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Bessel Subfusion Sequences and Subfusion Frames

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Abstract. Fusion frames are a generalized form of frames in Hilbert spaces. In the present paper we introduce Bessel subfusion sequences and subfusion frames and we investigate the relationship between their operation. Also, the definition of the orthogonal complement of subfusion frames and the definition of the completion of Bessel fusion sequences are provided and several results related with these notions are shown.

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1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [6]. The theory of frames has been generalized rapidly during last decades. Recently, for modelling some wider ranges of applications, various generalization forms of frames have been proposed [9, 12].

Casazza and Kutyniok [2] formulated a general method for piecing local frames in order to produce global frames. They introduced a new type of generalized frames as fusion frames. Some examples of fusion frames are; wireless sensor network [7], geophones in geophysics measurement and studies [5] and the physiological structure of visual and hearing system [10].

In this paper we introduce subfusion frames, completion of Bessel fusion sequences and fusion frame sequences in a Hilbert space. We also investigate the necessary and sufficient conditions for a Bessel fusion sequence to be a fusion frame.

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In the second section, we review the concept and properties of fusion frames. In the third section, subfusion frames are defined and the fusion frame properties of their orthogonal complements are shown. The last section is devoted to study of the completion of Bessel fusion sequences.
Throughout this paper, H is a separable Hilbert space, I is a countable index set and $\{V_i\}_{i\in I}$ is a sequence of closed subspaces of H.

2. Review of fusion frames

A brief review of some definitions and basic properties of fusion frames is provided here. For more information, the interested reader is referred to [2,4].

Definition 2.1. *Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H and let $\{\alpha_i\}_{i\in I}$ be a family of weights i.e., $\alpha_i > 0$ for all $i \in I$. Then $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ is a fusion frame, if there exist positive constants C and D (lower and upper fusion frame bounds, respectively) such that*

$$C\|f\|^2 \leq \sum_{i\in I} \alpha_i^2 \|\pi_{V_i}(f)\|^2 \leq D\|f\|^2, \quad for\, all f \in H, \tag{2.1}$$

where π_{V_i} is the orthogonal projection onto the subspace V_i.

1. *The optimal lower fusion frame bound is the supremum over all lower bounds, and the optimal upper fusion frame bound is the infimum over all upper bounds.*
2. *A fusion frame ν is called tight fusion frame with bound λ if $C = D = \lambda$, and α-uniform fusion frame if $\alpha = \alpha_i$ for all $i \in I$.*
3. *If the inequality to the right in (2.1) is satisfied, then ν is called a Bessel fusion sequence for H with Bessel bound D.*
4. *If $H = \oplus V_i$ and $\inf_{i\in I} \alpha_i > 0$, then ν is called an orthogonal basis of subspaces for H.*
5. *Similar to ordinary frames, the fusion frame operator S_ν is defined by*
$$S_\nu(f) = \sum_{i\in I} \alpha_i^2 \pi_{V}{}_i f, \qquad for\ all\ f \in H.$$

The operator S_ν is linear, bounded, positive, self-adjoint and invertible and the following condition is true.

$$CId_H \leq S_\nu \leq DId_H.$$

While the fusion frame operator could be defined for any Bessel fusion sequence, the inequalities related with S_ν in Definition 2.1 are true if and only if ν is a fusion frame. Note that, for selfadjoint operators U and V, $U \leq V$ if and only if $\langle Ux, x\rangle \leq \langle Vx, x\rangle$ for all $x \in H$.

Proposition 2.2. *Let $\{(V_i, \alpha_i)\}_{i\in I}$ be a Bessel fusion sequence for H, W_i be a closed subspace of V_i and $\beta_i \leq \alpha_i$ for all $i \in I$. Then $\{(W_i, \beta_i)\}_{i\in I}$ is a Bessel fusion sequence for H.*

Proof. Since W_i is a closed subspace of V_i,

$$\pi_{W_i}\pi_{V_i}f = \pi_{V_i}\pi_{W_i}f = \pi_{W_i}f \text{ and } \|\pi_{W_i}f\|^2 \le \|\pi_{V_i}f\|^2$$

for all $f \in H$ and for all $i \in I$. Hence

$$\sum_{i\in I}\beta_i^2\|\pi_{W}{}_i f\|^2 \le \sum_{i\in I}\alpha_i^2\|\pi_{V}{}_i f\|^2$$

implies that $\{(W_i, \beta_i)\}_{i\in I}$ is a Bessel fusion sequence. □

We can easily find examples such that $\{(V_i, \alpha_i)\}_{i\in I}$ is a fusion frame, W_i is a closed subspace of V_i and $\beta_i \le \alpha_i$ for all $i \in I$, while $\{(W_i, \alpha_i)\}_{i\in I}$ is not a fusion frame.

Example 2.3. *Let $\{(V_i, \alpha_i)\}_{i\in\mathbb{N}}$ be a fusion frame for H and $V_1 \ne H$. Define*

$$W_i = \begin{cases} V_1 & if \;\; i = 1, \\ 0 & otherwise. \end{cases}$$

Since $\overline{span}_{i\in\mathbb{N}}\{W_i\} = V_1 \ne H$, $\{(W_i, \alpha_i)\}_{i\in I}$ is not a fusion frame for H [2, Lemma 3.4].

3. Subfusion frames

Definition 3.1. *Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame for H, W_i be a closed subspace of V_i and $\beta_i \le \alpha_i$ for all $i \in I$. If $\omega = \{(W_i, \beta_i)\}_{i\in I}$ is a fusion frame for H, then ω is called a subfusion frame of ν. If ν and ω are Bessel fusion sequences for H, then ω is called a Bessel subfusion sequence of ν.*

A notion related to subfusion frames have been brought in [11], which is called frame of subspaces refinement (shortly:FSR). A subfusion frame $\omega = \{(W_i, \beta_i)\}_{i\in I}$ of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ is a FSR if $\alpha_i = \beta_i$ for all $i \in I$. Therefore a FSR is a special subfusion frame and the authors have studied the excess of FSR in [11].

As the following theorem shows, under some conditions, the orthogonal complement of a subfusion frame is a subfusion frame too.

Theorem 3.2. *Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame for H with lower bound A and upper bound B. Also, let $\{(W_i, \alpha_i)\}_{i\in I}$ be a FSR of $\{(V_i, \alpha_i)\}_{i\in I}$ with lower bound C and upper bound D such that $A > D$. Suppose that Y_i is the orthogonal complement of W_i with respect to V_i i.e., $Y_i = \{x \in V_i : \langle x, z\rangle = 0, for \;\; all \;\; z \in W_i\}$. Then $\{(Y_i, \alpha_i)\}_{i\in I}$ is a FSR of ν.*

Proof. By the orthogonality of Y_i and W_i,

$$\|\pi_{Y_i}f\|^2 + \|\pi_{W_i}f\|^2 = \|\pi_{V_i}f\|^2 \qquad \text{for all } f \in H.$$

Therefore,

$$\begin{aligned}\sum_{i\in I}\alpha_i^2\|\pi_{Y\,i}f\|^2 &= \sum_{i\in I}\alpha_i^2\|\pi_{V_i}f\|^2-\sum_{i\in I}\alpha_i^2\|\pi_{W_i}f\|^2\\ &\leq B\|f\|^2-C\|f\|^2\\ &= (B-C)\|f\|^2.\end{aligned}$$

Similarly, the following condition is true.

$$(A-D)\|f\|^2\leq\sum_{i\in I}\alpha_i^2\|\pi_{Y\,i}f\|^2.$$

Thus $\{(Y_i,\alpha_i)\}_{i\in I}$ is a fusion frame with frame bounds $B-C\geq A-D>0$. □

Lemma 3.3. *Let $\omega=\{(W_i,\beta_i)\}_{i\in I}$ be a Bessel subfusion sequence of $\nu=\{(V_i,\alpha_i)\}_{i\in I}$. Then $S_\omega\leq S_\nu$. Moreover, if ω is a fusion frame for H, so is ν.*

Proof. Since W_i is a closed subspace of V_i and $\beta_i\leq\alpha_i$,

$$\beta_i^2\|\pi_{W_i}f\|^2\leq\alpha_i^2\|\pi_{V_i}f\|^2,\text{ for all } i\in I.$$

Therefore $\sum_{i\in I}\beta_i^2\|\pi_{W\,i}f\|^2\leq\sum_{i\in I}\alpha_i^2\|\pi_{V\,i}f\|^2$ and $S_\omega\leq S_\nu$. □

Lemma 3.4. *Let $\omega=\{(W_i,\beta_i)\}_{i\in I}$ be a subfusion frame of the fusion frame $\nu=\{(V_i,\alpha_i)\}_{i\in I}$ with fusion frame operators S_ω and S_ν, respectively. Then the optimal fusion frame bounds of ω is not greater than of the optimal fusion frame bounds of ν.*

Proof. Upper and lower optimal bounds of ω are $\|S_\omega\|$ and $\|S_\omega^{-1}\|^{-1}$, respectively [3]. By Lemma 3.3. $0<S_\omega\leq S_\nu$ and $0<S_\nu^{-1}\leq S_\omega^{-1}$. Thus

$$\|S_\omega\|=\sup_{\|f\|=1}\langle S_\omega f,f\rangle\leq\sup_{\|f\|=1}\langle S_\nu f,f\rangle=\|S_\nu\|\text{ and }\|S_\omega^{-1}\|^{-1}\leq\|S_\nu^{-1}\|^{-1}.$$

□

Let $\nu=\{(V_i,\alpha_i)\}_{i\in I}$ and $\omega=\{(W_i,\beta_i)\}_{i\in I}$ be two Bessel fusion sequences for H. Then the frame operator for them is defined by

$$S_{\nu\omega}=\sum_{i\in I}\alpha_i\beta_i\pi_{V_i}\pi_{W_i}f,\quad\textit{for all } f\in H,$$

which is bounded and $S_{\nu\omega}^*=S_{\omega\nu}$ [8]. If ω is a Bessel subfusion sequence of ν, then $S_{\nu\omega}$ is self adjoint and positive.

Let $\{(W_i,\alpha_i)\}_{i\in I}$ be a fusion frame for H and let ${W_i}^\perp$ be the orthogonal complement of W_i . If the family $\{(W_i^\perp,\alpha_i)\}_{i\in I}$ is also a fusion frame, then $\{(W_i^\perp,\alpha_i)\}_{i\in I}$ is called the orthogonal fusion frame of $\{(W_i,\alpha_i)\}_{i\in I}$[1] .

Lemma 3.5. *Let $\omega=\{(W_i,\alpha_i)\}_{i\in I}$ be a subfusion frame of $\nu=\{(V_i,\alpha_i)\}_{i\in I}$. Then orthogonal fusion frame of ν is a subfusion frame of the orthogonal fusion frame of ω.*

Proof. The result is obtained by using $W_i\subset V_i$ and $V_i^\perp\subset W_i^\perp$ for all $i\in I$. □

Lemma 3.6. *Let $\omega = \{(W_i, \alpha_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ with optimal fusion frame bounds $A \leq B$ and $C \leq D$, respectively, and let $\bigcap_{i\in I} Vi = \{0\}$. Then orthogonal fusion frame of ν has the optimal bounds $\sum_{i\in I} \alpha_i^2 - D$ and $\sum_{i\in I} \alpha_i^2 - C$ and is a subfusion frame of the orthogonal fusion frame of ω with optimal bounds $\sum_{i\in I} \alpha_i^2 - B$ and $\sum_{i\in I} \alpha_i^2 - A$.*

Proof. Conclusion is obtained by using Lemma 3.6, [1,Theorem 2.2] and the fact that $\bigcap_{i\in I} W_i \subset \bigcap_{i\in I} V_i = \{0\}$. □

The preceding lemma shows that, the optimal bounds of a fusion frame and its orthogonal fusion frame could be obtained from each other.

Lemma 3.7. *Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a Bessel subfusion sequence of a Bessel fusion sequence $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ with bound D. If positive constant B is a lower bound of $S_{\nu\omega}$, then $S_{\nu\omega}$ is invertible and ω is a subfusion frame of fusion frame ν with the same lower bound $\frac{B^2}{D}$.*

Proof. Since $S_{\nu\omega}$ is bounded below, it is one-to-one and has a closed range. Therefore

$$Range(S_{\nu\omega}) = \overline{Range(S_{\nu\omega})} = N(S_{\nu\omega})^{\perp} = H.$$

For all $f, g \in H$

$$\langle S_{\nu\omega} f, g\rangle = \sum_{i\in I} \alpha_i \beta_i \langle \pi_{W_i} f, \pi_{V_i} g\rangle.$$

Now, by Cauchy-Schwartz inequality

$$|\langle S_{\nu\omega} f, g\rangle| \leq (\sum_{i\in I} \alpha_i^2 \|\pi_{V_i} g\|^2)^{\frac{1}{2}} (\sum_{i\in I} \beta_i^2 \|\pi_{W_i} f\|^2)^{\frac{1}{2}}.$$

Thus

$$\|S_{\nu\omega} f\| \leq \sqrt{D} (\sum_{i\in I} \beta_i^2 \|\pi_{W_i} f\|^2)^{\frac{1}{2}}$$

and

$$\frac{B^2\|f\|^2}{D} \leq \sum_{i\in I} \beta_i^2 \|\pi_{W_i} f\|^2 \leq \sum_{i\in I} \alpha_i^2 \|\pi_{V_i} f\|^2 \leq D\|f\|^2.$$

□

Lemma 3.8. *If $\omega = \{(W_i, \alpha_i)\}_{i\in I}$ is a Bessel subfusion sequence of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$, then $S_{\nu\omega} = S_\omega$.*

Proof. For all f, we have

$$S_{\nu\omega} f = \sum_{i\in I} \alpha_i^2 \pi_{V_i} \pi_{W_i} f = \sum_{i\in I} \alpha_i^2 \pi_{W_i} f = S_\omega f.$$

□

As a consequence of lemmas above , we have the following corollarie.

Corollary 3.9. *Let ω, ν, B and D be as Lemma 3.7. If S_ω is bounded below, then S_ω is invertible and ω is a subfusion frame of fusion frame ν with a lower bound given by $\frac{B^2}{D}$.*

4. Completion of Bessel fusion frames

Definition 4.1. *Let $\{(W_i, \alpha_i)\}_{i\in I}$ be a Bessel fusion sequence for H. If there are an index set J, a sequence of closed subspaces $\{W_i\}_{i\in J}$ and weights $\alpha_i, i \in J$ such that $\{(W_i, \alpha_i)\}_{i\in I\cup J}$ is a fusion frame for H, then $\{(W_i, \alpha_i)\}_{i\in J}$ is called a completion of $\{(W_i, \alpha_i)\}_{i\in I}$.*

In [2, Section 5.1], the authors have stated,
"Dealing with Bessel fusion sequences is important, since there are easy ways to turn such these families into fusion frames. One way is to just add the subspace $W_0 = H$ to the family. Another more careful method is the following one: Take any orthonormal basis for H and partition its elements into the subspaces $W_i, i \in I$. Then add the subspace spanned by the remaining elements to the Bessel family. This yields a fusion frame."
By a counter example, we show that the above second method is not true.

Example 4.2. *If $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for the Hilbert space $l^2(Z)$. Define $W_1 = \langle e_1\rangle, W_2 = \langle e_1, e_2\rangle, \ldots, W_i = \langle e_1, \ldots, e_i\rangle$ for $i = 1, 2, \ldots$. For all $f \in l^2(Z)$, we have $f = \sum_{i=-\infty}^{\infty} c_i e_i$. Thus*

$$\pi_{W_i}(f) = c_1e_1 + \ldots + c_ie_i,$$

and hence

$$\|\pi_{W_i}(f)\|^2 = \sum_{j=1}^{i} |c_j|^2 \quad for \quad i = 1, 2, \ldots.$$

Now, $\{(W_i, \sqrt{\frac{1}{2^i}})\}_{i=1}^{\infty}$ is a Bessel fusion sequence for H, because of,

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{1}{2^i}\|\pi_{W_i}(f)\|^2 &= \sum_{i=1}^{\infty} \frac{1}{2^i} \sum_{j=1}^{i} |c_j|^2 \\ &\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \sum_{j=1}^{\infty} |c_j|^2 \\ &\leq \|f\|^2 \sum_{i=1}^{\infty} \frac{1}{2^i} \\ &= \|f\|^2. \end{aligned}$$

Since $\overline{span}_{i\in\mathbb{N}}\{W_i\}$ is a proper subspace of H, $\{(W_i, \sqrt{\frac{1}{2^i}})\}_{i\in\mathbb{N}}$ is not a fusion frame for H [2 Lemma3.4].
Set $W_0 = \langle e_0, e_{-1}, e_{-2}, \ldots\rangle$, we show that $\{(W_i, \sqrt{\frac{1}{2^i}})\}_{i=0}^{\infty}$ is not a fusion frame.

Let $\{(W_i, \sqrt{\frac{1}{2^i}})\}_{i=0}^{\infty}$ *be a fusion frame for* $l^2(Z)$*, and* A *be one of its lower bounds. Applying* $f = e_1$ *in the inequality* $A\|f\|^2 \leq \sum_{i=0}^{\infty} \frac{1}{2^i}\|\pi_{W_i}(f)\|^2$ *implies that,*

$$A = A\|e_1\|^2 \leq \sum_{i=0}^{\infty} \frac{1}{2^i}\|\pi_{W_i}(e_1)\|^2 = \sum_{i=1}^{\infty} \frac{1}{2^i}\|e_1\|^2 = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$

Hence $A \leq 1$*. Also by substitution of* f *with* $= e_n$*, in the above inequality, we obtain* $A \leq \frac{1}{n}$ *for all* n*. Therefore* $A = 0$*, and this is a contradiction. Thus the sequence* $\{(W_i, \sqrt{\frac{1}{2^i}})\}_{i\in\mathbb{N}\cup\{\circ\}}$ *is not a fusion frame.*

In the following proposition we show a sufficient condition under which a Bessel fusion sequence admits a FSR with mutually orthogonal subspaces.

Proposition 4.3. *Let* $\{W_i\}_{i\in\mathbb{N}}$ *be a sequence of closed subspaces of* H *such that their corresponding orthogonal projections commute with each other and let* $V_0 = (\overline{span}_{j\in\mathbb{N}}\{W_j\})^{\perp}$*,* $V_1 = W_1$*,* $V_2 = W_2 \cap W_1^{\perp}$ *and*

$$V_i = W_i \cap (\overline{span}_{j\in\{1,..,i-1\}}\{W_j\})^{\perp}.$$

If $\{(W_i, \alpha_i)\}_{i\in\mathbb{N}}$ *is a Bessel fusion sequence with* $\alpha_i > B$*, for some* $B > 0$*, then* $\{(V_i, \alpha_i)\}_{i\in\mathbb{N}\cup\{0\}}$ *is a fusion frame for* H*, where* $\alpha_0 = B^2$*.*

Proof. The commutativity of π_{W_i} and π_{W_j} implies that W_i and $W_i^{\perp}$ are invariant under π_{W_j}. We can easily check that every $f \in H$ has a unique decomposition of the form,

$$f = f_1 + f_2 + \cdots + f_n + g_n, \quad f_i \in V_i, 1 \leq i \leq n \;\; \textit{and} \;\; g_n \in \cap_{i=1}^{n} W_i^{\perp}.$$

When $n \to \infty$, the above decomposition turns to

$$f = \sum_{i\in\mathbb{N}} f_i + g_\circ, \quad f_i \in V_i, i \in \mathbb{N} \;\; \textit{and} \;\; g_\circ \in V_0.$$

Now let D be a Bessel fusion bound of $\{(W_i, \alpha_i)\}_{i\in\mathbb{N}}$. Then for any $f \in H$ we have

$$\begin{aligned} B^2\|f\|^2 &= \sum_{i\in\mathbb{N}} B^2\|\pi_{V_i} f\|^2 + B^2\|\pi_{V_0} f\|^2 \\ &\leq \sum_{i\in\mathbb{N}} \alpha_i^2\|\pi_{W_i} f\|^2 + \alpha_0\|\pi_{V_0} f\|^2 \\ &\leq (D + \alpha_0)\|f\|^2, \end{aligned}$$

and the proof is completed. □

Definition 4.4. *A sequence* $\{W_i, \alpha\}_{i\in\mathbb{N}}$ *is called a fusion frame sequence in* H *if it is a fusion frame for* $\overline{span}_{i\in\mathbb{N}}\{W_i\}$*.*

Proposition 4.5. *A fusion Bessel sequence* $\omega = \{(W_i, \alpha_i)\}_{i\in\mathbb{N}}$ *for* H *is a fusion frame sequence if and only if* $(W^{\perp}, \alpha)$ *is a fusion frame completion of* ω*, where* α *is an arbitrary positive number and* $W = \overline{span}_{i\in\mathbb{N}}\{W_i\}$*.*

Proof. Let fusion Bessel sequence $\omega = \{(W_i, \alpha_i)\}_{i\in\mathbb{N}}$ for H is a fusion frame sequence and let positive constant α be given. Then there are positive numbers C and D such that

$$C\|f\|^2 \leq \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \qquad \text{for all } f \in W.$$

Since every $h \in H$ can be written in the form of $h = f + g$, where $f \in W$ and $g \in W^\perp$, thus

$$\begin{aligned} A\|h\|^2 &= A\|f\|^2 + A\|g\|^2 \\ &\leq \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 + \alpha\|\pi_{W^\perp}(g)\|^2 \\ &= \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(h)\|^2 + \alpha\|\pi_{W^\perp}(h)\|^2 \\ &\leq D\|f\|^2 + \alpha\|g\|^2 \\ &\leq B\|h\|^2, \end{aligned}$$

where $A = min\{C, \alpha\}$ and $B = max\{D, \alpha\}$. The Bessel property of ω with the above similar argument establishes the converse. □

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References

1. R. Calderbank, P. G. Casazza, A. Heinecke, G. Kutyniok, A. Pezeshki, *Constructing fusion frames with desired parameters*, Wavelets XIII (San Diego, CA, 2009), 744614-1 - 744614-11, SPIE Proc. 7446, SPIE, Bellingham, WA, 2009.
2. P. G. Casazza, G. Kutyniok, *Frames of subspaces.* In: Wavelets, Frames and Operator Theory, College Park, MD, 2003.
3. P. G. Casazza, G. Kutyniok, S. Li, Fusion frame and distributed processing, *Apple. Comput. Harmon. Anal.*, **25**, (2008), 114-132.
4. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhouser, Boston, 2003.
5. M. S. Craig, R. L. Genter, Geophone array formation and semblance evaluation, *Geophysics*, **71**, (2006), 1-8.
6. R. J. Duffin, A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.*, **72**, (1952), 341-366.
7. S. S. Iyengar, R. R. Brooks, *Distributed Sensor Networks,* Chapman Baton Rouge, 2005.
8. A. Khosravi, K. Musazadeh, Fusion frame and g-frames, *J. Math. Anal. Appl.*, **324**, (2008), 1068-1083.
9. R. Raisi Tousi, R.A. Kamyabi Gol, Shift invariant spaces and shift preserving operators on locally compact abelian groups, *Iranian Journal of Mathematical Sciences and Informatics*, **6**(2), (2011), 21-32.
10. C. J. Rozell, D. H. Johnson, Analysing the robustness of redundant population code in sonsory and feature extraction systems, *Neurocomputing*, **69**(10-12), (2006), 1215-1218.
11. M. A. Ruiz, D. Stojanoff, Some properties of frames of subspaces obtained by operator theory methods, *J. Math. Anal. Appl.*, **343**(1), (2008), 366-378.
12. A. Safapour, M. Shafiee, Constructing Finite frames via Platonic Solids, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(1), (2012), 35-41.