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## Higher rank Einstein solvmanifolds

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ABSTRACT. In this paper we study the structure of standard Einstein solvmanifolds of arbitrary rank. Also the validity of a variational method for finding standard Einstein solvmanifolds is proved.

**Keywords:** Nilpotent Lie algebra, Einstein, Solvmanifold, Critical point, Ricci soliton, Left invariant metric.

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# 1. INTRODUCTION

General form of standard Einstein solvmanifolds were determined by Jense Heber (see [2]). Later, Gorge Lauret deeply studied this kind of manifolds. Solvable Lie group endowed with the left invariant Riemanian metric is called solvmanifold. Let S be a simply connected Lie group with the corresponding Lie algebra s endowed with the inner product determined by  $\langle ., . \rangle$  and solvable Lie bracket [.,.]. We call S a higher rank solvmanifold if

$$s = n \oplus a; \quad n = [s, s], \ a = n^{\perp}$$

where n is a metric nilpotent Lie algebra of dimension k. The codimension n is called the rank of S. The solvable Lie group (S, [.,.], < ., . >) is called standard if a is abelian and it is said to be Einstein if its Ricci tensor  $ric_{[.,.]}$  satisfies ric < ., . >= c < ., . >, for some  $c \in \mathbb{R}$ . s is called a metric solvable extension of n if the restriction of the Lie bracket and inner product of s to n coincide respectively with the Lie bracket and inner product of n. Let  $\Lambda^2 n^* \otimes n$ 

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be the vector space of all bilinear skew-symmetric maps from  $n \times n$  to n. There is a natural action of GL(k) on  $\Lambda^2 n^* \otimes n$  which is given by

$$\phi.\mu(X,Y) = \phi\mu(\phi^{-1}X,\phi^{-1}Y); \ X,Y \in n, \ \phi \in GL(k), \ \mu \in \Lambda^2 n^* \otimes n.$$

Let N denote a simply connected nilpotent Lie group with Lie algebra  $(n, \mu)$ endowed with the left invariant Riemannian metric  $\langle ., . \rangle_n$ , where  $\mu$  is a nilpotent Lie algebra on n. The Ricci operator  $R_{\mu} : n \to n$  of N is defined by

$$< R_{\mu} \dots >_n = ric < \dots >_n$$
 .

This operator is reduced to

$$(1.1) < R_{\mu}X, Y >= -\frac{1}{2}\sum_{i,j} < \mu(X, X_i), X_j) >< \mu(Y, X_i), X_j) >$$
$$+\frac{1}{4}\sum_{i,j} < \mu(X_i, X_j), X >< \mu(X_i, X_j), Y >,$$

for all  $X, Y \in n$ , where  $\{X_1, X_2, ..., X_k\}$  is any orthonormal basis of n. The inner product  $\langle .., ..\rangle_n$  determines an inner product on  $\Lambda^2 n^* \otimes n$ , denoted by  $\langle .., ..\rangle$  and given by

$$\langle \mu, \lambda \rangle = \sum_{i,j,k} \langle \mu(X_i, X_j), X_k \rangle \langle \lambda(X_i, X_j), X_k \rangle$$

Also it naturally determines a norm on  $\Lambda^2 n^* \otimes n$  defined by

$$\forall \lambda \in \Lambda^2 n^* \otimes n \qquad \|\lambda\| = \sum_{ijk} < \lambda(X_i, X_j), X_v >^2 N$$

Consider the Riemannain function

$$F_k: \Lambda^2 n^* \otimes n \longrightarrow \mathbb{R}, \quad F_k(\mu) = tr R_{\mu}^2$$

and the sphere  $S_r$  given by

$$S_r = \{\mu \in \Lambda^2 n^* \otimes n; \|\mu\|^2 = 2r^2\},$$

for some  $r \in \mathbb{R}$ . Let  $\aleph_k$  be the vector space of all nilpotent Lie brackets on nand  $Der(\mu)$  be the Lie algebra of all derivations on n. Then  $\mu \in \aleph_k$  is called a Ricci soliton if  $R_{\mu} = cI + D$ , for some  $D \in Der(\mu)$  and  $c \in \mathbb{R}$ .

In [6], Jorge Lauret has proved that the standard Einstein solvmanifolds are exactly the critical points of modified scalar curvature function  $F_k|_{S_1}$ .

**Theorem 1.1.** [6]. For  $\mu \in \aleph_k \cap S_1$ , the following statements are equivalent:

(i)  $\mu$  is a critical point of  $F_k|_{S_1}$ .

(ii)  $\mu$  is a critical point of  $F_k|_{GL(k),\mu\cap S_1}$ .

(iii)  $\mu$  admits a rank-one extension which is Einstein.

(iv)  $\mu$  is a Ricci soliton.

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#### 2. Einstein solvmanifolds of rank $\geq 1$

In this section, the structure of standard Einstein solvmanifolds is introduced. The next lemma provides some useful properties of solvable Lie bracket and inner product of a solvable Lie group. Then, we extend the Ricci soliton in [4] for any arbitrary rank. We call it multiple Ricci soliton.

**Lemma 2.1.** [2]. Let  $(s = n \oplus a, [.,.], < .,. >)$  be a metric solvable extension of  $(n, \mu, < .,. >)$ , where for every  $0 \neq A \in a$ ,  $ad_A$  is nonzero and symmetric, then

 $\begin{array}{l} (i) < R_{[.,.]}A, B >= -tr(ad_A ad_B), \ for \ all \ A, B \in a. \\ (ii) < R_{[.,.]}A, X >= 0, \ for \ all \ A \in a, X \in n. \\ (iii) \ R_{[.,.]|_n} = -ad_Z|_n + R_{\mu}, \ where < Z, X >= tr(ad_X), \ for \ all \ X \in n. \end{array}$ 

**Definition 2.2.**  $0 \neq \mu \in \aleph_k$  is called a multiple Ricci soliton of degree r if

- (a)  $R_{\mu} = c_{\mu}I + D_{\mu}; \ D_{\mu} \in Der(\mu), \ c_{\mu} \in \mathbb{R}.$
- (b) There are nonzero symmetric derivations  $D_i$ ,  $1 \le i \le r$ , such that

$$D_{\mu} = D_1 + D_2 + \ldots + D_r, \quad tr D_i D_j = -\delta_{ij} c_{\mu} tr D_i$$

**Remark 2.3.** If  $\mu$  is a multiple Ricci soliton of degree r, then  $\mu$  is a multiple Ricci soliton of degree less than r. Therefore,  $\mu$  is the critical point of  $F_k|_{S_r \cap GL(k),\mu}$ .

Using Definition 2.1, we study the structure of standard Einstein solvmanifolds as follows.

**Proposition 2.4.** For  $0 \neq \mu \in \aleph_k \cap S_r$ , the following statements are equivalent:

- (i)  $\mu$  admits a metric extension which is Einstein.
- (ii)  $\mu$  is a multiple Ricci soliton.

Proof. Let the Lie algebra  $(n,\mu)$  admit an Einstein metric extension S with corresponding Lie algebra  $(s = n \oplus a, [.,.], < .., >)$  such that dim(a) = r. Let  $\{H_1, H_2, \ldots, H_r\}$  be an orthonormal basis for a and Z be the mean curvature vector field for the simply connected Lie group N with Lie algebra n. A straightforward calculation shows that  $D_{\mu} = D_1 + D_2 + \ldots + D_r$ , where  $D_{\mu} :=$  $ad_Z|_n$  and  $D_i = tr(ad_{H_i})ad_{H_i}|_n$ . [.,.] is the Lie bracket. Hence,  $D_i\mu(.,.) =$  $\mu(., D_i.) + \mu(D_i.,.)$ ; that is to say,  $D_i$ 's are derivations on n. Suppose that  $D_i$ 's and  $D_{\mu}$  are symmetric (see [2; 4.10]). Let  $Z_i = tr(ad_{H_i})H_i$ , then Lemma 2.1 implies that

$$trD_iD_j = tr(ad_{Z_i}ad_{Z_j}) = - \langle R_{[,,]}Z_i, Z_j \rangle = -c_{\mu} \langle Z_i, Z_j \rangle = -\delta_{ij}c_{\mu}trD_i.$$

Also  $R_{\mu} = c_{\mu}I + D_{\mu}$ , for some  $c_{\mu} \in \mathbb{R}$ . Therefore,  $\mu$  is a multiple Ricci soliton. Conversely, let  $\mu$  be a multiple Ricci soliton i.e.

(a)  $R_{\mu} = c_{\mu}I + D_{\mu}; \quad D_{\mu} \in Der(n,\mu), \ c_{\mu} \in \mathbb{R}.$ 

(b) There are symmetric derivations  $D_i$ ,  $1 \le i \le r$ , such that

$$D_{\mu} = D_1 + D_2 + \ldots + D_r, \quad tr D_i D_j = -\delta_{ij} c_{\mu} tr D_i.$$

Let  $(n, \mu, < ..., >_n)$  be a Lie algebra with orthonormal basis  $\{X_1, X_2, ..., X_k\}$ . We define Lie algebra s with a simply connected Lie group S as follows

$$s = n \oplus \sum_{i} \mathbb{R}Z_i,$$

endowed with the inner product  $\langle ., . \rangle$  defined by

$$< Z_i, Z_j >= \delta_{ij} tr D_i, \ < Z_i, n >= 0, \ < \ldots > |_{n \times n} = < \ldots >_n.$$

Also, Lie bracket [.,.] is defined by

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$$[Z_i, Z_j] = 0, \ [Z_i, X_j] = -[X_j, Z_i] = D_i X_j, \ [.,.]|_{n \times n} = \mu.$$

Clearly [.,.] is a Lie bracket, since  $D_i$  's are derivations.  $\{D_{\mu}X_1, D_{\mu}X_2, ..., D_{\mu}X_k\}$  is a linearly independent set which generates a subalgebra of [s, s]. Therefore, n = [s, s].  $\mu$  is nilpotent hence [.,.] is a solvable Lie bracket. Finally using Lemma 2.1, we have

$$\langle R_{[.,.]}Z_i, Z_j \rangle = -tr(D_iD_j) = \delta_{ij}c_{\mu}trD_i = c_{\mu} \langle Z_i, Z_j \rangle, \quad \langle R_{[.,.]}Z_i, n \rangle = 0,$$
  
$$\langle R_{[.,.]}X_i, X_j \rangle = \langle (-D_{\mu} + R_{\mu})X_i, X_j \rangle = \langle c_{\mu}X_i, X_j \rangle = c_{\mu} \langle X_i, X_j \rangle,$$
  
which implies that  $\langle .., .. \rangle_s$  is a Einstein metric. This completes the proof.  $\Box$ 

Using Proposition 2.1, we get a higher rank Einstein solvmanifold as the direct sum of the Lie algebras.

**Proposition 2.5.** If nonzero nilpotent Lie brackets  $\mu_1$  and  $\mu_2$  are Ricci solitons, then  $\mu = \mu_1 \oplus \mu_2$  is a multiple Ricci soliton of degree 2.

*Proof.*  $\mu_1$  and  $\mu_2$  are Ricci solitons i.e.

(2.1) 
$$R_{\mu_i} = c_{\mu_i} I + D_{\mu_i}; \quad D_{\mu_i} \in Der(\mu_i), \ c_{\mu_i} \in \mathbb{R}, \ i = 1, 2.$$

Up to isometry and scaling we can determine norms of  $\mu_1$  and  $\mu_2$  such that  $c_{\mu_1} = c_{\mu_2}$ . Set

$$D_{\mu} = \begin{bmatrix} D_{\mu_{1}} & 0\\ 0 & D_{\mu_{2}} \end{bmatrix},$$
$$R_{\mu} = \begin{bmatrix} R_{\mu_{1}} & 0\\ 0 & R_{\mu_{2}} \end{bmatrix},$$

Then  $R_{\mu} = c_{\mu_1}I + D_{\mu}$ ,  $R_{\mu} = R_{\mu_1} \oplus R_{\mu_2}$  and  $D_{\mu} = D_{\mu_1} \oplus D_{\mu_2}$ . Also, by Theorem 1.3,  $trD_{\mu_i}D_{\mu_j} = -\delta_{ij}c_{\mu_1}trD_{\mu_i}$ ; i = 1, 2. Therefore  $\mu$  is a multiple Ricci soliton which admits a 2-rank Einstein solvable extension.

**Corollary 2.6.** If nonzero nilpotent Lie brackets  $\mu_i$ 's,  $1 \leq i \leq r$ , are Ricci solitons, then  $\mu = \mu_1 \oplus \mu_2 \oplus ... \oplus \mu_r$  is a multiple Ricci soliton of degree r which admits an Einstein solvable extension of rank r.

**Remark 2.7.** There exist 31 Ricci soliton nonzero Lie algebras of dimension 6 (see [7]), which by direct sum of them, we can obtain a lot of multiple Ricci soliton nilpotent Lie algebras.

### 3. Standard methods

The goal of this section is to present certain results from [4] and [6]. In view of [4], Jorge Lauret has used a variational method for finding standard Einstein solvmanifolds. We will demonstrate this method in Theorem 3.1. We first give some preliminaries.

**Lemma 3.1.** (Lagrange multiplier theorem) [1]. Let P and M be smooth manifolds and  $g: M \longrightarrow P$  be a smooth submersion. Let  $f: M \longrightarrow \mathbb{R}$  be  $C^r$ ,  $m \in M$  and  $p \in P$  such that  $m \in g^{-1}(p)$ , then the following statements are equivalent:

- (i) m is a critical point of  $f|_{q^{-1}(p)}$ .
- (ii) There are  $\lambda \in T_p^*M$  such that  $T_m f = \lambda \circ T_m g$ .

The vector space  $\aleph_k$  is GL(k)-invariant, so we can refine Theorem 1.1 more accurately as follows.

**Lemma 3.2.** For  $0 \neq \mu \in \aleph_k$  and  $\psi \in GL(k)$  the following statements are equivalent:

- (i)  $\psi.\mu$  is a Ricci soliton.
- (ii)  $\psi.\mu$  is a critical point of  $F_k|_{S_r \cap GL(k).\mu}$ .
- (iii)  $\psi$  is a solution of the following system of equations:

$$\begin{cases} \|\phi.\mu\|^2 = 2r^2\\ \frac{\partial F_k(\phi.\mu)}{\partial \phi_{ij}} = t\frac{\partial(\|\phi.\mu\|)}{\partial \phi_{ij}} \end{cases}$$

where  $t \in \mathbb{R}$  and  $\phi \in GL(k)$ .

*Proof.* Let  $g(\lambda) = \frac{1}{2} ||\lambda||^2$  be a function on  $\Lambda^2 n^* \otimes n$  and use Theorem 1.1 and Lemma 3.1.

It is possible that the above system of equations is not solvable, hence we assume that for some  $k \in \mathbb{N}$  and every  $\phi \in GL(k)$  there exists  $\mu \in \aleph_k$  such that  $\phi.\mu$  isn't a Ricci soliton.

**Notation 3.3.** Suppose that  $DGL(k) := \{\phi \in GL(k); \phi \text{ is diagonal}\}$ , For any  $\mu \in \aleph_k \subseteq \Lambda^2 n^* \otimes n$ , set

$$\mu(X_i, X_j) = \sum_{v} c_{ijv} X_v, \ \phi = diag(\phi_1, \phi_2, ..., \phi_k),$$

then

$$(\phi.\mu)(X_i, X_j) = \sum_{x_{ijv} \neq 0} x_{ijv} X_v; \quad x_{ijv} = \frac{\phi_v c_{ijv}}{\phi_i \phi_j}$$

and for any i, j and v such that  $\langle \mu(X_i, X_j), X_v \rangle \neq 0$ , the diagonal elements of  $R_{\phi,\mu}$  are equal to

$$(R_{\phi,\mu})_{ii} = -\frac{1}{2}(-x_{ijv}^2 + \sum_{\substack{rst \\ x_{rst} \neq x_{ijv}}} \delta^{rst,i} x_{rst}^2),$$
  
$$(R_{\phi,\mu})_{jj} = -\frac{1}{2}(-x_{ijv}^2 + \sum_{\substack{rst \\ x_{rst} \neq x_{ijv}}} \delta^{rst,j} x_{rst}^2),$$
  
$$(R_{\phi,\mu})_{vv} = \frac{1}{2}(x_{ijv}^2 + \sum_{\substack{rst \\ x_{rst} \neq x_{ijv}}} \delta^{rst,v} x_{rst}^2),$$

where  $\delta^{rst,i}$ ,  $\delta^{rst,j}$  and  $\delta^{rst,v}$  are equal to 0, 1 or -1.

**Lemma 3.4.** [6]. Let  $(n, \mu, < .., >)$  be a Lie algebra and  $P_{\mu} = Sym(n) \cap Der(n)$ , then  $R_{\mu} \perp P_{\mu}$  with inner product tr(AB) on  $Sym(n) \times Sym(n)$ .

**Theorem 3.5.** (Lauret theory) For every  $\phi \in GL(k)$  and  $\mu \in \aleph_k$  if  $\phi.\mu \in S_r$ and  $R_{\mu}$  is diagonal, then for any i, j and v such that  $\langle \mu(X_i, X_j), X_v \rangle \neq 0$ , the following statements are equivalent:

(i)  $R_{\psi.\mu} = c_{\psi.\mu}I + D_{\psi.\mu}; \quad D_{\psi.\mu} \in Der(\psi.\mu)$ (ii)  $c_{\psi.\mu}$  and  $a_{ijv}^2$  is are solutions of the system

$$\begin{cases} \sum_{i,j,v} x_{ijv}^2 = r^2\\ \frac{\partial F_k(\phi,\mu)}{\partial u_{ijv}}|_{x_{ijv}:=a_{ijv}} = -c_{\psi,\mu} \end{cases}$$
$$)(X_i, X_j) = \sum_{x_{ijv} \neq 0} x_{ijv} X_v, \ u_{ijv} = x_{ijv}^2, \ \psi.\mu = \phi.\mu|_{x_{ijv}:=a_{ijv}} .$$

*Proof.* By Lemma 3.2 and the chain rule, it is easy to see that  $\psi.\mu$  is a Ricci soliton if and only if variation t and  $a_{ijv}^2$ 's are solutions of the system

$$\begin{cases} \sum_{i,j,v} x_{ijv}^2 = r^2 \\ \frac{\partial F_k(\phi,\mu)}{\partial u_{ijv}} |_{x_{ijv}:=a_{ijv}} = t \end{cases}$$

Now we shall obtain the Lagrangian coefficient. By Lemma 3.3 it is easy to see that  $trR_{\mu}^{2} = c_{\mu}trR_{\mu}$ . Also  $trR_{\mu} = -\frac{1}{2}\|\mu\|^{2}$ . Thus  $F_{k} = -c_{\mu}r^{2}$ . Consequently  $\frac{\partial F_{k}(\phi,\mu)}{\partial u_{ijv}}|_{x_{ijv}:=a_{ijv}} = -c_{\mu}$ .

Finally, we exhibit a rank-two Einstein solvmanifold of dimension 8 and a rank-three Einstein solvmanifold of dimension 15.

**Example 3.6.** Let  $\mu = \mu_1 \oplus \mu_2$ , where  $\mu_1(X_1, X_2) = X_5$ ,  $\mu_2(X_3, X_4) = X_6$ and  $\phi = diag(\phi_1, \phi_2, ..., \phi_6)$ , then

$$\phi.\mu_1(X_1, X_2) = \frac{\phi_5}{\phi_1\phi_2}X_5 \text{ and } \phi.\mu_2(X_3, X_4) = \frac{\phi_6}{\phi_3\phi_4}X_6.$$

where  $(\phi, \mu)$ 

Let  $x := \frac{\phi_5}{\phi_1\phi_2}$ ,  $y := \frac{\phi_6}{\phi_3\phi_4}$ . Using Theorem 3.1, it is easy to see  $x^2 = y^2 = 1$ . Therefore  $D_{\mu_1} = diag(1,1,0,0,2,0)$  and  $D_{\mu_2} = diag(0,0,1,1,0,2)$ . If  $(n_1,\mu_1,<\ldots,>_1)$  and  $(n_2,\mu_2,<\ldots,>_2)$  are nilradical Lie algebras, define the Lie algebras with simply connected Lie group S using the following direct sum

$$s = \mathbb{R}Z_1 \oplus n_1 \oplus \mathbb{R}Z_2 \oplus n_2$$

endowed with the inner product < ., . > defined by

$$< Z_i, Z_j >= 4\delta_{ij}, < Z_i, n >= 0, < ., . > |_{n_i \times n_i} = < ., . >_i; 1 \le i, j \le 2,$$

where the Lie bracket [.,.] on s is defined by

$$\begin{split} & [Z_1, X_1] = -[X_1, Z_1] = X_1, & [Z_1, X_2] = -[X_2, Z_1] = X_2, \\ & [Z_1, X_5] = -[X_5, Z_1] = 2X_5, & [Z_2, X_3] = -[X_3, Z_2] = X_3, \\ & [Z_2, X_4] = -[X_4, Z_2] = X_4, & [Z_2, X_6] = -[X_6, Z_2] = 2X_6, \\ & [Z_i, Z_j] = 0, \ [., .]_s|_{n_i \times n_i} = \mu_i; \ 1 \le i, j \le 2 \end{split}$$

and it is equal to zero otherwise. It is easy to see that  $R_{[.,.]_s} = -\frac{3}{2}I_{8\times 8}$  which implies that  $S_{\mu}$  is Einstein of rank 2.

**Example 3.7.** Let  $\mu = \mu_1 \oplus \mu_2 \oplus \mu_3$ , where  $\mu_1$  and  $\mu_2$  are Lie algebras given in Example 1 and  $\mu_3 \in \aleph_6$  is given by

$$\mu_3(X_7, X_i) = X_{i+1}; \ 8 \le i \le 11.$$

Every  $\phi.\mu_3 \in DGL(6).\mu_3 \cap S_{\sqrt{\frac{30}{13}}}$  is equal to

$$\phi.\mu_3(X_7, X_i) = a_{7,i,i+1}X_{i+1}; \ i = 8, 9, 10, 11.$$

By Lauret theory it is easy to see that a critical point of  $F_6$  restricted to the leaf  $\sum_{7 < i < 12} a_{7,i,i+1}^2 = \frac{20}{13}$  is equal to  $\mu_3\{a_{7,i,i+1}\}$  where

$$a_{7,8,9}^2 = \frac{12}{13}, \ a_{7,9,10}^2 = \frac{9}{13}, \ a_{7,10,11}^2 = \frac{3}{13}, \ a_{7,11,12}^2 = \frac{9}{13}, \ c_{\psi.\mu_3} = -\frac{3}{2}$$

and

$$D_{\mu_3(a_{7,i,i+1})} = diag(0,0,0,0,0,0,\frac{15}{26},\frac{42}{26},\frac{36}{26},\frac{51}{26},\frac{48}{26},\frac{51}{26}) \, .$$

Let  $(n_3, \mu_3(a_{7,i,i+1}), < ..., >_3)$  be a Ricci soliton. Define Lie algebra s with simply connected Lie group S using the following direct sum

$$s = \mathbb{R}Z_1 \oplus n_1 \oplus \mathbb{R}Z_2 \oplus n_2 \oplus \mathbb{R}Z_3 \oplus n_3$$

endowed with the inner product < ., . > which is defined by

$$\langle Z_1, Z_1 \rangle = 4, \langle Z_2, Z_2 \rangle = 4, \langle Z_3, Z_3 \rangle = \frac{243}{13}, \langle ., . \rangle|_{n_i \times n_i} = \langle ., . \rangle_i$$

and it is equal to zero otherwise. Lie bracket [.,.] on S defined by

$$\begin{split} & [Z_1, X_1] = -[X_1, Z_1] = X_1, \\ & [Z_1, X_5] = -[X_5, Z_1] = 2X_5, \\ & [Z_2, X_4] = -[X_4, Z_2] = X_4, \end{split} \quad \begin{array}{l} & [Z_1, X_2] = -[X_2, Z_1] = X_2, \\ & [Z_2, X_3] = -[X_3, Z_2] = X_3, \\ & [Z_2, X_6] = -[X_6, Z_2] = 2X_6 \end{split}$$

$$\begin{split} [Z_3, X_7] &= -[X_7, Z_3] = \frac{15}{26}X_7, \qquad [Z_3, X_8] = -[X_8, Z_3] = \frac{42}{26}X_8, \\ [Z_3, X_9] &= -[X_9, Z_3] = \frac{36}{26}X_9, \qquad [Z_3, X_{10}] = -[X_{10}, Z_3] = \frac{51}{26}X_{10}, \\ [Z_3, X_{11}] &= -[X_{11}, Z_3] = \frac{48}{26}X_{11}, \quad [Z_3, X_{12}] = -[X_{12}, Z_3] = \frac{51}{26}X_{12}, \\ [., .]|_{n_i \times n_i} = \mu_{n_i}; \quad i = 1, 2 \end{split}$$

and otherwise is equal to zero. It is easy to check that  $R_{[.,.]} = -\frac{3}{2}I_{15\times 15}$  which implies that  $S_{\mu}$  is Einstein of rank 3.

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