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Generalization of α-Centroidal Mean and its Dual

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Abstract. In this paper, the generalized α-centroidal mean and its dual form in 2 variables are introduced. Also, we will study some properties and prove their monotonicity. Further, shown that various means are particular cases of generalized α-centroidal mean.

Keywords: Monotonicity, Inequality, Power Oscillatory mean, Dual.

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1. Introduction

If a, $b > 0$ are two real numbers, then

$$A(a,b) = F_1(a,b) = \frac{a+b}{2} \tag{1.1}$$

$$G(a,b) = F_2(a,b) = \sqrt{ab} \tag{1.2}$$

$$H(a,b) = F_2(a,b) = \frac{2ab}{a+b}. \tag{1.3}$$

$$L(a,b) = \begin{cases} \frac{a-b}{\ln a - \ln b} & a \neq b \\ a & a = b \end{cases} \tag{1.4}$$

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$$I(a,b) = \begin{cases} e^{\left(\frac{a\ln a - b\ln b}{a-b} - 1\right)} & a \neq b \\ a & a = b \end{cases} \tag{1.5}$$

$$M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}} & r \neq 0 \\ \sqrt{ab} & r = 0 \end{cases} \tag{1.6}$$

$$H_p(a,b) = \left\{ \frac{a^p + (\sqrt{ab})^p + b^p}{3} \right\}^{1/p} \tag{1.7}$$

are respectively called arithmetic mean, geometric mean, harmonic mean, logarithmic mean, identric mean, power mean and power type heron mean thoroughly studied by various researchers having their own importance in the literature. In [12], the definition of contra-harmonic mean is given by;

$$C(a,b) = \frac{a^2+b^2}{a+b} \tag{1.8}$$

and its generalization form is gives as;

$$C_n(a,b) = \frac{a^n+b^n}{a^{n-1}+b^{n-1}} \tag{1.9}$$

All these means defined above have been studied extensively. For more details the interested reader is referred to [1-10]. In [11], the authors established new Čebyš type integral inequalities involving functions whose derivatives belong to L_p spaces via certain integral identities.

In [7,10], the authors defined oscillatory mean and its dual forms. Also, established some new inequalities involving logarithmic mean, identric mean and power mean. Further, the authors also obtained best possible values for the equalities as follows:

Definition 1.1. For a,$b > 0$ and $\alpha \in (0,1)$, then Oscillatory mean and its dual form are defined as follows;

$$O(a,b;\alpha) = \alpha G(a,b) + (1-\alpha)A(a,b) \tag{1.10}$$

and

$$O^{(d)}(a,b;\alpha) = G(a,b)^{\alpha} A(a,b)^{1-\alpha}. \tag{1.11}$$

Definition 1.2. For a,$b > 0$ and $\alpha \in (0,1)$, then r^{th} Oscillatory mean and its dual form are defined as follows;

$$O(a,b;\alpha,r) = \alpha M_r(a,b) + (1-\alpha)A(a,b) \tag{1.12}$$

and

$$O^{(d)}(a,b;\alpha,r) = M_r(a,b)^{\alpha} A(a,b)^{1-\alpha}. \tag{1.13}$$

In [8], K. M. Nagaraja and P. Siva Kota Reddy introduced α-centroidal mean and its dual and studied some important results.

Definition 1.3. For $a, b > 0$ and $\alpha \in (0,1)$, α-centroidal mean and its dual form are defined as follows;

$$CT(a,b;\alpha) = \alpha H(a,b) + (1-\alpha)C(a,b) \tag{1.14}$$

and

$$CT^{(d)}(a,b;\alpha) = H(a,b)^{\alpha}C(a,b)^{1-\alpha}. \tag{1.15}$$

and the extended mean values given as below:

$$S_{s,t}(a,b) = \begin{cases} \left(\frac{t(a^s-b^s)}{s(a^t-b^t)}\right)^{\frac{1}{s-t}} & if (s-t)st \neq 0, a \neq b \\ exp\left(-\frac{1}{s} + \frac{a^s loga - b^s logb}{a^s - b^s}\right) & if s = t \neq 0, a \neq b \\ exp\left(\frac{a^s - b^s}{s(a^s loga - b^s logb)}\right)^{\frac{1}{s}} & if s \neq 0, t = 0, a \neq b \\ \sqrt{ab} & if s = t = 0 \\ a & if a = b \end{cases} \tag{1.16}$$

By the motivation of work done by the authors [8], in this paper we make an attempt to generalize the α-centroidal mean and its dual. In the next section, we established monotonicity results for generalized α-centroidal mean and some inter-related inequalities among several means. In concluding section, some consequent examples are appended. The new initiated mean in this paper is the generalization of oscillatory mean, r^{th} oscllatory mean, α-centroidal mean and some other means.

2. Definitions and Properties

In this section, the generalized α-centroidal mean and its dual are defined as follows:

Definition 2.1. For $a, b > 0$, r, n are real numbers and $\alpha \in (0,1)$, then the generalized α-centroidal mean and its dual form are defined as follows:

$$CT(a,b;\alpha,r,n) = \alpha M_r(a,b) + (1-\alpha)C_n(a,b) \tag{2.1}$$

and

$$CT^{(d)}(a,b;\alpha,r,n) = M_r(a,b)^{\alpha}C_n(a,b)^{1-\alpha}. \tag{2.2}$$

For $\alpha \in (0,1)$, the generalized α-centroidal mean and its dual satisfy the following properties:

Property 2.2. The generalized α-centroidal mean and its dual are means. That is

$$Min\{a,b\} \leq \{CT(a,b;\alpha,r,n), CT^d(a,b;\alpha,r,n)\} \leq Max\{a,b\}.$$

Property 2.3. The means $CT(a,b;\alpha,r,n)$ and $CT^d(a,b;\alpha,r,n)$ are:

(1) **Symmetric** :
$CT(a,b;\alpha,r,n) = CT(b,a;\alpha,r,n))$ and $CT^{(d)}(a,b;\alpha,r,n) = CT^{(d)}(b,a;\alpha,r,n)$.

(2) **Homogeneous** :
$CT(at, bt; \alpha, r, n) = tCT(a, b; \alpha, r, n)$ and $CT^{(d)}(at, bt; \alpha, r, n) = tCT^{(d)}(a, b; \alpha, r, n)$.

According to Definition 2.1, the following characteristic properties for $CT(a, b; \alpha, r, n)$ and $CT^{(d)}(a, b; \alpha, r, n)$ are straightforward, this result gives that the various means are particular cases of the generalized α-centroidal mean and its dual.

Proposition 2.4. For $\alpha \in (0, 1)$, then

(1) $CT(a, b; \alpha, 0, 1) = O(a, b; \alpha)$ is the oscillatory mean
(2) $CT^d(a, b; \alpha, 0, 1) = O^d(a, b; \alpha)$ is dual oscillatory mean
(3) $CT(a, b; \alpha, r, 1) = O(a, b; \alpha, r)$ is the r^{th} oscillatory mean
(4) $CT^d(a, b; \alpha, r, 1) = O^d(a, b; \alpha, r)$ is the r^{th} dual oscillatory mean
(5) $CT(a, b; \alpha, -1, 2) = O(a, b; \alpha)$ is the α-centroidal mean
(6) $CT^d(a, b; \alpha, -1, 2) = O^d(a, b; \alpha)$ is the dual α-centroidal mean
(7) $CT(a, b; \frac{1}{3}, 0, 1) = H_e(a^2, b^2)$ is the Heron mean.
(8) $CT^d(a, b; \frac{1}{3}, 0, 1) = G^{\frac{1}{3}}(a, b)A^{\frac{2}{3}}(a, b) = O^d(a, b; \frac{1}{3}, 0)$,
(9) $min(a, b) \leq CT^{(d)}(a, b; \alpha, r, n) \leq CT(a, b; \alpha, r, n) \leq Max(a, b)$.
(10) $M_r(a, b) \leq CT^{(d)}(a, b; \alpha, r, n) \leq CT(a, b; \alpha, r, n) \leq C_n(a, b)$.
(11) $CT(a, b; 1, -1, n) = H(a, b) = CT^d(a, b; 1, -1, n)$.
(12) $CT(a, b; 1, 0, n) = G(a, b) = CT^d(a, b; 1, 0, n)$.
(13) $CT(a, b; 1, 1, n) = A(a, b) = CT^d(a, b; 1, 1, n)$.
(14) $CT(a, b; 0, r, 1) = A(a, b) = CT^d(a, b; 0, r, 1)$.
(15) $CT(a, b; 0, r, 2) = A(a, b) = CT^d(a, b; 0, r, 2)$.
(16) $CT(a, b; 0, r, \frac{3}{2}) = \frac{a^{3/2}+b^{3/2}}{a^{1/2}+b^{1/2}} = CT^d(a, b; 0, r, \frac{3}{2})$.

From the above results, the following remarks are drawn:

Remark 2.5.

(1) $CT(a, b; 1, -1, n) + CT(a, b; 0, r, 2) = 2CT(a, b; 1, 1, n)$
(2) $CT^d(a, b; 1, -1, n) + CT^d(a, b; 0, r, 2) = 2CT^d(a, b; 1, 1, n)$
(3) $CT(a, b; 1, 0, n) + CT(a, b; 0, r, \frac{3}{2}) = 2CT(a, b; 1, 1, n)$
(4) $CT^d(a, b; 1, 0, n) + CT^d(a, b; 0, r, \frac{3}{2}) = 2CT^d(a, b; 1, 1, n)$.
(i.e., $H(a, b) + C_2(a, b) = G(a, b) + C_{3/2}(a, b) = 2A(a, b)$)

3. Monotonic Results

In this section, the monotonicity and behavior of the generalized α-centroidal mean and its dual are studied.

Theorem 3.1. For $\alpha \in (0, 1)$ and for $a, b > 0$, then $CT^{(d)}(a, b; \alpha, r, n) \leq CT(a, b; \alpha, r, n)$.

Proof. The proof follows from the well known power mean inequality:

$$M_r(a,b) = \begin{cases} \left(\dfrac{a^r+b^r}{2}\right)^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases} \tag{3.1}$$

□

Lemma 3.2. For a real number r and $a, b > 0$, we have $M_{r+1}(a,b) \geq M_r(a,b)$.

Theorem 3.3. The generalized α-centroidal mean $CT(a,b;\alpha,r,n)$ is an increasing function with respect to r, for $a, b > 0$ and $\alpha \in (0,1)$. That is,

$$CT(a,b;\alpha,r+1,n) \geqslant CT(a,b;\alpha,r,n) \tag{3.2}$$

Proof. From Definition 2.1,

$$\begin{aligned} CT(a,b;\alpha,r+1,n) &= \alpha M_{r+1}(a,b) + (1-\alpha)C_n(a,b) \\ &\geqslant \alpha M_r(a,b) + (1-\alpha)C_n(a,b) \\ &= CT(a,b;\alpha,r,n). \end{aligned}$$

This completes the proof. □

Theorem 3.4. The generalized α-centroidal mean $CT(a,b;\alpha,r,n)$ is an increasing function with respect to r, for $a, b > 0$ and $\alpha \in (0,1)$. That is,

$$CT^d(a,b;\alpha,r+1,n) \geqslant CT^d(a,b;\alpha,r,n) \tag{3.3}$$

Proof. From Definition 2.1,

$$\begin{aligned} CT^d(a,b;\alpha,r+1,n) &= M_{r+1}(a,b)^{\alpha} + C_n(a,b)^{1-\alpha} \\ &\geqslant M_r(a,b)^{\alpha} + C_n(a,b)^{1-\alpha} \\ &= CT^d(a,b;\alpha,r,n). \end{aligned}$$

This completes the proof. □

Lemma 3.5. For a real number n and $a, b > 0$, we have $C_{n+1}(a,b) \geq C_n(a,b)$.

Theorem 3.6. The generalized α-centroidal mean $CT(a,b;\alpha,r,n)$ an increasing function with respect to n, for $a, b > 0$ and $\alpha \in (0,1)$. That is,

$$CT(a,b;\alpha,r,n+1) \geqslant CT(a,b;\alpha,r,n) \tag{3.4}$$

Proof. From Definition 2.1,

$$\begin{aligned} CT(a,b;\alpha,r,n+1) =& \alpha M_r(a,b) + (1-\alpha)C_{n+1}(a,b) \\ \geqslant& \alpha M_r(a,b) + (1-\alpha)C_n(a,b) \\ =& CT(a,b;\alpha,r,n). \end{aligned}$$

This completes the proof. □

Theorem 3.7. The generalized α-centroidal mean $CT(a,b;\alpha,r,n)$ an increasing function with respect to n, for $a,b>0$ and $\alpha \in (0,1)$. That is,

$$CT^d(a,b;\alpha,r,n+1) \geqslant CT^d(a,b;\alpha,r,n) \tag{3.5}$$

Proof. From Definition 2.1,

$$\begin{aligned} CT^d(a,b;\alpha,r,n+1) =& M_r(a,b)^\alpha + C_{n+1}(a,b)^{1-\alpha} \\ \geqslant& M_r(a,b)^\alpha + C_n(a,b)^{1-\alpha} \\ =& CT^d(a,b;\alpha,r,n). \end{aligned}$$

This completes the proof. □

Theorem 3.8. The generalized α-centroidal mean and its dual are varies from $C_n(a,b)$ to $M_(a,b)$ with α varies from 0 and 1.

Proof. The proof is follows from Definition 2.1. □

Remark 3.9.
For $r=n$, the generalized α-centroidal mean and its dual are respectively given by:

$$CT(a,b;\alpha,r,r) = \alpha M_r(a,b) + (1-\alpha)C_r(a,b)$$

and

$$CT^d(a,b;1,r,n) = M_r(a,b)^\alpha + C_r(a,b)^{1-\alpha}.$$

Example 3.10.
When $r=n=0$, $CT(a,b;\alpha,0,0) = \alpha G(a,b) + (1-\alpha)H(a,b)$, this mean is increasing with request to α, then

$$H(a,b) \leq \frac{3H+G}{4} \leq \frac{H+G}{2} \leq \frac{H+3G}{4} \leq G(a,b).$$

Theorem 3.11. For $a,b>0$, $\alpha\in(0,1)$ and $n=r$, the generalized α-centroidal mean and its dual are increasing with respect to r.

Proof. The proof is follows from Lemma 3.2 and 3.5. □

Example 3.12. From Remark 3.9 and Theorem 3.11, the following holds:

$$H(a,b) \leq \frac{M_{1/4}(a,b) + C_{1/4}(a,b)}{4} \leq \frac{3G+A}{4} \leq \frac{M_{3/4}(a,b) + C_{3/4}(a,b)}{4} \leq A(a,b).$$

Lemma 3.13. For a real number r and $a > b > 0$, we have

(1) $M_r(a,b) > C_r(a,b)$ if $r < 1$
(2) $M_r(a,b) < C_r(a,b)$ if $r > 1$
(3) $M_r(a,b) = C_r(a,b)$ if $r = 1$.

Proof. Put $a = t, b = 1$, in $M_r(a,b)$ and $C_r(a,b)$. Consider

$$f(t) = \frac{1}{r}[ln(t^r + 1) - ln2] - [ln(t^r + 1) - ln(t^{r-1} + 1)]$$

then

$$f^{|}(t) = (1-r)(t-1)\frac{t^{r-2}}{(t^r + 1)(t^{r-1} + 1)} > 0$$

if $r < 1$, $f(t)$ is increasing (i.e., $M_r(a,b) > C_r(a,b)$). The result is clear. □

Theorem 3.14. If $a, b > 0$, $\alpha \in (0,1)$ and $n = r$, then the generalized α-centroidal mean and its dual monotonically increasing with respect to α if $r < 1$, monotonically decreasing with respect to α if $r > 1$ and inequality turns out to be equality if $r = 1$.

Proof. The proof follows from Theorem 3.11 and Lemma 3.13. □

4. Some Inequalities

From the definitions of oscillatory mean, the α-centroidal mean and its dual forms for $\alpha \in (0,1)$ and $a, b > 0$, the following inequalities hold:

$$CT^d(a,b;1,r,n) \leq ... \leq O^d(a,b;1) \leq ... \leq O(a,b;0) \leq ... \leq CT(a,b;0,r,n).$$

By replacing $a = t+1, b = 1$ in Definition 2.1, the Taylor's series expansion of $O(a,b;\alpha,k)$ and $O^{(d)}(a,b;\alpha,k)$ are as follows:

$$L(a,b) = L(t,1) = 1 + \frac{t}{2} - \frac{1}{12}t^2 + ... \tag{4.1}$$

$$I(a,b) = I(t,1) = 1 + \frac{t}{2} - \frac{1}{24}t^2 + ... \tag{4.2}$$

$$H_p(a,b) = H_p(t,1) = 1 + \frac{t}{2} + \frac{2p-3}{24}t^2 + ... \tag{4.3}$$

$$M_r(a,b) = M_r(t,1) = 1 + \frac{t}{2} + \frac{r-1}{8}t^2 + ... \tag{4.4}$$

$$CT(t,1;\alpha,n,n) = 1 + \frac{1}{2}t + \frac{3n-3}{8}t^2 + ... \tag{4.5}$$

$$CT^{(d)}(t,1;\alpha,n,n) = 1 + \frac{1}{2}t + \frac{3n-3}{8}t^2 + ... \tag{4.6}$$

From the above Taylor's series expansions of various means, we compute the following inequalities.

If $a,b > 0$, $\alpha \in (0,1)$ and n_1, n_2 are real numbers, then we have following results.

Proposition 4.1. For $n_1 \leq \frac{7}{9} \leq n_2$, the following double inequality holds

$$CT^{(d)}(a,b;\alpha,n_1,n_1) \leq L(a,b) \leq CT(a,b;\alpha,n_2,n_2). \tag{4.7}$$

Further, $n_1 = \frac{7}{9} = n_2$ is the best possible for (4.7).

Proof. From eqs 4.1 to 4.6, we have

$$CT^{(d)}(a,b;\alpha,n_1,n_1) \leq L(a,b) \leq CT(a,b;\alpha,n_2,n_2) \tag{4.8}$$

whenever, $\frac{3n_1-3}{4} \leq \frac{-1}{12} \leq \frac{3n_2-3}{4}$.

□

Proposition 4.2. For $n_1 \leq \frac{8}{9} \leq n_2$, the following double inequality holds

$$CT^{(d)}(a,b;\alpha,n_1,n_1) \leq I(a,b) \leq CT(a,b;\alpha,n_2,n_2). \tag{4.9}$$

Further, $n_1 = \frac{8}{9} = n_2$ is the best possible for (4.9).

Proof. From eqs 4.1 to 4.6, we have

$$CT^{(d)}(a,b;\alpha,n_1,n_1) \leq I(a,b) \leq CT(a,b;\alpha,n_2,n_2) \tag{4.10}$$

whenever, $\frac{3n_1-3}{4} \leq \frac{-1}{24} \leq \frac{3n_2-3}{4}$.

□

Proposition 4.3. For $n_1 \leq \frac{r+2}{3} \leq n_2$, the following double inequality holds

$$CT^{(d)}(a,b;\alpha,n_1,n_1) \leq M_r(a,b) \leq CT(a,b;\alpha,n_2,n_2). \tag{4.11}$$

Further, $n_1 = \frac{r+2}{3} = n_2$ is the best possible for (4.11).

Proof. From eqs 4.1 to 4.6, we have

$$CT^{(d)}(a,b;\alpha,n_1,n_1) \leq I(a,b) \leq CT(a,b;\alpha,n_2,n_2) \tag{4.12}$$

whenever, $\frac{3n_1-3}{4} \leq \frac{r-1}{8} \leq \frac{3n_2-3}{4}$.
Or $n_1 \leq \frac{r+2}{3} \leq n_2$.

□

Proposition 4.4. For $n_1 \leq \frac{2p+6}{3} \leq n_2$, the following double inequality holds

$$CT^{(d)}(a,b;\alpha,n_1,n_1) \leq M_r(a,b) \leq CT(a,b;\alpha,n_2,n_2). \tag{4.13}$$

Further, $n_1 = \frac{2p+6}{3} = n_2$ is the best possible for (4.13).

Proof. From eqs 4.1 to 4.6, we have

$$CT^{(d)}(a,b;\alpha,n_1,n_1) \leq I(a,b) \leq CT(a,b;\alpha,n_2,n_2) \tag{4.14}$$

whenever, $\frac{3n_1-3}{4} \leq \frac{2p-3}{24} \leq \frac{3n_2-3}{4}$.

Or $n_1 \leq \frac{2p+6}{3} \leq n_2$. □

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