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Derivations in Hyperrings and Prime Hyperrings

A. Asokkumar
Department of Mathematics, Aditanar College of Arts and Science,
Tiruchendur-628216, Tamilnadu, India

E-mail: ashok_a58@yahoo.co.in

Abstract. In this paper we introduce derivations in Krasner hyperrings and derive some basic properties of derivations. We also prove that for a strongly differential hyperring R and for any strongly differential hyperideal I of R, the factor hyperring R/I is a strongly differential hyperring. Further we prove that a map $d : R \rightarrow R$ is a derivation of a hyperring R if and only if the induced map φ_d is a homomorphism.

Keywords: Canonical hypergroup, hyperring, reduced hyperring, derivation, differential hyperring, differential hyperideal, hderivation.

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1. Introduction

The theory of hyperstructures was introduced in 1934 by Marty [14] at the 8^{th}congress of Scandinavian Mathematicians. Then several researchers have worked on this new field and developed it. Mittas [16] introduced the notion of canonical hypergroups. Corsini [4] studied the Canonical Hypergroups [6], Feebly Canonical Hypergroups [5], Quasi- Canonical Hypergroups [7]. Krasner [13] introduced the notion of hyperrings and hyperfields. G.G Massouros [15] introduced the theory of hypercompositional structures into the theory of automata. Asokkumar [2] studied the idempotent elements of Krasner hyperrings. Babaei et al. [3] studied R-parts in hyperrings.

The notion of derivations of rings plays a significant role in algebra [11]. The study of derivations in rings got interested after Posner [17], who gave

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striking results on derivations of prime rings. Then the notion of derivations has been developed by many authors in various directions like Jordan derivation, generalised derivation in rings and in near-rings. Ebadian et al. [10] studied Jordan derivations on Banach algebras. From the motivation of derivations, Vougiouklis [19] introduced a hyperoperation called *theta* hyperoperation and studied H_ν-structures. Jan Chvalina et al. [12], introduced a hyperoperation $*$ on a differential ring R so that $(R, *)$ is a hypergroup.

In this paper we introduce derivations in Krasner hyperrings and give examples. Also we derive some basic properties of derivations. Further we prove that for a strongly differential hyperring R and for a strongly differential hyperideal I of R, the factor hyperring R/I is a strongly differential hyperring. We also prove that a map $d : R \to R$ is a derivation of a hyperring R if and only if the induced map φ_d is a homomorphism.

2. **Basic definitions and notations**

This section explains some basic definitions that have been used in the sequel. A *hyperoperation* $*$ on a non-empty set H is a mapping of $H \times H$ into the family of non-empty subsets of H (i.e., $x * y \subseteq H$ for every $x, y \in H$). In the sense of Marty [14], a *hypergroup* $(H, *)$ is a non-empty set H equipped with a hyperoperation $*$ which satisfies the following axioms:
(i) $x * (y * z) = (x * y) * z$ for every $x, y, z \in H$(the associative axiom).
(ii) $x * H = H * x = H$ for every $x \in H$(the reproductive axiom).
A comprehensive review of the theory of hypergroups appears in [4]. The basic results of hyperstructures and hyperrings are found in [8], [9] and [20].

Definition 2.1. A non-empty set R with a hyperaddition $+$ and a multiplication $\cdot$ is called an *additive hyperring or Krasner hyperring* if it satisfies the following:
(1) $(R, +)$ is a *canonical hypergroup*, i.e.,
(i) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$,
(ii) for every $x, y \in R$, $x + y = y + x$,
(iii) there exists $0 \in R$ such that $0 + x = x$ for all $x \in R$,
(iv) for every $x \in R$ there exists an unique element denoted by $-x \in R$ such that $0 \in x + (-x)$,
(v) for every $x, y, z \in R$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.
(2) $(R, \cdot)$ is a *semigroup* having 0 as a bilaterally absorbing element, i.e.,
(i) for every $x, y, z \in R$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(ii) $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$.
(3) The multiplication $\cdot$ is *distributive* with respect to the hyperoperation $+$. i.e., for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

A non-empty subset I of a canonical hypergroup R is called a *canonical subhypergroup* of R if I itself is a canonical hypergroup under the same hyperoperation as that of R. Equivalently, a non-empty subset I of a canonical hypergroup R is a canonical subhypergroup of R if for every $x, y \in I$, $x - y \subseteq I$. Here after we denote xy instead of $x \cdot y$. Moreover, for $A, B \subseteq R$ and $x \in R$, by $A + B$ we mean the set $\bigcup_{a\in A, b\in B}(a + b)$ and $AB = \bigcup_{a\in A, b\in B}(ab)$, $A + x = A + \{x\}$, $x + B = \{x\} + B$ and also $-A = \{-a : a \in A\}$.

The following elementary facts in a hyperring easily follow from the axioms: (i) $-(-a) = a$ for every $a \in R$; (ii) 0 is the unique element such that for every $a \in R$, there is an element $-a \in R$ with the property $0 \in a+(-a)$ and $-0 = 0$; (iii) $-(a + b) = -a - b$ for all $a, b \in R$; (iv) $-(ab) = (-a)b = a(-b)$ for all $a, b \in R$.

In a hyperring R, if there exists an element $1 \in R$ such that $1a = a1 = a$ for every $a \in R$, then the element 1 is called the *identity element* of the hyperring R. In fact, the element 1 is unique. Further, if $ab = ba$ for every $a, b \in R$ then the hyperring R is called a *commutative* hyperring. Throughout this paper, by a hyperring we mean the Krasner hyperring.

Example 2.2. *The set R =$\{0,1\}$ with the following hyperoperations is a hyperring.*

$+$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

$\cdot$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$

Example 2.3. *M.Krasner [13] constructed a class of hyperrings as follows. Let $(R, +, .)$ be a ring and G be a normal subgroup of the multiplicative semigroup $(R, .)$, that is, $\{xG = Gx$ for every $x \in R\}$. Consider the set $\overline{R} = \{\overline{x} = xG : x \in R\}$ of classes modulo G. If we define hyperaddition $\oplus$ and multiplication $\odot$ on $\overline{R}$ as $\overline{x} \oplus \overline{y} = xG \oplus yG = \{(xp + yq)G : p, q \in G\}$ and $\overline{x} \odot \overline{y} = xG \odot yG = xyG$ for all $\overline{x}, \overline{y} \in \overline{R}$, then $(\overline{R}, \oplus, \odot)$ is a hyperring.*

Definition 2.4. Let R be a hyperring. A non-empty subset S of R is called a *subhyperring* of R if $x - y \subseteq S$ and $xy \in S$ for all $x, y \in S$.

Definition 2.5. Let R be a hyperring and I be a non-empty subset of R. I is called a *left* (resp. *right*) *hyperideal* of R if (i) $(I, +)$ is a canonical subhypergroup of R, i.e., for every $x, y \in I, x - y \subseteq I$ and (ii) for every $a \in I, r \in R, ra \subseteq I$ (resp. $ar \subseteq I$). A *hyperideal* of R is one which is a left as well as a right hyperideal of R.

Definition 2.6. Let R and S be hyperrings, where both additions and multiplications are hyperoperations. A mapping $\phi : R \to S$ is called a *homomorphism* from R to S if for all $x, y \in R$, (i) $\phi(x+y) \subseteq \phi(x)+\phi(y)$, (ii) $\phi(xy) \subseteq \phi(x)\phi(y)$ and (iii) $\phi(0) = 0$ hold. If R is a Krasner hyperring, then the condition (ii) becomes $\phi(xy) \in \phi(x)\phi(y)$. If both R and S are Krasner hyperrings, then the condition (ii) is $\phi(xy) = \phi(x)\phi(y)$.

Definition 2.7. A hyperring R is said to be a prime hyperring if $aRb = 0$ for $a, b \in R$ implies either $a = 0$ or $b = 0$.

Definition 2.8. A hyperring R is said to be a reduced hyperring if it has no nilpotent elements. That is, if $x^n = 0$ for $x \in R$ and a natural number n, then $x = 0$.

Definition 2.9. A hyperring R is said to be 2-torsion free if $0 \in x + x$ for $x \in R$ implies $x = 0$.

3. Derivation of hyperrings and examples

In this section we define derivation and strong derivation of hyperrings and give examples.

Definition 3.1. Let R be a hyperring. A map $d : R \to R$ is said to be a *derivation of* R if d satisfies:

(i) $d(x + y) \subseteq d(x) + d(y)$ and

(ii) $d(xy) \in d(x)y + xd(y)$ for all $x, y \in R$.

The hyperring R equipped with a derivation d is called a *d-differential hyperring.* If the map d is such that $d(x + y) = d(x) + d(y)$ for all $x, y \in R$ and satisfies the condition (ii), then d is called a *strong derivation* of R. In this case, the hyperring is called *strongly d-differential* hyperring. Since the hyperaddition of the hyperring R is commutative, the condition (ii) of the derivation is equivalent to $d(xy) \in xd(y) + d(x)y$ for all $x, y \in R$.

When there is no confusion regarding the derivation d, we simply write differential hyperring instead of d-differential hyperring. For $a \in R$, we call the element $d(a)$ by the *derivative* of a and we write $d^2(a), d^3(a), \cdots, d^n(a)$ for the successive derivatives of a.

Proposition 3.2. *Let R be a hyperring and $d : R \to R$ be a derivation of R. Then (i) $d(0) = 0$, (ii) $d(-a) = -d(a)$ for all $a \in R$, (iii) if 1 is the identity element of R, then $d(1) \in d(1) + d(1)$.*

Proof. It is clear that $d(0) = 0$. Now, for each $a \in R$, $0 = d(0) \in d(a - a) = d(a + (-a)) \subseteq d(a) + d(-a)$. That is, $d(a) \in 0 - d(-a)$. Hence $d(a) = -d(-a)$. Therefore, $-d(a) = -(-d(-a)) = d(-a)$. Also, $d(1) = d(1.1) \in d(1).1 + 1.d(1) = d(1) + d(1)$. That is, $d(1) \in d(1) + d(1)$. □

Remark 3.3. Let R be a hyperring $c \in R$ and $d : R \to R$ is a map defined by $d(x) = c$ for all $x \in R$. Then d is a derivation if and only if $c = 0$. This derivation is a strong derivation, called the *trivial derivation.*

Example 3.4. *Let R be a hyperring such that $x \in x+x$ for every $x \in R$. Then the identity map, $i(x) = x$ for every $x \in R$, is a strong derivation of R.*

Example 3.5. *In the hyperring given in the example 2.2, we have two derivations namely (i) $d(x) = 0$ for all $x \in R$ and (ii) $d(x) = x$ for all $x \in R$.*

Example 3.6. *Consider the hyperring R =\{0, a, b\} with the hyperaddition and the multiplication defined as follows.*

+	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a,b\}$	R
b	$\{b\}$	R	$\{a,b\}$

.	0	a	b
0	0	0	0
a	0	b	a
b	0	a	b

Define a map $d : R \to R$ by $d(0) = 0, d(a) = b, d(b) = a$. Clearly, d is a well defined map. Now, $d(a + b) = d(R) = R = b + a = d(a) + d(b)$ and $d(ab) = d(a) = b \in b+b = bb+aa = d(a)b+ad(b)$. Thus $d(ab) \in d(a)b+ad(b)$. Also, $d(a + a) = d(\{a, b\}) = \{a, b\} = d(a) + d(a)$ and $d(aa) = d(b) = a \in \{a, b\} = a+a = ba+ab = d(a)a+ad(a)$. Further, $d(b+b) = d(\{a, b\}) = \{a, b\} = d(b) + d(b)$ and $d(bb) = d(b) = a \in \{a, b\} = a + a = ab + ba = d(b)b + bd(b)$. Hence d is a derivation of R. Here d is a strong derivation of R.

Example 3.7. *Let R be a hyperring and $M(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R \right\}$ be a collection of 2×2 matrices over R. A hyperaddition $\oplus$ is defined on $M(R)$ by $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x \in a + c, y \in b + d \right\}$ for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M(R)$. Clearly, this hyperaddition is well defined and $(M(R), \oplus)$ is a canonical hypergroup. The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the additive identity of $M(R)$. Also, for each matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ of $M(R)$, there exists a unique matrix $\begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix} \in M(R)$, such that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix}$. Now, a multiplication $\otimes$ is defined on $M(R)$ by*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M(R).$$

Clearly, the multiplication $\otimes$ is well defined and associative. Therefore, $(M(R), \otimes)$ is a semigroup.

Let $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} \in M(R)$. Then

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} \right\} &= \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \left\{ \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} : r \in x + p, s \in y + q \right\} \\ &= \left\{ \begin{pmatrix} ar & as \\ 0 & 0 \end{pmatrix} : r \in x + p, s \in y + q \right\}. \end{aligned}$$

Also, $\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} ax & ay \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} ap & aq \\ 0 & 0 \end{pmatrix}$
$= \left\{\begin{pmatrix} l & m \\ 0 & 0 \end{pmatrix} : l \in ax + ap, m \in ay + aq\right\}$. *By the left distributive axiom of* R,
$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}\right\} = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix}\right\}$.
Similarly, we can show that the right distributive law is also satisfied on $M(R)$. *Thus* $M(R)$ *is a Krasner hyperring.*

Now, define a function d *on* $M(R)$ *by* $d\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. *Clearly, this map is well defined. We shall now show that* d *is a derivation.*
For $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M(R)$,*the set* $d\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\}$ *and the set* $d\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \oplus d\left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right)$ *are equal and equal to the set* $\left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in b + d\right\}$.
Also, $d\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 0 & ad \\ 0 & 0 \end{pmatrix}$
$= \left\{d\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \otimes \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \otimes d\left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right)\right\}$.
Thus d *is a derivation on* $M(R)$. *Here* d *is a strong derivation of* R.

Proposition 3.8. *In any hyperring with a derivation, the elements with derivative 0, form a subhyperring.*

Proof. Let R be a hyperring and d be a derivation of R. Let $S = \{x \in R : d(x) = 0\}$. Since $d(0) = 0$, we see that S is non-empty. Let $a, b \in S$. Then $d(a) = 0, d(b) = 0$. Now, $d(a + b) \subseteq d(a) + d(b) = 0 + 0 = 0$. Further, for any $a \in S$, $d(-a) = -d(a) = 0$. Also, $d(ab) \in d(a)b + ad(b) = 0 + 0 = 0$. Thus for any $a, b \in A$, $a + b \subseteq S, -a \in S, ab \in S$. So, S is a subhyperring of R. $\square$

4. Properties of derivations of prime hyperrings and differential hyperideals

In this section we prove two simple properties of derivations in prime hyperrings. The results are analogues results of Posner [17]. Also, we define differential hyperideal and prove the existence of a differential hyperideal in a differential hyperring. Further, we prove that if R is a strongly differential hyperring and I is a strongly differential hyperideal of R, then the factor hyperring R/I is a strongly differential hyperring.

Proposition 4.1. *Let* d *be a derivation of a prime hyperring* R *and* $a \in R$ *such that* $ad(u) = 0$ *(or* $d(u)a = 0$*) for all* $u \in R$. *Then either* $a = 0$ *or* $d = 0$.

Proof. Let $x, y \in R$. Suppose $ad(u) = 0$ for every $u \in R$, then

$$\begin{aligned} 0 = ad(xy) &\in a(d(x)y + xd(y)) \\ &= ad(x)y + axd(y) \\ &= 0 + axd(y) \\ &= axd(y). \end{aligned}$$

Thus $axd(y) = 0$. Since R is a prime hyperring, $a = 0$ or $d(y) = 0$. If $a \neq 0$, then $d(y) = 0$ for every $y \in R$. That is, $d = 0$. Suppose $d(u)a = 0$ for every $u \in R$, then

$$\begin{aligned} 0 = d(yx)a &\in (d(y)x + yd(x))a \\ &= d(y)xa + yd(x)a \\ &= d(y)xa. \end{aligned}$$

Thus $d(y)xa = 0$. Now, the primeness of R implies that $a = 0$ or $d(y) = 0$. If $a \neq 0$, then $d(y) = 0$ for every $y \in R$. That is, $d = 0$. □

Proposition 4.2. *Let d be a derivation of a 2-torsion free prime hyperring R. If $d^2 = 0$, then $d = 0$.*

Proof. Let $d^2 = 0$. Suppose $d \neq 0$, then there exists an element $a \in R$ such that $d(a) \neq 0$. Then for every $y \in R$,

$$\begin{aligned} d^2(ay) = 0 &= d(d(ay)) \\ &\in d(d(a)y + ad(y)) \\ &\subseteq d(d(a)y) + d(ad(y)) \\ &\subseteq d^2(a)y + d(a)d(y) + d(a)d(y) + ad^2(y) \\ &= d(a)d(y) + d(a)d(y). \end{aligned}$$

Since R is 2-torsion free hyperring, $d(a)d(y) = 0$. Since R is prime hyperring, by the Proposition 4.1, $d(y) = 0$ for every $y \in R$. That is, we get $d = 0$, which is a contradiction to the assumption. Hence $d = 0$. □

Proposition 4.3. *Let d_1, d_2 be derivations of a 2-torsion free prime hyperring R. If $d_1d_2 = 0$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. For $x, y \in R$ we have

$$\begin{aligned} d_1d_2(xy) = 0 &= d_1(d_2(xy)) \\ &\in d_1(d_2(x)y + xd_2(y)) \\ &\subseteq d_1(d_2(x)y) + d_1(xd_2(y)) \\ &\subseteq d_1d_2(x)y + d_2(x)d_1(y) + d_1(x)d_2(y) + xd_1d_2(y) \\ &= d_2(x)d_1(y) + d_1(x)d_2(y). \end{aligned}$$

Replace x by $d_2(x)$, we get $0 \in d_2(d_2(x))d_1(y) + d_1(d_2(x))d_2(y) = d_2^2(x)d_1(y)$. Now, by the Proposition 4.1, one can obtain $d_1 = 0$ or $d_2^2 = 0$. If $d_2^2 = 0$, then by the Proposition 4.2, we have $d_2 = 0$. This completes the proof of the Proposition. $\square$

Definition 4.4. Let d be a non-trivial derivation (resp. strong derivation) of a hyperring R. A hyperideal I of R is said to be a *d-differential (resp. strongly d-differential) hyperideal* of R if $d(I) \subseteq I$.

Let S be a non-empty subset of a hyperring R. The set $Ann_l(S) = \{x \in R : xS = 0\}$ is called the *left annihilator* of S in R. Similarly, we have the *right annihilator* $Ann_r(S)$ of S in R. In a reduced hyperring R, if $ab = 0$ for all $a, b \in R$, then $ba = 0$ and therefore, there is no distinction from a left annihilator of S and a right annihilator of S in R. In this case, we just call it by the *annihilator* of S in R and is denoted by $Ann(S)$. The following results of reduced hyperrings follows from [1].

Proposition 4.5. *Let R be a reduced hyperring. (i) If S is a non-empty subset of R,then $Ann(S)$ is a hyperideal of R. (ii) If S_1 and S_2 are subsets of R such that $S_1 \subseteq S_2$, then $Ann(S_2) \subseteq Ann(S_1)$.*

Proof. The proof of (i) is obvious. Let $x \in Ann(S_2)$. Then $S_2x= 0$. That is, $s_2x = 0$ for all $s_2 \in S_2$. This means that x annihilates all elements of S_2. In particular, x annihilates all elements of S_1. Therefore, $x \in Ann(S_1)$. This completes the proof of (ii). $\square$

Corollary 4.6. *Let R be a reduced hyperring and I be a differential hyperideal of R, then $Ann(I) \subseteq Ann(d(I))$.*

Proof. Since I is a differential ideal of R, we have $d(I) \subseteq I$, by the Proposition 4.5, we get $Ann(I) \subseteq Ann(d(I))$. $\square$

Theorem 4.7. *Let d be a derivation of a reduced hyperring R. Then for any subset S of R, $d(Ann(S)) \subseteq Ann(S)$.*

Proof. If $x \in Ann(S)$, then $Sx = 0$. Now, for $s \in S$, $0 = d(sx) \in d(s)x + sd(x)$. Multiplying by s from the right, we get $0 \in d(s)xs + sd(x)s$. Since $sx = 0$, we have $xs = 0$. Therefore, $sd(x)s = 0$. Multiply by $d(x)$ from the right, we get $sd(x)sd(x) = 0$. That is, $(sd(x))^2 = 0$. Since R is reduced, we obtain $sd(x) = 0$. This means that $d(x) \in Ann(S)$. Thus we have $d(Ann(S)) \subseteq Ann(S))$. $\square$

Theorem 4.7 shows the existence of a differential hyperideal in a differential hyperring. As an illustration we have the following example.

Example 4.8. *Consider the reduced hyperring $R = \{0, a, b, c, \}$ with the hyper-addition $\oplus$ and the multiplication $\odot$ defined as follows.*

$\oplus$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0, b\}$	$\{a, c\}$	$\{b\}$
b	$\{b\}$	$\{a, c\}$	$\{0, b\}$	$\{a\}$
c	$\{c\}$	$\{b\}$	$\{a\}$	$\{0\}$

$\odot$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
b	$\{0\}$	$\{b\}$	$\{b\}$	$\{0\}$
c	$\{0\}$	$\{c\}$	$\{0\}$	$\{c\}$

It is clear that the map $d : R \to R$ defined by $d(0) = 0, d(a) = b, d(b) = b, d(c) = 0$, is a derivation of R. Now, $Ann(0, c) = \{0, b\}$ is a hyperideal of R. Since $d(Ann(0, c)) = d(\{0, b\}) = \{0, b\} = (Ann(0, c))$, we see that $Ann(0, c) = \{0, b\}$ is a differential hyperideal of R.

Let H be a canonical hypergroup and N be a canonical subhypergroup of H. For any two elements $a, b \in H$, we define $a \sim b$ if $a \in b+N$. Then $\sim$ is an equivalence relation on H. We denote the equivalence class determined by the element $x \in H$ by the equivalence relation $\sim$ by $\overline{x}$. It is clear that $\overline{x} = x+N$. We denote the collection of all equivalence classes $\{\overline{x} : x \in H\}$ induced by the equivalence relation $\sim$ by H/N. If we define $\overline{x} \oplus \overline{y} = \{\overline{z} : z \in x+y\}$ for all $\overline{x}, \overline{y} \in H/N$, then H/N is a canonical hypergroup. Let R be a hyperring and I be a hyperideal of R. Since I is a canonical subhypergroup of R, $R/I = \{\overline{x} : x \in R\}$ is a canonical hypergroup under the hyperaddition defined above. Now, we define $\overline{x} \otimes \overline{y} = \overline{xy} = xy + I$ for all $\overline{x}, \overline{y} \in R/I$, then R/I is a Krasner hyperring [18].

Proposition 4.9. *Let R be a strongly differential hyperring. Then for any strongly differential hyperideal I of R, the factor hyperring R/I is a strongly differential hyperring.*

Proof. Let d be a strong derivation of R. Let us define a map $D : R/I \to R/I$ by $D(a+I) = d(a)+I$ for every $a+I \in R/I$. If $a, b \in R$ such that $a+I = b+I$, then $a \in b+I$. Since $d(I) \subseteq I$, we get $d(a) \in d(b)+I$. That is, $d(a)+I = d(b)+I$. Hence $D(a + I) = D(b + I)$. Therefore, D is a well defined map.

Let $r + I, s + I \in R/I$. Now, $D((r + I) + (s + I)) = D(\{x + I : x \in r + s\}) = \{d(x) + I : x \in r + s\}$. Further, $D(r + I) + D(s + I) = (d(r) + I) + (d(s) + I) = \{x+I : x \in d(r)+d(s)\} = \{x+I : x \in d(r+s)\} = \{d(y)+I : \text{for } y \in r+s\}$. Since d is a strong derivation of R, we get $D((r + I) + (s + I)) = D(r + I) + D(s + I)$.

Also, we have $D((r+I)(s+I)) = D(rs+I) = d(rs)+I$. But, $D(r+I)(s+I)+(r+I)D(s+I) = (d(r)+I)(s+I)+(r+I)(d(s)+I) = \{x+I : x \in d(r)s+rd(s)\}$. Since $d(rs) \in d(r)s + rd(s)$, we get $d(rs) + I \in \{x + I : x \in d(r)s + rd(s)\}$. That is, $D((r + I)(s + I)) \in D(r + I)(s + I) + (r + I)D(s + I)$. Thus D is a strong derivation of R/I. □

Proposition 4.10. *Let R be a differential hyperring and I a differential hyperideal of R. Then there exists a one-to-one correspondence between the set of all differential hyperideals of R containing I and the the set of all differential hyperideals of R/I.*

Proof. Let $\mathcal{A}$ be the set of all differential hyperideals of R containing I and $\mathcal{F}$ be the set of all differential hyperideals of R/I. Define a map $f : \mathcal{A} \to \mathcal{F}$ by $f(J) = J/I$, where J is a differential hyperideals of R containing I. Since J is a differential hyperideal of R, it is clear that J/I is a differential hyperideal of R/I. Therefore, the map f is well defined.

Let J, K be two differential hyperideals of R containing I such that $f(J) = f(K)$. Then $J/I = K/I$. Now, for $x \in J$, we have $x+I \in J/I = K/I$. Therefore, $x + I = y + I$ for some $y \in K$. That is, $x \in y + I \subseteq K$. Hence $J \subseteq K$. Similarly, we can prove that $K \subseteq J$ and hence $K = J$. Therefore, the function f is one-to-one. Clearly, the map f is on to. Hence f is a bijective map. $\square$

Let R be a Krasner hyperring. Then, $M_2(R)$, the set of all 2×2 matrices over R, is a hyperring under the usual hyperaddition of matrices and the usual hypermultiplication of matrices.

Proposition 4.11. *Let R be a hyperring and d be a map from R to R. Let φ_d be a map from R to $M_2(R)$ defined by $\varphi_d(r) = \begin{pmatrix} r & d(r) \\ 0 & r \end{pmatrix}$. The map d is a derivation of R if and only if the map φ_d is a homomorphism.*

Proof. Suppose that d is a derivation of R and $x, y \in R$. If $a \in \varphi_d(x + y)$, then $a = \varphi_d(r) = \begin{pmatrix} r & d(r) \\ 0 & r \end{pmatrix}$ for some $r \in x + y$. Since $r \in x + y$, we get $d(r) \in d(x + y) \subseteq d(x) + d(y)$. Thus the element $a = \begin{pmatrix} r & d(r) \\ 0 & r \end{pmatrix}$, where $d(r) \in d(x) + d(y)$. But, $\varphi_d(x) + \varphi_d(y)$

$$= \begin{pmatrix} x & d(x) \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & d(y) \\ 0 & y \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} : u, w \in x + y, v \in d(x) + d(y) \right\}.$$

Therefore, $a \in \varphi_d(x) + \varphi_d(y)$ and hence $\varphi_d(x + y) \subseteq \varphi_d(x) + \varphi_d(y)$.

Now, $\varphi_d(xy) = \begin{pmatrix} xy & d(xy) \\ 0 & xy \end{pmatrix}$. But, $\varphi_d(x)\varphi_d(y) = \begin{pmatrix} x & d(x) \\ 0 & x \end{pmatrix} \begin{pmatrix} y & d(y) \\ 0 & y \end{pmatrix}$

$$= \left\{ \begin{pmatrix} xy & c \\ 0 & xy \end{pmatrix} : c \in xd(y) + d(x)y, \right\}.$$

Since $d(xy) \in d(x)y + xd(y)$, we get $\varphi_d(xy) \in \varphi_d(x)\varphi_d(y)$. Thus the map φ_d is a hyperring homomorphism. Conversely, assume that φ_d is a hyperring homomorphism. Let $u \in d(x + y)$. Then there exists some $r \in x + y$ such that $u = d(r)$. Now, $\begin{pmatrix} r & d(r) \\ 0 & r \end{pmatrix} = \varphi_d(r) \in \varphi_d(x + y) \subseteq \varphi_d(x) + \varphi_d(y)$

$= \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \in x + y, b \in d(x) + d(y) \right\}$. This means that,$u = d(r) \in d(x) + d(y)$. Thus $d(x + y) \subseteq d(x) + d(y)$. Now, $\begin{pmatrix} xy & d(xy) \\ 0 & xy \end{pmatrix} = \varphi_d(xy) \in$

$\varphi_d(x)\varphi_d(y) = \left\{ \begin{pmatrix} xy & c \\ 0 & xy \end{pmatrix} : c \in xd(y) + d(x)y \right\}$. Thus $d(xy) \in xd(y)+d(x)y$. Thus d is a derivation. □

5. **Hyperderivations of hyperrings**

In a ring R, for a fixed $r \in R$, the map $f_r : R \to R$ for all $x \in R$ defined by $f_r(x) = xr - rx$ is a derivation which is called an inner derivation of the ring R. If R is a hyperring, then for $x, r \in R$, $xr - rx$ is a set. That is, we associate a set for each element x of the hyperring R. This motivates to define hyperderivation on a hyperring R. In this section we define hyperderivation and give examples. Further, we prove that the collection of all hyperderivations of a hyperring R is an additive semigroup.

Definition 5.1. A map f from a hyperring R to $\mathcal{P}^*(R)$ is said to be a *hyperderivation or hderivation* if it satisfies:

(i) $f(x+y) \subseteq f(x) + f(y)$ and

(ii) $f(xy) \in f(x)y + xf(y)$ for all $x, y \in R$.

If the map f is such that $f(x+y) = f(x) + f(y)$ for all $x \in R$ and satisfies the condition (ii) then f is called a *strong hderivation of* R. The collection of all hderivations of R is denoted by $\mathcal{D}$.

Example 5.2. *Let R be a hyperring. Now, select $r \in R$ and fix it. Define a map $f_r : R \to \mathcal{P}^*(R)$ by $f_r(x) = xr - rx$ for every $x \in R$. That is, for every $x \in R$, we associate a non-empty set $xr - rx$ by f_r. Clearly, this map is well defined.*

If $a \in f_r(x+y)$, then $a \in f_r(s)$ for some $s \in x+y$. So, $a \in sr - rs \subseteq (x+y)r - r(x+y) = xr + yr - rx - ry = xr - rx + yr - ry = f_r(x) + f_r(y)$. Therefore, $f_r(x+y) \subseteq f_r(x) + f_r(y)$.

Also, if $a \in f_r(xy) = xyr - rxy = xyr - rxy + (0) \subseteq xyr - rxy + xry - xry = xry - rxy + xyr - xry = (xr - rx)y + x(yr - ry) = f_r(x)y + xf_r(y)$. Thus $f_r(xy) \subseteq f_r(x)y + xf_r(y)$. That is, f_r is a hderivation. This f_r is called an inner hderivation of R.

Example 5.3. *Let $d : R \to R$, be a derivation of a hyperring R. Now, define a map D from R to $\mathcal{P}^*(R)$ by $D(x) = d(x) + d(x)$ for all $x \in R$. That is, for every $x \in R$, we associate a non-empty set $d(x) + d(x)$ by D.*

Let $r \in D(x+y)$. Then there exists $s \in x+y$ such that $r \in D(s) = d(s) + d(s) \subseteq d(x+y)+d(x+y) \subseteq d(x)+d(y)+d(x)+d(y) = d(x)+d(x)+d(y)+d(y) = D(x) + D(y)$. Thus $D(x+y) \subseteq D(x) + D(y)$.

Also, $D(xy) = d(xy) + d(xy) \subseteq d(x)y + xd(y) + d(x)y + xd(y) = d(x)y + d(x)y + xd(y) + xd(y) = D(x)y + xD(y)$ for all $x, y \in R$. This D is a hderivation induced by d.

Remark 5.4. In the Example 3.6, $D(a+a) = D(\{a,b\}) = D(a) \cup D(b) = (d(a) + d(a)) \cup (d(b) + d(b)) = (b+b) \cup (a+a) = \{a,b\} \cup \{a,b\} = \{a,b\}$.

Moreover, $D(a)+D(a)=(d(a)+d(a))+(d(a)+d(a))=\{a,b\}+\{a,b\}=R$. So this hderivation D is not a strong hderivation even though d is a strong derivation.

Example 5.5. *Let d be a derivation of a hyperring R. If we define D' from R to $\mathcal{P}^*(R)$ by $D'(x)=d(x)-d(x)$ for all $x\in R$, then $D'(x+y)\subseteq D'(x)+D'(y)$ and $D'(xy)\in xD'(y)+D'(x)y$ for all $x,y\in R$. This D' is a hderivation induced by d. The collection of all hderivations induced by a derivation d is denoted by $\mathcal{D}_d$. It is clear that $\mathcal{D}_d\subseteq\mathcal{D}$.*

Proposition 5.6. *The collection of all hderivations of a hyperring R is an additive semigroup.*

Proof. Suppose that $D_1,D_2\in\mathcal{D}$. Now, for any $x\in R$ we define $(D_1+D_2)x=D_1(x)+D_2(x)$. Let $x,y\in R$ and $r\in(D_1+D_2)(x+y)$. Then $r\in(D_1+D_2)(s)$ for some $s\in x+y$. But, $(D_1+D_2)(s)=D_1(s)+D_2(s)\subseteq D_1(x+y)+D_2(x+y)\subseteq D_1(x)+D_1(y)+D_2(x)+D_2(y)=D_1(x)+D_2(x)+D_1(y)+D_2(y)=(D_1+D_2)(x)+(D_1+D_2)(y)$. So $r\in(D_1+D_2)(x)+(D_1+D_2)(y)$. That is,$(D_1+D_2)(x+y)\subseteq(D_1+D_2)(x)+(D_1+D_2)(y)$.

Also, $(D_1+D_2)(xy)=D_1(xy)+D_2(xy)\subseteq D_1(x)y+xD_1(y)+D_2(x)y+xD_2(y)=D_1(x)y+D_2(x)y+xD_1(y)+xD_2(y)=(D_1(x)y+D_2(x))y+x(D_1(y)+D_2(y))=((D_1+D_2)(x))y+x((D_1+D_2))(y)$. Thus $(D_1+D_2)(xy)\subseteq((D_1+D_2)(x))y+x((D_1+D_2)(y))$ Hence $D_1+D_2\in\mathcal{D}$.

Moreover, if $D_1,D_2,D_3\in\mathcal{D}$, then $(D_1+(D_2+D_3))x=D_1(x)+(D_2+D_3)x=D_1(x)+D_2(x)+D_3(x)$ for all $x\in R$. Similarly, $((D_1+D_2)+D_3)x=D_1(x)+D_2(x)+D_3(x)$ for every $x\in R$. Thus $D_1+(D_2+D_3)=(D_1+D_2)+D_3$. Further, the map $0:R\longrightarrow\mathcal{P}^*(R)$ defined by $0(x)=\{0\}$ for all $x\in R$ is the additive identity of $\mathcal{D}$. □

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