $s_0 \in R$ such that $r = u \pm s_0$ and $g(s_0) = 0$. Then $s = \theta(r) = \theta(u \pm s_0) = \theta(u) \pm \theta(s_0)$ with $\theta(u) \in U(S)$. But $\theta'(g(\theta(s_0))) = \theta(a_0) + \theta(a_1)\theta(s_0) + ... + \theta(a_n)\theta(s_0^n) = \theta(a_0 + a_1 s_0 + ... + a_n s_0^n) = \theta(g(s_0)) = \theta(0) = 0$, we have $s$ is weakly $\theta'(g(x))$-clean. Therefore $S$ is weakly $\theta'(g(x))$-clean.

□

**Corollary 2.5.** *If $R$ is weakly $g(x)$-clean, then for any ideal $I$ of $R$, $R/I$ is weakly $\bar{g}(x)$-clean where $\bar{g}(x) \in C(R/I)[x]$.*

**Proposition 2.6.** *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i\in I}$ be a family of rings. Then $\prod_{i\in I} R_i$ is weakly $g(x)$-clean if and only if for all $i \in I$, $R_i$ is weakly $g(x)$-clean.*

*Proof.* Let $\prod_{i\in I} R_i$ be a weakly $g(x)$-clean. Define $\pi_j : \prod_{i\in I} R_i \longrightarrow R_j$ by $\pi_j(\{a_i\}_{i\in I}) = a_j$. Since for all $j \in I$, $\pi_j$ is a ring epimorphism, so by Proposition 2, for every $i \in I$, each $R_i$ is a weakly $g(x)$-clean ring.
For the converse, let $x = \{x_i\}_{i\in I} \in R = \prod_{i\in I} R_i$. In $R_{i_0}$, we can write $x_i = u_{i_0} + s_{i_0}$ or $x_i = u_{i_0} - s_{i_0}$ where $u_{i_0} \in U(R_{i_0})$ and $g(s_{i_0}) = 0$. If $x_{i_0} = u_{i_0} + s_{i_0}$, for $i \neq i_0$, let $x_i = u_i + s_i$ where $u_i \in U(R_i)$, $g(s_i) = 0$; while if $x_{i_0} = u_{i_0} - s_{i_0}$, for $i \neq i_0$, let $x_i = u_i - s_i$ where $u_i \in U(R_i)$, $g(s_i) = 0$. Then $u = \{u_i\}_{i\in I} \in U(R)$ and

$$\begin{aligned} g(s = \{s_i\}_{i\in I}) &= a_0\{1_{R_i}\}_{i\in I} + a_1\{s_i\}_{i\in I} + ... + a_n\{s_i^n\}_{i\in I} \\ &= \{a_0\}_{i\in I} + \{a_1 s_i\}_{i\in I} + ... + \{a_n s_i^n\}_{i\in I} \\ &= \{a_0 + a_1 s_i + ... + a_n s_i^n\}_{i\in I} \\ &= \{g(s_i)\}_{i\in I} = 0 \end{aligned}$$

That is, $\prod_{i\in I} R_i$ is weakly g(x)-clean. □

Define $\pi_n : C(R) \longrightarrow M_n(R)$ by $a \longmapsto aI_n$ with $I_n$ being the identity matrix of $M_n(R)$ and $a \in C(R)$. Then $M_n(R)$ is a $C(R)$-algebra.

**Theorem 2.7.** *Let $R$ be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then $R$ is weakly $g(x)$-clean if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is weakly $g(x)$-clean.*

*Proof.* Let $R$ be weakly $g(x)$-clean and $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ for $1 \leq j < i \leq n$. Since $R$ is weakly $g(x)$-clean, for any $1 \leq i \leq n$, there exist $s_{ii} \in R$ and $u_{ii} \in U(R)$ such that $a_{ii} = u_{ii} \pm s_{ii}$ with $g(s_{ii}) = 0$. So we have

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}=\begin{bmatrix} u_{11}\pm s_{11} & a_{12} & \dots & a_{1n} \\ 0 & u_{22}\pm s_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn}\pm s_{nn} \end{bmatrix}$$

In $R$ for any $0 \le i \le n$, we can write $a_{ii} = u_{ii} + s_{ii}$ or $a_{ii} = u_{ii} - s_{ii}$ where $u_{ii} \in U(R)$ and $g(s_{ii}) = 0$. If $a_{ii} = u_{ii} + s_{ii}$ for $j \neq i$, let $a_{jj} = u_{jj} + s_{jj}$ where ($u_{jj} \in U(R)$, $g(s_{jj}) = 0$); while if $a_{ii} = u_{ii} - s_{ii}$, for $j \neq i$, let $a_{jj} = u_{jj} - s_{jj}$ such that ($u_{jj} \in U(R)$, $g(s_{jj}) = 0$). Then by elementary row and column operations we can see that,

$$U=\begin{bmatrix} u_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & u_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix} \in GL_n(R).$$

Suppose $g(x) = \sum_{i=0}^{m} a_i x^i \in C(R)[x]$, then

$$g(S=\begin{bmatrix} s_{11} & 0 & \dots & 0 \\ 0 & s_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_{nn} \end{bmatrix}) = a_0 I_n + a_1 S + \dots + a_n S^n$$

$$=\begin{bmatrix} a_0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 \end{bmatrix}+\begin{bmatrix} a_1 s_{11} & 0 & \dots & 0 \\ 0 & a_1 s_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 s_{nn} \end{bmatrix}+\dots$$

$$+\begin{bmatrix} a_m s_{11}^m & 0 & \dots & 0 \\ 0 & a_m s_{22}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m s_{nn}^m \end{bmatrix}$$

$$=\begin{bmatrix} g(s_{11}) & 0 & \dots & 0 \\ 0 & g(s_{22}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g(s_{nn}) \end{bmatrix}=0.$$

So $\mathbb{T}_n(R)$ is weakly $g(x)$-clean.

Now let $\mathbb{T}_n(R)$ be weakly $g(x)$-clean. Define $\theta : \mathbb{T}_n(R) \longrightarrow R$ by $\theta(A) = a_{11}$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$. Then $\theta$ is a ring epimorphism. For any $a \in R$, let $B$ be the diagonal matrix diag$(a, ...a)$. Then $a = \theta(B) = \theta(U \pm S) = \theta(U) \pm \theta(S)$ where $U \in GL_n(R)$ and

$$\begin{aligned} g(\theta(S)) &= a_0 + a_1\theta(S) + ... + a_n\theta(S^n) \\ &= \theta(B_0) + \theta(B_1)\theta(S) + ... + \theta(B_n)\theta(S^n) \\ &= \theta(B_0 + B_1 S + ... + B_n S^n) \\ &= \theta(a_0 I_n + (a_1 I_n)S + ... + (a_n I_n)S^n) \\ &= \theta(g(S)) = 0. \end{aligned}$$

Thus $a$ is weakly $g(x)$-clean, i.e., $R$ is a weakly $g(x)$-clean ring.

□

*Remark* 2.8. Let $R$ be a ring with identity, then the following hold:

(1) $f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a unit if and only if $a_0$ is a unit of $R$.

(2) $U(R[t]) = \{r_0 + r_1 t + ... + r_n t^n \mid r_0 \in U(R),\ r_i \in \sqrt{(0)}\ for\ i = 0, 1, ..., n\}$

**Proposition 2.9.** *Let $R$ be a ring and $g(x) \in C(R)[x]$. Then the formal power series ring $R[[t]]$ is weakly $g(x)$-clean if and only if $R$ is weakly $g(x)$-clean.*

*Proof.* Let $R$ be weakly $g(x)$-clean and $f = \sum_{i\geq 0} a_i t^i \in R[[t]]$. Since $R$ is weakly $g(x)$-clean, $a_0 = u \pm s$ for some $s \in R$ and $u \in U(R)$ and $g(s) = 0$. Then $f = (u + \sum_{i\geq 1} a_i t^i) \pm s$. By Remark 6, $u + \sum_{i\geq 1} a_i t^i \in U(R[[t]])$. So $f$ is weakly $g(x)$-clean, i.e., $R[[t]]$ is weakly $g(x)$-clean.
For the converse, let $R[[t]]$ be weakly $g(x)$-clean. Since $\theta : R[[t]] \longrightarrow R$ with $\theta(f) = a_0$ is a ring epimorphism where $f = \sum_{i\geq 0} a_i t^i \in R[[t]]$, by Proposition 2, $R$ is weakly $g(x)$-clean.

□

*Remark* 2.10. Generally, the polynomial ring $R[t]$ is not weakly $g(x)$-clean for non-zero polynomial $g(x) \in C(R)[x]$. For example let $R$ be a commutative ring and also let $g(x) = x$, we show that $t$ is not weakly $g(x)$-clean. If $t = u \pm s$ then it must be that $s = 0$ and so $t = u$. But, by Remark 6, clearly $t \notin U(R[t])$, i.e., $R[t]$ is not weakly $g(x)$-clean.

For more examples of weakly $g(x)$-clean rings, we consider the method of idealization. Let $R$ be a commutative ring and $M$ an $R$-module. The idealization of $R$ and $M$ is the ring $R(M) = R \bigoplus M$ with product $(r,m)(r',m') = (rr', rm' + r'm)$ and addition $(r,m)(r',m') = (r+r', m+m')$.

**Theorem 2.11.** *Let $R$ be a commutative ring, $M$ an $R$-module and $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. If $R$ is a weakly $g(x)$-clean ring, then the idealization $R(M)$ of $R$ and $M$ is weakly $g(x)$-clean.*

*Proof.* Let $(r,m) \in R(M)$. Since $R$ is a weakly $g(x)$-clean ring, we have $r = u \pm s$ where $u \in U(R)$ and $g(s) = 0$. So $(r,m) = (u \pm s, m) = (u,m) \pm (s,0)$. We have $(u,m)(u^{-1}, -u^{-1}mu^{-1}) = (uu^{-1}, u(-u^{-1}mu^{-1}) + mu^{-1}) = (1,0)$. Therefore $(u,m) \in U(R(M))$. Also we have

$$\begin{aligned} g((s,0)) &= a_0(1,0) + a_1(s,0) + ... + a_n(s,0)^n \\ &= a_0(1,0) + a_1(s,0) + ... + a_n(s^n,0) \\ &= (a_0,0) + (a_1 s, 0) + ... + (a_n s^n, 0) \\ &= (a_0 + a_1 s + ... + a_n s^n, 0) = (g(s),0) = (0,0). \end{aligned}$$

Thus $(r,m)$ is weakly $g(x)$-clean and so $R(M)$ is a weakly $g(x)$-clean ring. □

## 3. WEAKLY $(x^n - x)$-CLEAN RINGS

In this section we consider the weakly $(x^n - x)$-clean rings and weakly 2-clean rings.

**Theorem 3.1.** *Let $R$ be a ring, $n \in \mathbb{N}$ and $a, b \in R$. Then $R$ is weakly $(ax^{2n} - bx)$-clean if and only if $R$ is weakly $(ax^{2n} + bx)$-clean.*

*Proof.* Suppose $R$ is weakly $(ax^{2n} - bx)$-clean. Then for any $r \in R$, $-r = u \pm s$ where $(as^{2n} - bs) = 0$ and $u \in U(R)$. So $r = (-u) \pm (-s)$ where $(-u) \in U(R)$ and $a(-s)^{2n} + b(-s) = 0$. Hence, $r$ is weakly $(ax^{2n} + bx)$-clean. Therefore, $R$ is weakly c$(ax^{2n} + bx)$-clean. Now suppose $R$ is weakly $(ax^{2n} + bx)$-clean. Let $r \in R$. Then there exist $s$ and $u$ such that $-r = u \pm s$, $as^{2n} + bs = 0$ and $u \in U(R)$. So $r = (-u) \pm (-s)$ satisfies $(as^{2n} - bs) = 0$. Hence, $R$ is weakly $(ax^{2n} - bx)$-clean.

□

**Proposition 3.2.** *Let $2 \leq n \in \mathbb{N}$. If for every $a \in R$, $a = u \pm v$ where $u \in U(R)$ and $v^{n-1} = 1$, then $R$ is weakly $(x^n - x)$-clean.*

The following Lemma is well-known.

**Lemma 3.3.** *Let $a \in R$. The following statements are equivalent for $n \geq 1$:*
(1) *$a = a(ua)^n$ for some $u \in U(R)$;*
(2) *$a = ve$ for some $e^{n+1} = e$ and some $v \in U(R)$;*
(3) *$a = fw$ for some $f^{n+1} = f$ and some $w \in U(R)$.*

*Proof.* See Lemma 4.3 of [3].

□

**Proposition 3.4.** *Let $R$ be a weakly $(x^n - x)$-clean ring where $n \geq 2$ and $a \in R$. Then either (i) $a = u \pm v$ where $u \in U(R)$ and $v^{n-1} = 1$ or (ii) both $aR$ and $Ra$ contain nontrivial idempotents.*

*Proof.* Since $R$ is weakly $(x^n - x)$-clean, write $a = u \pm e$ where $u \in U(R)$ and $e^n = e$. Then $ae^{n-1} = ue^{n-1} \pm e$. So $a(1 - e^{n-1}) = u(1 - e^{n-1})$. Since $1 - e^{n-1}$ is an idempotent, by Lemma 12, $u(1 - e^{n-1}) = fw$ where $w \in U(R)$ and $f^2 = f \in R$. So $f = a(1 - e^{n-1})w^{-1} \in aR$. Suppose (i) does not hold. Then $1 - e^{n-1} \neq 0$, hence $f \neq 0$. Thus, $aR$ contains a nontrivial idempotent. Similarly, $Ra$ contains a nontrivial idempotent.

□

**Definition 3.5.** An element $r \in R$ is called weakly $n$-clean if $r = u_1 + u_2 + ... + u_n \pm e$ with $e^2 = e \in R$ and $u_i \in U(R)$ for $1 \leq i \leq n$ and $R$ is called weakly $n$-clean if every element of $R$ is weakly $n$-clean.

**Definition 3.6.** An element $a \in R$ is called right $\pi$-regular if it satisfies the following equivalent conditions,
(1) $a^n \in a^{n+1}R$ for some integer $n \geq 1$;
(2) $a^nR = a^{n+1}R$ for some integer $n \geq 1$;
(3) The chain $aR \supseteq a^2R \supseteq$ ...terminates.

The left $\pi$-regular elements are defined analogously. An element $a \in R$ is called strongly $\pi$-regular if it is both left and right $\pi$-regular, and R is called strongly $\pi$-regular if every element is strongly $\pi$-regular [10].

**Proposition 3.7.** *Let $n \in \mathbb{N}$, if the ring $R$ is weakly $(x^n - x)$-clean, then $R$ is weakly 2-clean.*

*Proof.* Let $r \in R$. Then $r = u \pm t$ for some $t^n = t \in R$ and $u \in U(R)$. Since $t$ is a strongly $\pi$-regular element and strongly $\pi$-regular elements are strongly clean [10] (it is of course clean and weakly clean), $t = v \pm e$ for some $e^2 = e$ and

$v \in U(R)$. Then $r = u \pm v \pm e$ is weakly 2-clean. Hence, $R$ is weakly 2-clean. $\square$

In fact, all weakly $(x^2 - x)$-clean rings and weakly $(x^2 + cx + d)$-clean rings with $d \in U(R)$ discussed above, are weakly 2-clean rings.

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## References

[1] D. D. Anderson and V. P. Camillo, Cammutative rings whose elements are a sum of a unit and an idempotent, *Comm. Algebra*, **30**, (2002), 3327-3336.

[2] V. P. Camillo and J. J. Simon, The Nicholson-Varadarajan theorem on clean linear transformations, *Glasg. Math. J.*, **44**, (2002), 365-369.

[3] L. Fan and X. Yang, On rings whose elements are the sum of a unit and a root of a fixed polynomial, *Comm. Algebra*, **36**(1), (2008), 269-278.

[4] J. Han and W. K. Nicholson, Extensions of clean rings, *Comm. Algebra*, **29**, (2001), 2589-2595.

[5] M. Henriksen, Two classes of rings generated by their units, *J. Algebra*, **31**, (1974), 182-193.

[6] R. Jahani-Nezhad, The dual of a strongly prime ideal, *Iranian Journal of Mathematical Sciences and Informatics*, **5**(1), (2010), 19-26.

[7] G. R. Mohtashei Borzabaran, A not via diagonality of 2 by 2 bhattacharyya matrices, *Iranian Journal of Mathematical Sciences and Informatics*, **1**(2), (2006), 79-84.

[8] A. Myung-Sook, and D. D. Anderson, Weakly clean rings and almost clean rings, *Rocky Mountain J. Math.*, **36**(3), (2006), 783-798.

[9] W. K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.*, **229**, (1977), 269-278.

[10] W. K. Nicholson, Strongly clean rings and Fitting's lemma, *Comm. Algebra*, **27**(3), (1999), 3583-3592.

[11] W. K. Nicholson and Y. Zhou, Endomorphism that are the sum of a unit and a root of a fixed polynomial, *Canad. Math. Bull.*, **49**, (2006), 265-269.