Iranian Journal of Mathematical Sciences and Informatics
Vol. 8, No. 2 (2013), pp 49-55

On the domination polynomials of non P_4-free graphs

Saeid Alikhani
Department of Mathematics, Yazd University, 89195-741, Yazd, Iran

E-mail: alikhani@yazduni.ac.ir

Abstract. A graph G is called P_4-free, if G does not contain an induced subgraph P_4. The domination polynomial of a graph G of order n is the polynomial $D(G,x) = \sum_{i=1}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i. Every root of $D(G,x)$ is called a domination root of G. In this paper we state and prove formula for the domination polynomial of non P_4-free graphs. Also, we pose a conjecture about domination roots of these kind of graphs.

Keywords: Domination polynomial; Simple path; Root.

2000 Mathematics subject classification: 05C15;05C60.

1. Introduction

Graph polynomials are a well-developed area useful for analyzing properties of graphs. There are some polynomials associated to graphs. Chromatic polynomial, clique polynomial, characteristic polynomial and Tutte polynomial are some examples of these polynomials. Also there are some graphs polynomials related to a molecular graph (see [9, 13]). Domination polynomial of a graph is a new graph polynomial. Let to define domination polynomial of a graph.
Let $G = (V,E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V|uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every

Received 10 December 2011; Accepted 4 September 2012

vertex in $V\backslash S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. For a detailed treatment of this parameter, the reader is referred to [10]. Let $\mathcal{D}(G,i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. The domination polynomial $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $\gamma(G)$ is the domination number of G ([2, 6]).
A root of $D(G,x)$ is called a *domination root* of G ([3]). A *simple path* is a path where all its internal vertices have degree two. A graph G is called P_4-free, if G does not contain an induced subgraph P_4. The non P_4-free graphs are in the form of one of the following specific graphs:
Let P_{n+1} be a path with vertices labeled by $y_1, y_2, \ldots, y_{n+1}$, for $n \geq 1$ and let v_0 be a specific vertex of a graph G. Denote by $G_{v_0}(n)$ (or simply $G(n)$ if there is no likelihood of confusion) a graph obtained from G by identifying the vertex v_0 of G with an end vertex y_1 of P_{n+1} ([7]).
Let P_n be a path with vertices labeled $y_1, \ldots, y_n$ and let a, b be two specific vertices of a graph G. Denote by $G'_{a,b}(n)$ (or simply $G'(n)$, if there is no likelihood of confusion) a graph obtained from G and P_n by adding edges ay_1 and by_n ([7]).

In the next section, we investigate and recall some background materials related to recursive families of polynomials. In Section 3, we give a formula for the domination polynomials of non P_4-free graphs. Also we investigate the domination roots and state some open problems of these kind of graphs in the last section.

2. Recursive families of polynomials

Before we proceed to a discussion of the roots of domination polynomials, we need to state (in detail) an analytic result on particular families of polynomials (namely, *recursive families*). We begin with the following definition.

Definition 2.1. If $\{f_n(x)\}$ is a family of (complex) polynomials, we say that a number $z \in C$ is a limit of roots of $\{f_n(x)\}$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathcal{R}(f_n(x))$, where $\mathcal{R}(f_n(x))$ is the union of the roots of the $f_n(x)$.

Now (as in [8]) a family of polynomials $\{f_n(x)\}$ is a *recursive family of polynomials* if $f_n(x)$ satisfy a homogenous recurrence

$$f_n(x) = \sum_{i=1}^{k} a_i(x) f_{n-i}(x). \tag{2.1}$$

where the $a_i(x)$ for $1 \leq i \leq k$, are fixed polynomials, with $a_k(x) \not\equiv 0$. The number k is the order of the recurrence.

From (2.1), we can form its associated *characteristic equation*

$$\lambda^k - a_1(x)\lambda^{k-1} - a_2(x)\lambda^{k-2} - \cdots - a_k(x) = 0, \quad (2.2)$$

whose roots $\lambda = \lambda(x)$ are algebraic functions, and there are exactly k of them counting multiplicity (see [1] or [11]).
If these roots, say $\lambda_1(x), \lambda_2(x), \ldots, \lambda_k(x)$, are distinct, then the general solution to (2.1) is known (see [8]) to be

$$f_n(x) = \sum_{i=1}^{k} \alpha_i(x)\lambda_i(x)^n, \quad (2.3)$$

with the usual variant (see [8]) if some of the $\lambda_i(x)$ were repeated. The functions $\alpha_1(x), \alpha_2(x), \ldots, \alpha_k(x)$ are determined from the initial conditions, that is, the k linear equations in the $\alpha_i(x)$ obtained by letting $n = 0, 1, \ldots, k-1$ in (2.3) or its variant. The details are found in [8].

3. Domination polynomial of non P_4-free graphs

In this section, we shall use the results in previous section to study the domination polynomials of graphs containing a simple path of length at least three (or simply non P_4-free graphs). Every graphs G containing a simple path of length at least three, is in the form of one of the graphs $G(n)$ or $G'(n)$. In [7] the following recurrence formulas was proved in terms of edge contraction.

Theorem 3.1. ([7]) *For every $n \geq 3$,*

(i) $D(G(n), x) = x\Big[D(G(n-1), x) + D(G(n-2), x) + D(G(n-3), x)\Big]$,

(ii) $D(G'(n), x) = x\Big[D(G'(n-1), x) + D(G'(n-2), x) + D(G'(n-3), x)\Big]$.

Let us to give a simple proof for these recurrence relations. First we give a formula for the domination polynomial of a graph in terms of the domination polynomials of several other graphs which have fewer vertices or edges. The vertex contraction G/v of a graph G by a vertex v is the operation under which all vertices in $N(v)$ are joined to each other and then v is deleted (see [14]).

Theorem 3.2. *For any vertex v in a graph G we have*

$$D(G, x) = xD(G/v, x) + D(G - v, x) + xD(G - N[v], x) - (x + 1)p_v(G, x)$$

where $p_v(G, x)$ is the polynomial counting those dominating sets for $G - N[v]$ which additionally dominate the vertices of $N(v)$ in G.

Proof. Any dominating set S of $G - v$ is a dominating set for G unless v is not dominated by S, that is $N(v) \cap S = \emptyset$. In this case all elements of $N(v)$ will be dominated by S. Similarly, every dominating set T for G/v will give rise to a dominating set $T \cup v$ of G. However, if $N(v) \cap T = \emptyset$ and some elements of $N(v)$ are not dominated by T then $T \cup v$ will be a dominating set of G, nonetheless. Therefore, we can partition the domination polynomial of $G - N[v]$ into two polynomials based upon whether the vertices of $N(v)$ are dominated. Define $p_v(G,x)$ as the polynomial counting the dominating sets for $G-N[v]$ which dominate $N(v)$. Using this we see that the dominating sets for G that don't include v are counted by $D(G-v,x)-p_v(G,x)$. The dominating sets for G that do include v will be counted by $xD(G/v,x)+xD(G-N[u],x)-xp_v(G,x)$. Adding these two polynomials gives the required formula for $D(G,x)$. □

Now we can obtain the following result using Theorem 3.2 ([7]).

Theorem 3.3. *Let G be a graph which contains a simple path of length at lease three. Then*

$$D(G,x)=x\Big[D(G*e_1,x)+D(G*e_1*e_2,x)+D(G*e_1*e_2*e_3,x)\Big]$$

where e_1,e_2 and e_3 are three edges of the simple path, $G*e$ is the graph obtained from G by contracting the edge e, and $G*e_1*e_2=(G*e_1)*e_2$ and $G*e_1*e_2*e_3=\Big((G*e_1)*e_2\Big)*e_3$.

Proof. Suppose the five vertices in the induced path are u,v,w,r and s in order along the path. We apply Theorem 3.2 to the central vertex w:

$$D(G,x)=xD(G/w,x)+D(G-w,x)+xD(G-v-w-r,x)-(1+x)p_w(G,x)$$

We shall prove that

$$\begin{aligned}xD(G-v-w-r,x)-(1+x)p_w(G,x)+D(G-w,x)=\\ xD(G/v/w,x)+xD(G/v/w/r,x).\end{aligned} \tag{3.1}$$

Let $G'=G-N[w]$, and let A be a dominating set for G'. We like to extend A to a dominating set for each of the graphs in above equation by considering whether or not v and/or r must or may be added to A. For A to dominate $N(w)$ it must include both u and s. We consider 3 cases, dependent on how many of u and s are in A. When u, say, is in A it will dominate v and so v can either be in A or out of it, giving a factor of $(1+x)$ to multiply the domination polynomial of G'. If u is not in A then v must be in A in order for A to be a dominating set, giving a factor of x for G'. We tabulate the respective contributions for vertices v and y in the different graphs, substituting $q(x):=xD(G-v-w-r,x)-(1+x)p_w(G,x)$.

TABLE 1. Table of contributions from vertices v and r.

$\lvert A\cap\{w,r\}\rvert$	$q(x)$	$D(G-w,x)$	$xD(G/v/w,x)$	$xD(G/v/w/r,x)$
2	-1	$(1+x)^2$	$x(1+x)$	x
1	x	$x(1+x)$	$x(1+x)$	x
0	x	x^2	x^2	x

For each of these rows we can see that Equation (3.1) is satisfied by adding both pairs of columns. Since all of the possibilities for A fall into one of these three cases, the proof is complete. □

By Theorem 3.1, the characteristic equation of the recursive family of polynomials $\{D(G(n),x)\}$ and $\{D(G'(n),x)\}$ is

$$\lambda^3 - x\lambda^2 - x\lambda - x = 0.$$

Let

$$p(x) = \sqrt[3]{\frac{x^3}{27} + \frac{x^2}{6} + \frac{x}{2} + \sqrt{\frac{x^4}{36} + \frac{7x^3}{54} + \frac{x^2}{4}}}$$

and

$$q(x) = \sqrt[3]{\frac{x^3}{27} + \frac{x^2}{6} + \frac{x}{2} - \sqrt{\frac{x^4}{36} + \frac{7x^3}{54} + \frac{x^2}{4}}}.$$

By Cardan's formula (see [12]),we have

$$\lambda_1(x) = \frac{x}{3} + p(x) + q(x), \tag{3.2}$$

$$\lambda_2(x) = \Big(\frac{x}{3} - \frac{p(x)}{2} - \frac{q(x)}{2}\Big) + \frac{\sqrt{3}}{2}\Big(p(x) - q(x)\Big)i \tag{3.3}$$

and

$$\lambda_3(x) = \Big(\frac{x}{3} - \frac{p(x)}{2} - \frac{q(x)}{2}\Big) + \frac{\sqrt{3}}{2}\Big(q(x) - p(x)\Big)i \tag{3.4}$$

Now we state the following corollary:

Corollary 3.4. *Let $\lambda_1(x), \lambda_2(x)$ and $\lambda_3(x)$ are the above algebraic functions which satisfy in $\lambda^3 - x\lambda^2 - x\lambda - x = 0$. For every graph $H(n)$ of order n containing a simple path of length at least three*

$$D(H(n),x) = \sum_{i=1}^{3} \alpha_i(x)\lambda_i(x)^n,$$

where the functions $\alpha_1(x), \alpha_2(x), \alpha_3(x)$ are the following functions:

$\alpha_1(x) = \frac{\frac{-4x^2}{9} - 2x - p(x)^2 - q(x)^2 - \frac{2}{3}xp(x) - \frac{2}{3}xq(x) + p(x)q(x)}{-3(p(x)^2 + q(x)^2 + p(x)q(x))}$

$\alpha_2(x) = \frac{x - \alpha_1(x)(\lambda_1(x) - \lambda_3(x)) - \lambda_3(x)}{\lambda_2(x) - \lambda_3(x)}$,

$\alpha_3(x) = 1 - \alpha_1(x) - \alpha_2(x)$.

Proof. By (2.3), we have $D(H(n),x) = \sum_{i=1}^{3} \alpha_i(x)\lambda_i(x)^n$. Since $D(H(0),x) = 1, D(H(1),x) = x$ and $D(H(2),x) = x^2 + 2x$, then by substituting $n = 0,1,2$ in (2.3) we have

$$\alpha_1(x) = \frac{x(\lambda_2(x) + \lambda_3(x)) - x^2 - 2x - \lambda_2(x)\lambda_3(x)}{\lambda_1(x)\lambda_2(x) + \lambda_1(x)\lambda_3(x) - \lambda_2(x)\lambda_3(x) - (\lambda_1(x))^2},$$

$$\alpha_2(x) = \frac{x - \alpha_1(x)(\lambda_1(x) - \lambda_3(x)) - \lambda_3(x)}{\lambda_2(x) - \lambda_3(x)},$$

$$\alpha_3(x) = 1 - \alpha_1(x) - \alpha_2(x).$$

By substituting $\lambda_1(x), \lambda_2(x)$ and $\lambda_3(x)$, we have the result. □

Table 2. Real and complex roots of $D(P_n, x)$.

	Real roots	**Complex roots**
P_1	0	
P_2	0,-2	
P_3	0,-2.6180339,-0.381966	
P_4	0,0,-2,-2	
P_5	0,0,-0.5344287681	$-2.232785616 \pm 0.7925519925i$
P_6	0,0,-0.1167964941,-1.46899949	$-2.207106781 \pm 0.97831834i$
P_7	0,0,0,-2,-0.6467900358	$-2.176604982 \pm 1.202820919i$
P_8	0,0,0,-2 -0.2164290187,-1.393915461	$-2.194827760 \pm 1.3461759996i$
P_9	0,0,0,-0.527421508 -0.7078172217	$-1.916274378 \pm 0.4225236912i$ $-2.203443403 \pm 1.449357720i$
P_{10}	0,0,0,0 -0.3253111791,-1.298477136	$-1.97101181 \pm 0.5464779336i$ $-2.216404662 \pm 1.529772651i$
P_{11}	0,0,0,0,-0.1132672109 -0.7554905703,-1.733110206	$-1.969841869 \pm 0.7862099224i$ $-2.229224138 \pm 1.589095474i$
P_{12}	0,0,0,0 -0.02876579218,-0.4006431579 -1.260363791,-1.824483038	$-2.002790548 \pm 0.9457511977i$ $-2.240081562 \pm 1.635233587i$
P_{13}	0,0,0,0,0 -0.1902335768,-0.7871330112	$-1.732458303 \pm 0.2597377308i$ $-2.028923421 \pm 1.070514191i$ $-2.249934982 \pm 1.671696359i$

4. Open problem

By study of domination roots of graphs we are able to obtain some information about the structure of graphs (see [2, 3, 4, 5]). Using Maple we computed the domination roots of some non P_4-free graphs such as paths P_n and cycles C_n. We denote the roots of $D(P_n, x)$ and $D(C_n, x)$ by $Z(P_n)$ and $Z(C_n)$, respectively. We show $Z(P_n)$ for $1 \leq n \leq 14$ in Table 2. By observation from the tables, we think that the following conjecture is true for domination roots of the families of paths (and also for cycles):

Conjecture 4.1. *Let $r(P(x))$ be the number of real roots of the polynomial $P(x)$, then*

(i) *For every natural number k, $r(D(P_{4k-1}, x)) = r(D(P_{4k+1}, x)) = 2k+1$ and $r(D(P_{4k}, x)) = r(D(P_{4k+2}, x)) = 2k+2$, and all these real roots are in $[\frac{-3-\sqrt{5}}{2}, 0]$,*

(ii) *For every natural number k, $r(D(C_{4k+1}, x)) = r(D(C_{4k+3}, x)) = 2k+1$ and $r(D(C_{4k+2}, x)) = r(D(C_{4k+4}, x)) = 2k+2$,*

(iii) *Real roots of the families $D(P_n, x)$ and $D(C_n, x)$ are dense in the interval $[-2, 0]$, for $n \geq 4$.*

Acknowledgement. The author would like to express his gratitude to the referee for helpful comments.

References

1. L. Ahlfors, *Complex Analysis*, 3rd edition, McGraw-Hill, New York, (1979).
2. S. Akbari, S. Alikhani, and Y.H. Peng, Characterization of graphs using domination polynomial, *European Journal of Combinatorics*, **31**, (2010), 1714-1724.
3. S. Akbari, S. Alikhani, M.R. Oboudi and Y.H. Peng, On the zeros of domination polynomial of a graph, *Contemporary Mathematics*, **531**, (2010), 109-115.
4. S. Alikhani, The domination polynomial of a graph at −1, *Graphs and Combinatorics*, **29**, (2013), 11751181.
5. S. Alikhani, On the graphs with four distinct domination roots, *international journal of computer mathematics*, **88**(13), (2011), 2717-2720.
6. S. Alikhani, Y. H. Peng, Introduction to domination polynomial of a Grpah, *Ars Combinatoria*, In press. Available at `http://arxiv.org/abs/0905.2251`.
7. S. Alikhani, Y.H. Peng, Dominating sets and domination polynomial of certain graphs, II, *Opuscula Mathematica*, **30**(1), (2010), 37-51.
8. S. Beraha, J. Kahane, and N. Weiss, *Limits of zeros of recursively defined families of polynomials*, in studies in foundations and combinatorics: advanced in Math., Supplementrary Studies, vol. 1, G. Rota (Ed.), Academic Press, New York, 1978.
9. G. H. Fath-Tabar and A. R. Ashrafi, The Hyper-Wiener polynomial of graphs, *Iranian Journal of Mathematical Sciences and Informatics* **6**(2), (2011), 67-74.
10. T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, NewYork, 1998.
11. E. Hille, *Analytic function theory*, Ginn Co., Boston, 1959.
12. R. W. D. Nickalls, A new approach to solving the cubic: Cardan's solution revealed, *The Mathematical Gazette*, **77**, (1993), 354-359.
13. S. Sedghi, N. Shobeb and M. A. Salahshoor, The polynomials of a graph, *Iranian Journal of Mathematical Sciences and Informatics*, **3**(2), (2008), 55-67.
14. M. Walsh, The hub number of a graph, *Int. J. Math. Comput. Sci.*, **1**, (2006), 117-124.