Iranian Journal of Mathematical Sciences and Informatics
Vol. 7, No. 2 (2012), pp 93-102

The Best Uniform Polynomial Approximation of Two Classes of Rational Functions

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Abstract. In this paper we obtain the explicit form of the best uniform polynomial approximations out of P_n of two classes of rational functions using properties of Chebyshev polynomials. In this way we present some new theorems and lemmas. Some examples will be given to support the theoretical results.

Keywords: Best polynomial approximation, Alternating set, Shifted Chebyshev polynomials, Uniform norm.

2000 Mathematics subject classification: 41A50, 41A52.

1. Introduction

As we know, in 1962, Rivlin [6] studied the class of functions given by $f(x) = \sum_{j=0}^{\infty} t^j T_{aj+b}(x)$ where a and b are integers, $a > 0$, $b \geq 0$ and $|t| < 1$. He introduced the best uniform polynomial approximation p_n^* for f on $[-1, 1]$. Also in 1978, Ollin [5] found the best uniform polynomial approximation of the class of functions given by $f(x) = \sum_{j=0}^{\infty} t^j U_{2j+b}(x)$, where b is a nonnegative integer and $|t| < 1$, on $[-1, 1]$.
In this paper, using Chebyshev expansion we obtain the best uniform polynomial approximations of two classes of rational functions:

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Received 5 March 2012; Accepted 15 July 2012

In section 2, we will introduce the best uniform approximation out of P_n for

$$f(x) = \sum_{j=1}^{\infty} j t^j T_j^*(x) + \frac{M(x,t,a,b)}{N(x,t,a,b)}, \tag{1}$$

and in section 3, we will determine the best uniform polynomial approximation out of P_{2n} for

$$f(x) = \sum_{j=0}^{\infty} t^{2j+1} (U_{2j+1}^*(x))' + \frac{K(x,t,a,b)}{L(x,t,a,b)}, \tag{2}$$

on $[a,b]$, where $|t| < 1$ and M, N, K, L are the polynomials of the variables x, t, a, b. In the following, we give some preliminaries.

Theorem 1.1. [7]. Given $f(x) \in C[a,b]$, there exists a polynomial $p_n^* \in P_n$, such that

$$\|f - p_n^*\| \leq \|f - p_n\| \quad \text{for all} \quad p_n \in P_n,$$

where $\|.\|$ is the uniform norm over the interval $[a,b]$, that is $\|g\| = \max_{a \leq x \leq b} |g(x)|$, for any $g \in C[a,b]$.

Definition 1.2. A set of $x_1, x_2, \ldots, x_{n+2}$, satisfying $a \leq x_1 < x_2 < \ldots < x_{n+2} \leq b$ is called an alternating set for the error function $f - p_n$ if

$$|f(x_j) - p_n(x_j)| = \|f - p_n\|, \qquad j = 1, 2, ..., n+2,$$

and

$$[f(x_j) - p_n(x_j)] = -[f(x_{j+1}) - p_n(x_{j+1})], \qquad j = 1, 2, ..., n+1.$$

Definition 1.3. We define

$$E_n(f;[a,b]) = E_n(f) = \max_{a \leq x \leq b} |f(x) - p_n^*(x)|. \tag{3}$$

Theorem 1.4. [7]. (**Chebyshev Alternation Theorem**) Suppose $f(x) \in C[a,b]$. Then p_n^* is the best uniform polynomial approximation for f out of p_n if and only if there exists an alternating set for $f - p_n$ consisting of $n+2$ points. One can see [1, 2] for more information about the best approximation theory.

Definition 1.5. The first kind Chebyshev polynomial in $[-1,1]$ of degree n is denoted by T_n and is defined by

$$T_n(x) = \cos(n\theta), \quad \text{where} \quad x = \cos(\theta). \tag{4}$$

Also Chebyshev polynomials satisfy in the following relations:

$$T_0(x) = 1; \quad T_1(x) = x; \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots. \tag{5}$$

Definition 1.6. The shifted first kind Chebyshev polynomial on the interval $[a,b]$ of degree n is denoted by T_n^* and is defined by

$$T_n^*(x) = \cos(n\theta), \quad \text{where} \quad \cos(\theta) = \frac{2x-(a+b)}{b-a},\ \theta \in [0,\pi]. \tag{6}$$

Indeed for $x \in [a,b]$, if we put $s = \frac{2x-(a+b)}{b-a}$ that $s \in [-1,1]$ then $T_n^*(x) = T_n(s)$.

Definition 1.7. The second kind Chebyshev polynomial in $[-1,1]$ of degree n is denoted by $U_n(x)$ and is defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{where} \quad x = \cos(\theta). \tag{7}$$

Chebyshev polynomials of the second kind satisfy in the following relation:

$$U_0(x) = 1; \quad U_1(x) = 2x; \quad U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \dots. \tag{8}$$

Definition 1.8. The shifted second kind Chebyshev polynomial on the interval $[a,b]$ of degree n is denoted by U_n^* and is defined as

$$U_n^*(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{where} \quad \cos(\theta) = \frac{2x-(a+b)}{b-a}, \quad \theta \in [0,\pi]. \tag{9}$$

Indeed for $x \in [a,b]$, if we put $s = \frac{2x-(a+b)}{b-a}$ that $s \in [-1,1]$ then $U_n^*(x) = U_n(s)$. The interested reader is referred to [4] for further information about properties of the Chebyshev polynomials.

2. Best uniform polynomial approximation of $f(x) = \sum_{j=1}^{\infty} jt^j T_j^*(x) + \frac{M(x,t,a,b)}{N(x,t,a,b)}$

Let us start this section with the following lemma, and then introduce one of the main theorems of this paper.

Lemma 2.1.

$$\begin{aligned}\sum_{j=n}^{\infty} jt^j T_j^*(x) &= n\sum_{j=n}^{\infty} t^j T_j^*(x) + \frac{t^{n+1}(T_{n+1}^*(x) - 2tT_n^*(x) + t^2 T_{n-1}^*(x))}{(1+t^2-2tT_1^*(x))^2} \\ &= \frac{nt^n(\cos(n\theta) - t\cos((n-1)\theta)}{1+t^2-2t\cos(\theta)} + \frac{t^{n+1}(\cos((n+1)\theta - 2t\cos(n\theta) + t^2\cos((n-1)\theta)}{(1+t^2-2t\cos(\theta))^2}.\end{aligned} \tag{10}$$

Proof. Since [6] we can write:

$$\sum_{j=n}^{\infty} t^j T_j^*(x) = \frac{t^n(\cos(n\theta) - t\cos((n-1)\theta)}{1+t^2-2t\cos(\theta)}, \tag{11}$$

then we can conclude

$$\begin{aligned}\sum_{j=n}^{\infty} jt^j T_j^*(x) &= t\frac{d}{dt}(\sum_{j=n}^{\infty} t^j T_j^*(x)) \\ &= \frac{nt^n(\cos(n\theta) - t\cos(n-1)\theta)}{1+t^2-2t\cos(\theta)} - \frac{t^{n+1}\cos(n-1)\theta}{1+t^2-2t\cos(\theta)} \\ &\quad - \frac{t^{n+1}(2t-2\cos\theta)(\cos(n\theta) - t\cos(n-1)\theta)}{(1+t^2-2t\cos\theta)^2} \\ &= \frac{nt^n(\cos(n\theta) - t\cos(n-1)\theta)}{1+t^2-2t\cos(\theta)} \\ &\quad + \frac{t^{n+1}(\cos((n+1)\theta - 2t\cos(n\theta) + t^2\cos((n-1)\theta)}{(1+t^2-2t\cos(\theta))^2}.\end{aligned}$$

□

Theorem 2.2. The best uniform polynomial approximation out of P_n for

$$f(x)=\sum_{j=1}^{\infty} jt^jT_j^*(x)-\frac{t^2(T_2^*(x)-2tT_1^*(x)+t^2)}{(1+t^2-2tT_1^*(x))^2}, \tag{12}$$

on $[a,b]$, where $|t|<1$ is

$$p_n^*(x)=\sum_{j=1}^{n} t^jT_j^*(x)+\frac{t^{n+2}}{1-t^2}T_n^*(x). \tag{13}$$

Proof. From Lemma 2.1 and Definition 1.5 we have:

$$\begin{aligned}
e_{n-1}(x) &= f(x)-P_n^*(x)\\
&=\sum_{j=1}^{\infty} jt^jT_j^*(x)-\frac{t^2(T_2^*(x)-2tT_1^*(x)+t^2)}{(1+t^2-2tT_1^*(x))^2}-\left(\sum_{j=1}^{n} t^jT_j^*(x)+\frac{t^{n+2}}{1-t^2}T_n^*(x)\right)\\
&=\left(\sum_{j=1}^{\infty} t^jT_j^*(x)+\frac{t^2(T_2^*(x)-2tT_1^*(x)+t^2)}{(1+t^2-2tT_1^*(x))^2}\right)\\
&\quad-\frac{t^2(T_2^*(x)-2tT_1^*(x)+t^2)}{(1+t^2-2tT_1^*(x))^2}-\left(\sum_{j=1}^{n} t^jT_j^*(x)+\frac{t^{n+2}}{1-t^2}T_n^*(x)\right)\\
&=\sum_{j=n+1}^{\infty} t^jT_j^*(x)-\frac{t^{n+2}}{1-t^2}T_n^*(x))\\
&=Re\left[\sum_{j=n+1}^{\infty}(te^{i\theta})^j-\frac{t^{n+2}}{1-t^2}e^{in\theta}\right]\\
&=\frac{t^{n+1}(\cos(n+1)\theta-2t\cos(n\theta)+t^2\cos(n-1)\theta}{(1-t^2)(1+t^2-2t\cos\theta)}
\end{aligned}$$

$$=\frac{t^{n+1}}{(1-t^2)}\left(\cos(n\theta)\frac{(1+t^2)\cos\theta-2t}{1+t^2-2t\cos(\theta)}-\sin(n\theta)\frac{(1-t^2)\sin\theta}{1+t^2-2t\cos(\theta)}\right). \tag{14}$$

Let us define ϕ by

$$\begin{cases}\cos\phi=\frac{(1+t^2)\cos\theta-2t}{1+t^2-2t\cos(\theta)},\\ \sin\phi=\frac{(1-t^2)\sin\theta}{1+t^2-2t\cos(\theta)}.\end{cases} \tag{15}$$

Then from (14) and (15) we can write:

$$e_n(x)=\frac{t^{n+1}}{(1-t^2)}(\cos(n\theta)\cos\phi-\sin(n\theta)\sin\phi)=\frac{t^{n+1}}{(1-t^2)}\cos(n\theta+\phi). \tag{16}$$

When θ varies from 0 to π, ϕ varies from 0 to π, then $n\theta+\phi$ increases continuously from 0 to $(n+1)\pi$, hence $\cos(n\theta+\phi)$ possesses at least $n+2$ extermal points, where it assumes alternately the values ±1. Thus

$$E_n(f,[a,b])=\frac{|t|^{n+1}}{(1-t^2)}. \tag{17}$$

$\square$

Remark 2.3. Considering (17) with condition $|t|<1$, we have:

$$\lim_{n\to\infty} E_n(f,[a,b]) = \lim_{n\to\infty} \frac{|t|^{n+1}}{(1-t^2)} = 0.$$

In other words, we prove that $p_n^*(x)$ approximates $f(x)$ uniformly as $n\to\infty$.

3. Best uniform polynomial approximation of class of mentioned functions in (2)

Let us start this section with the following Lemma.

Lemma 3.1 [3]. Let $(U_n^*(x))'$ be the derivative of $(U_n^*(x))$. Then we have

$$(U_n^*(x))' = \begin{cases} \frac{8}{b-a}\sum_{k=0}^{s-1}(k+1)U_{2k+1}^*(x); & n=2s, \\ \frac{4}{b-a}\sum_{k=0}^{s}(2k+1)U_{2k}^*(x)\ ; & n=2s+1. \end{cases} \tag{18}$$

Lemma 3.2. For $|t|<1$,we have:

$$\sum_{j=0}^{\infty} t^{2j+1}(U_{2j+1}^*(x))' = \frac{4t}{(1-t^2)(b-a)}\left\{\sum_{j=1}^{\infty}(2j)\,t^{2j}U_{2j}^*(x) + \sum_{j=0}^{\infty} t^{2j}U_{2j}^*(x)\right\}. \tag{19}$$

Proof. From Lemma 2.1 we can write

$$\begin{aligned}
\sum_{j=0}^{\infty} t^{2j+1}(U_{2j+1}^*(x))' &= \frac{4}{b-a}\sum_{j=0}^{\infty} t^{2j+1}\sum_{k=0}^{j}(2k+1)\,U_{2k}^*(x) \\
&= \frac{4}{b-a}\sum_{j=0}^{\infty}(2j+1)\,U_{2j}^*(x)\sum_{k=j}^{\infty} t^{2k+1} \\
&= \frac{4}{(1-t^2)(b-a)}\sum_{j=0}^{\infty}(2j+1)\,t^{2j+1}U_{2j}^*(x) \\
&= \frac{4t}{(1-t^2)(b-a)}\left\{\sum_{j=1}^{\infty}(2j)\,t^{2j}U_{2j}^*(x) + \sum_{j=0}^{\infty} t^{2j}U_{2j}^*(x)\right\}.
\end{aligned}$$

□

Lemma 3.3. For $|t|<1$, and an integer n, we have:

$$\sum_{j=n}^{\infty}(2j)t^{2j}U_{2j}^*(x) = 2n\sum_{j=n}^{\infty} t^{2j}U_{2j}^*(x) + \frac{2t^{2n+2}(U_{2n+2}^*(x) - 2t^2U_{2n}^*(x) + t^4U_{2n-2}^*(x))}{(1+t^4-2t^2T_2^*(x))^2}. \tag{20}$$

Proof.

$$\begin{aligned}
\sum_{j=n}^{\infty}(2j)\,t^{2j}U_{2j}^*(x) &= t\frac{d}{dt}\left(\sum_{j=n}^{\infty} t^{2j}U_{2j}^*(x)\right) = \left(\frac{t}{\sin\theta}\right)\frac{d}{dt}\left(Im\left(\sum_{j=n}^{\infty} t^{2j}e^{(2j+1)i\theta}\right)\right) \\
&= \left(\frac{t}{\sin\theta}\right)\frac{d}{dt}\left(\frac{t^{2n}\sin(2n+1)\theta - t^{2n+2}\sin(2n-1)\theta}{1+t^4-2t^2\cos(2\theta)}\right) \\
&= \frac{2nt^{2n}(U_{2n}^*(x) - t^2U_{2n-2}^*(x))}{1+t^4-2t^2T_2^*(x)} - \frac{2t^{2n+2}U_{2n-2}^*(x)}{1+t^4-2t^2T_2^*(x)}
\end{aligned}$$

$$-4t^{2n+2}\frac{(t^2-T_2^*(x))(U_{2n}^*(x)-t^2U_{2n-2}^*(x))}{(1+t^4-2t^2T_2^*(x))^2}$$
$$=2n\sum_{j=n}^{\infty}t^{2j}U_{2j}^*(x)+\frac{2t^{2n+2}(U_{2n+2}^*(x)-2t^2U_{2n}^*(x)+t^4U_{2n-2}^*(x))}{(1+t^4-2t^2T_2^*(x))^2}.$$

Proof. **Theorem 3.4.** The best uniform polynomial approximation out of P_{2n} for

$$f(x)=\sum_{j=0}^{\infty}t^{2j+1}(U_{2j+1}^*(x))'-\frac{8t^5(U_4^*-2t^2U_2^*+t^4)}{(1-t^2)(b-a)(1+t^4-2t^2T_2^*(x))^2}\,, \tag{21}$$

on $[a,b]$, where $|t|<1$, is

$$p_{2n}(x)=-\tfrac{8t}{(1-t^2)(b-a)} +\tfrac{12t}{(1-t^2)(b-a)}\left\{\sum_{j=0}^{n}t^{2j}U_{2j}^*(x)-\tfrac{t^{2n+4}U_{2n-2}^*(x)}{(1-t^2)^2(1+t^2)}-\tfrac{t^{2n+2}(t^4-t^2-1)U_{2n}^*(x)}{(1-t^2)^2(1+t^2)}\right\}. \tag{22}$$

Proof. From Lemmas 3.2 and 3.3, we have:

$$\begin{aligned}
e(x)&=f(x)-p_{2n}(x)\\
&=\tfrac{4t}{(1-t^2)(b-a)}\left\{\sum_{j=1}^{\infty}(2j)\,t^{2j}U_{2j}^*(x)+\sum_{j=0}^{\infty}t^{2j}U_{2j}^*(x)\right\}\\
&\quad-\tfrac{8t^5(U_4^*-2t^2U_2^*+t^4)}{(1-t^2)(b-a)(1+t^4-2t^2T_2^*(x))^2}+\tfrac{8t}{(1-t^2)(b-a)}\\
&\quad-\tfrac{12t}{(1-t^2)(b-a)}\left\{\sum_{j=0}^{n}t^{2j}U_{2j}^*(x)-\tfrac{t^{2n+4}U_{2n-2}^*(x)}{(1-t^2)^2(1+t^2)}-\tfrac{t^{2n+2}(t^4-t^2-1)U_{2n}^*(x)}{(1-t^2)^2(1+t^2)}\right\}\\
&=\tfrac{4t}{(1-t^2)(b-a)}\left\{3\sum_{j=0}^{\infty}t^{2j}U_{2j}^*(x)+\tfrac{2t^4(U_4^*-2t^2U_2^*+t^4)}{(1+t^4-2t^2T_2^*(x))^2}-2\right\}\\
&\quad-\tfrac{8t^5(U_4^*-2t^2U_2^*+t^4)}{(1-t^2)(b-a)(1+t^4-2t^2T_2^*(x))^2}+\tfrac{8t}{(1-t^2)(b-a)}\\
&\quad-\tfrac{12t}{(1-t^2)(b-a)}\left\{\sum_{j=0}^{n}t^{2j}U_{2j}^*(x)-\tfrac{t^{2n+4}U_{2n-2}^*(x)}{(1-t^2)^2(1+t^2)}-\tfrac{t^{2n+2}(t^4-t^2-1)U_{2n}^*(x)}{(1-t^2)^2(1+t^2)}\right\}\\
&=\tfrac{12t}{(1-t^2)(b-a)}\left\{\sum_{j=n+1}^{\infty}t^{2j}U_{2j}^*(x)+\tfrac{t^{2n+4}U_{2n-2}^*(x)}{(1-t^2)^2(1+t^2)}+\tfrac{t^{2n+2}(t^4-t^2-1)U_{2n}^*(x)}{(1-t^2)^2(1+t^2)}\right\}.
\end{aligned} \tag{23}$$

Since [5] we have:

$$\sum_{j=n+1}^{\infty}t^{2j}U_{2j}^*(x)=t^{2n+2}\frac{U_{2n+2}^*-t^2U_{2n}^*}{1+t^4-2t^2T_2^*(x)}. \tag{24}$$

Thus

$$\begin{aligned}
e(x)&=\frac{12t}{(1-t^2)(b-a)\sin\theta}\left\{\frac{t^{2n+2}\left(\sin(2n+3)\theta-t^2\sin(2n+1)\theta\right)}{1+t^4-2t^2T_2^*(x)}\right.\\
&\quad\left.+\frac{t^{2n+4}\sin(2n-1)\theta}{(1-t^2)^2(1+t^2)}+\frac{t^{2n+2}(t^4-t^2-1)\sin(2n+1)\theta}{(1-t^2)^2(1+t^2)}\right\}\\
&=\frac{24t^{2n+3}}{(1-t^2)^3(1+t^2)(b-a)}\{\cos(2n+1)\theta\cos\psi-\sin(2n+1)\theta\sin\psi\}\\
&=\frac{24t^{2n+3}}{(1-t^2)^3(1+t^2)(b-a)}\cos((2n+1)\theta+\psi),
\end{aligned} \tag{25}$$

where

$$\begin{aligned} \cos\psi &= \tfrac{(1-2t^2-t^4)\sin(2\theta)+t^4\sin(4\theta)}{2\sin\theta(1+t^4-2t^2\cos(2\theta))}, \\ \sin\psi &= \tfrac{(1+2t^2)-(1+t^2)^2\cos(2\theta)+t^4\cos(4\theta)}{2\sin\theta(1+t^4-2t^2\cos(2\theta))}. \end{aligned} \tag{26}$$

Now if ψ varies from 0 to π, as θ varies from 0 to π. Therefore the argument $\cos((2n+1)\theta+\psi)$ increases continuously from 0 to $(2n+2)\pi$ as θ increases from 0 to π, so the error takes its extreme values $\left|\frac{24t^{2n+3}}{(1-t^2)^3(1+t^2)(b-a)}\right|$ with alternating sign at the $2n+2$ points in $0 \le \theta \le \pi$ at which $\cos((2n+1)\theta+\psi)$will be ± 1. Using the Chebyshev Alternation Theorem, $p_{2n}(x)$ is $p^*_{2n}(x)$, with

$$E_{2n}(f,[a,b]) = \frac{24\,|t|^{2n+3}}{(1-t^2)^3(1+t^2)(b-a)}. \tag{27}$$

□

Remark 3.5. Considering (27) with condition $|t|<1$, we can write

$$\lim_{n\to\infty} E_{2n}(f,[a,b]) = 0.$$

In other words, we prove that $p_{2n}(x)$ approximates $f(x)$ uniformly as $n\to\infty$.
Corollary 3.6. The best uniform polynomial approximation out of P_{2n} for

$$f(x) = \sum_{j=1}^{\infty}(2j)\,t^{2j}U^*_{2j}(x) + \sum_{j=0}^{\infty} t^{2j}U^*_{2j}(x) - \frac{2t^4(U^*_4(x)-2t^2U^*_4(x)+t^4)}{(1+t^4-2t^2T^*_1(x))^2}, \tag{28}$$

on $[a,b]$ where $|t|<1$, is

$$p_{2n}(x) = -2+3\left\{\sum_{j=0}^{n} t^{2j}U^*_{2j}(x) - \frac{t^{2n+4}U^*_{2n-2}(x)}{(1-t^2)^2(1+t^2)} - \frac{t^{2n+2}(t^4-t^2-1)U^*_{2n}(x)}{(1-t^2)^2(1+t^2)}\right\}. \tag{29}$$

Proof. We can prove this corollary similar to Theorem 3.4. □

Corollary 3.7. The zeros of the polynomial

$$r_1(x) = \left((1+t^2)\,(2x-(a+b)) - 2t(b-a)\right)\,(T^*_n(x))' + 2n\,(1-t^2)T^*_n(x)\,, \tag{30}$$

and -1, $+1$ are the alternating points of (12).

Proof. From the Theorem 2.1 we know the alternants of (12) are the extermal points of $\cos((2n+1)\theta+\psi)$ where $\cos(\theta) = \frac{2x-(a+b)}{b-a}$ and $\cos\psi = \frac{(1+t^2)\cos\theta-2t}{1+t^2-2t\cos\theta}$. But for every extermal of $\cos((2n+1)\theta+\psi)$ we have $\sin((2n+1)\theta+\psi)=0$. Therefore

$$\begin{aligned} 0 = \sin(n\theta+\psi) &= \frac{(T^*_n(x))'\sin(\theta)(b-a)}{2n}\cos\psi + T^*_n(x)\sin\psi \\ &= \frac{(T^*_n(x))'\sin(\theta)(b-a)}{2n} \times \frac{(1+t^2)\cos\theta-2t}{1+t^2-2t\cos\theta} + T^*_n(x)\frac{(1-t^2)\sin\theta}{1+t^2-2t\cos\theta} \\ &= \frac{\sqrt{(b-a)^2-(2x-(a+b))^2}}{2n\ ((1+t^2)(b-a)-4tx+2t(a+b))} \\ &\quad \times \left(\left((1+t^2)\,(2x-(a+b))-2t(b-a)\right)\,(T^*_n(x))' + 2n\,(1-t^2)T^*_n(x)\right). \end{aligned}$$

Hence the roots of $r_1(x)$ and -1,$+1$ are the alternants of (12). □

Corollary 3.8. The zeros of the polynomial

$$r_2(x) = 2(2n+1)T^*_{2n+1}(x)\Big\{(1+t^2)^2(b-a) - 4t^4(2x-(a+b))^2\Big\}$$
$$+(2x-(a+b))(T^*_{2n+1}(x))'\Big\{(1-t^2)^2(b-a) - 4t^4\left((2x-(a+b))^2 - (b-a)^2\right)\Big\}, \tag{31}$$

and -1 and $+1$ are the alternating points of (21).

Proof. From Theorem 3.4 we know the alternants of (27) are the extermal points of $\cos((2n+1)\theta+\psi)$ where $\cos(\theta) = \frac{2x-(a+b)}{b-a}$, $\cos\psi = \frac{(1-2t^2-t^4)\sin(2\theta)+t^4\sin(4\theta)}{2\sin\theta(1+t^4-2t^2\cos(2\theta))}$. But for every extermal of $\cos((2n+1)\theta+\psi)$ we have $\sin((2n+1)\theta+\psi) = 0$. Therefore

$$0 = \sin((2n+1)\theta+\psi) = \frac{\sqrt{(b-a)^2-(2x-(a+b))^2}}{2(2n+1)(b-a)^2\Big((1+t^2)^2(b-a)^2 - 4t^4(2x-(a+b))^2\Big)}$$
$$\times\Big\{2(2n+1)T^*_{2n+1}(x)\Big\{(1+t^2)^2(b-a)^2 - 4t^4(2x-(a+b))^2\Big\}$$
$$+(2x-(a+b)){T^*_{2n+1}}'(x)\Big\{(1-t^2)^2(b-a)^2 - 4t^4\Big((2x-(a+b))^2 - (b-a)^2\Big)\Big\}\Big\},$$

Hence the roots of $r_2(x)$ and -1, $+1$ are the alternants of (21). □

4. Examples

In this section, we present some examples to demonstrate the theoretical results of this study.

Example 4.1. In Figure 4.1, we show both functions $f_1(x)$ and its best uniform polynomial approximation of degree 5 (the dashed points), on the interval $[-1,3]$ where $t = 0.25$, and

$$f_1(x) = \sum_{j=1}^{\infty} j(0.25)^j T^*_j(x) - \frac{8x^2-20x-3}{(4x-21)^2}. \tag{32}$$

Also in Figure 4.2 we show the error function. From Theorem 2.1, we have $E_5(x) \approx 2.6\times10^{-4}$.

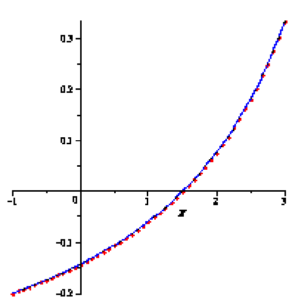

FIGURE 1. The best uniform approximation of $f_1(x)$.

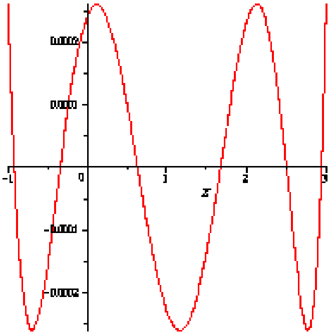

FIGURE 2. The error of best aproximation of $f_1(x)$.

Example 4.2. Putting $t = 0.2$, $n = 3$, $[a, b] = [-2, 2]$ in Theorem 3.1, we show both the function $f_2(x)$ and it's best uniform polynomial approximation $p_6^*(x)$ (the dashed points) in Figure 4.3. Also in Figure 4.4, we show the error function and we have $E_6(x) \approx 3.3 \times 10^{-6}$.

$$f_2(x) = \sum_{j=0}^{\infty}(0.2)^{2j+1}(U_{2j+1}^*(x))' - \frac{2(0.2)^5(U_4^* - 2(0.2)^2U_2^* + (0.2)^4)}{(1-(0.2)^2)(1+(0.2)^4 - 2(0.2)^2T_2^*(x))^2}. \quad (33)$$

$$p_6^*(x) = \frac{-1}{2.4} + \frac{1}{1.6}\left\{\sum_{j=0}^{3}(0.2)^{2j}U_{2j}^*(x) - \frac{(0.2)^8\left\{(0.2)^2U_4^*(x) + \left((0.2)^4 - (0.2)^2 - 1\right)U_6^*(x)\right\}}{\left(1-(0.2)^2\right)^2\left(1+(0.2)^2\right)}\right\}.$$

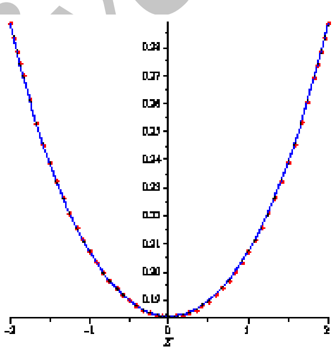

FIGURE 3. The best approximation of $f_2(x)$.

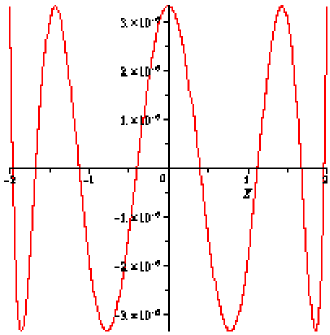

FIGURE 4. The error of best aproximation of $f_2(x)$.

5. Conclusion

In this study, we determined the best uniform polynomial approximation out of P_n to $f(x) = \sum_{j=1}^{\infty} jt^j T_j^*(x) - \frac{t^2(T_2^*(x)-2tT_1^*(x)+t^2)}{(1+t^2-2tT_1^*(x))^2}$, on $[a,b]$, where $|t| < 1$. Also we could find the best uniform polynomial approximation out of P_{2n} to another class of functions of the form $f(x) = \sum_{j=0}^{\infty} t^{2j+1}(U_{2j+1}^*(x))' - \frac{8t^5(U_4^*-2t^2U_2^*+t^4)}{(1-t^2)(b-a)(1+t^4-2t^2T_2^*(x))^2}$, on $[a,b]$, where $|t| < 1$. In this way we presented some new theorems and lemmas about the best uniform polynomial approximations of these classes of rational functions. Furthermore we obtained the alternating set of the mentioned classes.

Acknowledgments. The authors are very grateful to the referees of this paper for their comments and suggestions which have improved the paper.

References

[1] S. Chandok, T. D. Narang, Common Fixed Points and Invariant Approximations for Cq-commuting Generalized nonexpansive mappings, *Iranian Journal of Mathematical Sciences and Informatics*, **7** (1), (2012), 21-34.

[2] N. Eftekhari, The Basic Theorem and its Consequences, *Iranian Journal of Mathematical Sciences and Informatics*, **4** (1), (2009), 27-35.

[3] M. R. Eslahchi, M. Dehghan, S. Amani, The third and fourth kinds Chebyshev polynomials and best uniform approximation, *Mathematical and Computer Modelling*, **55**, (2012), 1746-1762.

[4] J. C. Mason, D. C. Handscomb, *Chebyshev Polynomials*, Chapman & Hall/CRC, 2003.

[5] H. Z. Ollin, Best polynomial approximation to certain rational functions, *J. Approx. Theory*, **26**, (1979), 389-392.

[6] T. J. Rivlin, Polynomials of best uniform approximation to certain rational functions, *Numerical Mathematik*, **4**, (1962), 345-349.

[7] T. J. Rivlin, *An Introduction to the Approximation of Functions*, Dover, New York, 1981.