Iranian Journal of Mathematical Sciences and Informatics
Vol. 8, No. 2 (2013), pp 75-84

Generalized weakly contractive multivalued mappings and common fixed points

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Abstract. In this paper we introduce the concept of generalized weakly contractiveness for a pair of multivalued mappings in a metric space. We then prove the existence of a common fixed point for such mappings in a complete metric space. Our result generalizes the corresponding results for single valued mappings proved by Zhang and Song [14], as well as those proved by D. Doric [4].

Keywords: multivalued mapping; weakly contractive mapping; common fixed point.

2000 Mathematics subject classification: 47H10, 47H09.

1. Introduction

A fundamental result in fixed point theory is the Banach contraction principle. Over the years, this result has been generalized in different directions and different spaces by mathematicians.

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Received 20 January 2012; Accepted 5 June 2012

In 1997, Alber and Guerre-Delabriere [1] introduced the concept of weak contraction:

Definition 1.1. Let (E, d) be a metric space. A mapping $T : E \to E$ is said to be weakly contractive provided that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$$

where $x, y \in E$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Using the concept of weakly contractiveness, Alber and Guerre-Delabriere succeeded to establish the existence of fixed points for such mappings in Hilbert spaces. Later on Rhoades [9] proved that the result of [1] is also valid in complete metric spaces. Rhoades [9] also proved the following fixed point theorem which is a generalization of the Banach contraction principle, because it contains contractions as special cases when we assume $\phi(t) = (1 - k)t$ for some $0 \leq k < 1$.

Theorem 1.2. Let (E, d) be a complete metric space and let $T : E \to E$ be a weakly contractive mapping. Then T has a fixed point.

In 2008, Dutta and Choudhury [5] proved the following theorem which in turn generalizes Rhoades' theorem.

Theorem 1.3. Let (E, d) be a complete metric space and $T : E \to E$ be a self-mapping satisfying the inequality

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

where $\phi, \psi : [0, \infty) \to [0, \infty)$ are two continuous and monotone nondecreasing functions with $\phi(t) = 0 = \psi(t)$ if and only if $t = 0$. Then T has a fixed point.

During the last few decades, a number of hybrid contractive mapping results have been obtained by many researchers; see [2, 3, 7, 8, 10, 11, 12] and the references therein. Recently Zhang and Song [14] have proved the following theorem.

Theorem 1.4. Let (E, d) be a complete metric space, and $T, S : E \to E$ be two mappings such that for all $x, y \in E$ we have

$$d(Tx, Sy) \leq M(x, y) - \phi(M(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function and $\phi(t) = 0$ if and only if $t = 0$, and

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Sy, y), \frac{d(y, Tx) + d(x, Sy)}{2}\right\}.$$

Then there exists a unique point $u \in E$ such that $u = Tu = Su$.

This theorem was generalized by D. Doric [4] in the following way:

Theorem 1.5. Let (E, d) be a complete metric space, and $T, S : E \to E$ be two mappings such that for all $x, y \in E$ we have

$$\psi(d(Tx, Sy)) \leq \psi(M(x, y)) - \phi(M(x, y))$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ and ϕ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$, and ψ is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, and

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Sy, y), \frac{d(y, Tx) + d(x, Sy)}{2}\right\}.$$

Then there exists a unique point $u \in E$ such that $u = Tu = Su$.

Let (E, d) be a metric space, and let $B(E)$ denote the family of all nonempty bounded subsets of E. Then for $A, B \in B(E)$, define the distance between A and B by

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

and the diameter of A and B by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

Let $T : E \to B(E)$ be a multivalued mapping, then an element $x \in E$ is called a fixed point of T provided that $x \in T(x)$.

For $T : E \to B(E)$, we define

$$Q_T(x) = \{y \in T(x) : d(x, y) = \delta(x, T(x)\}.$$

In the present paper we shall establish a common fixed point theorem for generalized weakly contractive multivalued mappings. The result we obtain generalizes recent results of Zhang and Song [14], as well as those of D. Doric [4].

2. The Main Result

This section is devoted to the main result of this paper. In the sequel, we shall define

$$N(x, y) = \max\left\{d(x, y), \delta(Tx, x), \delta(y, Sy), \frac{D(y, Tx) + D(x, Sy)}{2}\right\}. \tag{2.1}$$

Now we state the main result of this paper.

Theorem 2.1. Let (E, d) be a complete metric space, and let $T, S : E \to B(E)$ be two mappings such that for all $x, y \in E$

$$\psi(\delta(Tx, Sy)) \leq \psi(N(x, y)) - \phi(N(x, y)) \tag{2.2}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$, and $\psi : [0, \infty) \to [0, \infty)$ is a continuous and monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$. We further assume that for each $x \in E$, both $Q_T(x)$ and $Q_S(x)$ are nonempty. Then S and T have a unique common fixed point $z \in E$. Moreover $Sz = Tz = \{z\}$.

Proof. We choose $x_0 \in E$. Since by assumption for each $x \in E$, both $Q_T(x)$ and $Q_S(x)$ are nonempty, we can define a sequence in the following way:
$x_{2n+1} \in Tx_{2n}$ such that $\delta(Tx_{2n}, x_{2n}) = d(x_{2n}, x_{2n+1})$ and
$x_{2n+2} \in Sx_{2n+1}$ such that $\delta(Sx_{2n+1}, x_{2n+1}) = d(x_{2n+1}, x_{2n+2})$.
Now we have

$$\begin{aligned} N(x_{2n}, x_{2n+1}) = \max\{&d(x_{2n}, x_{2n+1}), \delta(Tx_{2n}, x_{2n}), \delta(Sx_{2n+1}, x_{2n+1}), \\ &\frac{D(Tx_{2n}, x_{2n+1}) + D(Sx_{2n+1}, x_{2n})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

Similarly

$$N(x_{2n+1}, x_{2n+2}) = \max\{d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+3})\}.$$

If for some n we have either $x_{2n} = x_{2n+1}$ or $x_{2n+1} = x_{2n+2}$, then we conclude that the sequence $\{x_n\}$ is constant and thus it is a Cauchy sequence. Suppose $x_n \neq x_{n+1}$ for each n. If

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2}),$$

then

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &\leq \psi(\delta(Tx_{2n}, Sx_{2n+1})) \\ &\leq \psi(N(x_{2n}, x_{2n+1})) - \phi(N(x_{2n}, x_{2n+1})) \\ &= \psi(d(x_{2n+1}, x_{2n+2})) - \phi(d(x_{2n+1}, x_{2n+2})) \end{aligned}$$

which is a contradiction. Hence $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$ and

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n}, x_{2n+1})) - \phi(d(x_{2n}, x_{2n+1})).$$

Similarly $d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2})$ and

$$\psi(d(x_{2n+2}, x_{2n+3})) \leq \psi(d(x_{2n+1}, x_{2n+2})) - \phi(d(x_{2n+1}, x_{2n+2})).$$

So for each n we have $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$. Therefore the sequence $\{d(x_{n+1}, x_n)\}$ is monotone decreasing and bounded below. Thus there exists

$r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Because of

$$d(x_{n+1}, x_n) \leq N(x_n, x_{n-1}) \leq d(x_n, x_{n-1}),$$

we conclude that $\lim_{n\to\infty} N(x_{n+1}, x_n) = r$. Then (by the lower semicontinuity of ϕ), we have

$$\phi(r) \leq \liminf_{n\to\infty} \phi(N(x_n, x_{n-1})).$$

We now claim that $r = 0$. In fact taking upper limits as $n \to \infty$ on either sides of the inequality

$$\psi(d(x_n, x_{n+1})) \leq \psi(N(x_n, x_{n-1})) - \phi(N(x_n, x_{n-1})),$$

we obtain, by the continuity of ψ, that

$$\psi(r) \leq \psi(r) - \liminf_{n\to\infty} \phi(N(x_n, x_{n-1})) \leq \psi(r) - \phi(r),$$

i.e. $\phi(r) \leq 0$. Thus $\phi(r) = 0$ (by the property of the function ϕ), and furthermore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.3}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. In view of (2.3) it suffices to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose not. Then there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k > m_k \geq k$, such that

$$d(x_{2m_k}, x_{2n_k)} \geq \varepsilon. \tag{2.4}$$

Furthermore, assume that for each k, n_k is the smallest positive integer greater than m_k for which (2.4) holds; this implies that

$$d(x_{2m_k}, x_{2n_k-2)} < \varepsilon.$$

Therefore we have

$$\begin{aligned}\varepsilon \leq d(x_{2m_k}, x_{2n_k}) &\leq d(x_{2m_k}, x_{2n_k-2}) + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) \\ &< \varepsilon + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k})\end{aligned}$$

Now, letting $k \to \infty$ we obtain $d(x_{2m_k}, x_{2n_k}) \to \varepsilon$. We note that

$$|d(x_{2m_k}, x_{2n_k+1}) - d(x_{2m_k}, x_{2n_k})| \leq d(x_{2n_k}, x_{2n_k+1})$$

and

$$|d(x_{2m_k-1}, x_{2n_k}) - d(x_{2m_k}, x_{2n_k})| \leq d(x_{2m_k}, x_{2m_k-1}),$$

from which it follows that

$$\lim_{n\to\infty} d(x_{2m_k-1}, x_{2n_k}) = \lim_{n\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \varepsilon.$$

It is not difficult to see that

$$\begin{aligned}d(x_{2m_k}, x_{2n_k+1}) - d(x_{2m_k+1}, x_{2m_k}) - d(x_{2n_k+2}, x_{2n_k+1}) &\leq d(x_{2m_k+1}, x_{2n_k+2}) \\ \leq d(x_{2m_k}, x_{2n_k+1}) + d(x_{2m_k+1}, x_{2m_k}) &+ d(x_{2n_k+2}, x_{2n_k+1}).\end{aligned}$$

Thus

$$\lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k+2}) = \varepsilon.$$

Now, it can be verified that

$$\begin{aligned} &N(x_{2m_k+1}, x_{2n_k+2}) \\ &\quad= \max\{d(x_{2m_k+1}, x_{2n_k+2}), \delta(Tx_{2n_k+2}, x_{2n_k+2}), \delta(x_{2m_k+1}, Sx_{2m_k+1}), \\ &\qquad \frac{D(Tx_{2n_k+2}, x_{2m_k+1}) + D(Sx_{2m_k+1}, x_{2n_k+2})}{2}\} \end{aligned}$$

tends to ε as $k \to \infty$. Finally, by letting $k \to \infty$, we conclude from

$$\begin{aligned} \psi(d(x_{2m_k+2}, x_{2n_k+3})) &\leq \psi(\delta(Tx_{2n_k+2}, Sx_{2m_k+1})) \\ &\leq \psi(N(x_{2n_k+2}, x_{2m_k+1})) - \phi(N(x_{2n_k+2}, x_{2m_k+1})) \end{aligned}$$

that $\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$, or equivalently $\phi(\varepsilon) \leq 0$ which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. Notice that E is complete, hence $\{x_n\}$ is convergent. Let us write $\lim_{n\to\infty} x_n = z$ for some $z \in E$. Now we prove that $\delta(Tz, z) = 0$. Suppose that this is not true, then $\delta(Tz, z) > 0$. For large enough n, we claim that the following equations are true:

$$\begin{aligned} N(z, x_{2n+1}) = \max\{&d(z, x_{2n+1}), \delta(z, Tz), \delta(Sx_{2n+1}, x_{2n+1}), \\ &\frac{D(Tz, x_{2n+1}) + D(Sx_{2n+1}, z)}{2}\} = \delta(z, Tz). \end{aligned}$$

Indeed, since $\lim_{n\to\infty} d(z, x_{2n+1}) = 0$, and

$$\lim_{n\to\infty} \delta(Sx_{2n+1}, x_{2n+1}) = \lim_{n\to\infty} d(x_{2n+2}, x_{2n+1}) = 0,$$

it follows that

$$\begin{aligned} &\lim_{n\to\infty} \frac{D(Tz, x_{2n+1}) + D(Sx_{2n+1}, z)}{2} \\ &\quad\leq \lim_{n\to\infty} \frac{\delta(Tz, z) + d(z, x_{2n+1}) + \delta(Sx_{2n+1}, x_{2n+1}) + d(x_{2n+1}, z)}{2} \\ &\quad= \frac{\delta(Tz, z)}{2}. \end{aligned}$$

Therefore, there exists $k \in N$ such that $N(z, x_{2n+1}) = \delta(z, Tz)$ for $n > k$. Note that

$$\psi(\delta(Tz, x_{2n+2})) \leq \psi(\delta(Tz, Sx_{2n+1})) \leq \psi(N(z, x_{2n+1})) - \phi(N(z, x_{2n+1})).$$

Letting $n \to \infty$, we have

$$\psi(\delta(Tz, z)) \leq \psi(\delta(Tz, z)) - \phi(\delta(Tz, z))$$

i.e, $\phi(\delta(Tz,z)) \leq 0$. This is a contradiction, therefore $\delta(Tz,z)=0$ i.e., $Tz=\{z\}$. Since

$$N(z,z) = \max\left\{d(z,z), \delta(Tz,z), \delta(z,Sz), \frac{D(Tz,z)+D(Sz,z)}{2}\right\}$$
$$= \max\{\delta(Sz,z), \frac{D(Sz,z)}{2}\} = \delta(Sz,z),$$

we conclude that

$$\psi(\delta(z,Sz)) \leq \psi(\delta(Tz,Sz))$$
$$\leq \psi(N(z,z)) - \phi(N(z,z))$$
$$\leq \psi(\delta(z,Sz)) - \phi(\delta(Sz,z)).$$

which in turn implies that $Sz=\{z\}$. Hence the point z is a common fixed point of S and T.
Now let $y \in E$ be another common fixed point of S and T. Note that

$$N(y,y) = \max\{d(y,y), \delta(Ty,y), \delta(y,Sy), \frac{D(Ty,y)+D(Sy,y)}{2}\}$$
$$= \max\{\delta(Sy,y), \delta(y,Ty)\}.$$

Hence

$$\psi(\delta(y,Ty)) \leq \psi(\delta(Sy,Ty)) \leq \psi(N(y,y)) - \phi(N(y,y))$$
$$\leq \psi(max\{\delta(y,Sy), \delta(y,Ty)\}) - \phi(\max\{\delta(y,Sy), \delta(y,Ty)\}.$$

Similarly, we have

$$\psi(\delta(y,Sy)) \leq \psi(\delta(Ty,Sy)) \leq \psi(N(y,y)) - \phi(N(y,y))$$
$$\leq \psi(\max\{\delta(y,Sy), \delta(y,Ty)\} - \phi(\max(\delta(y,Sy), \delta(y,Ty)\}.$$

Therefore

$$\psi(\max\{\delta(y,Sy), \delta(y,Ty)\}) \leq$$
$$\psi(\max\{\delta(y,Sy), \delta(y,Ty)\}) - \phi(\max\{\delta(y,Sy), \delta(y,Ty)\})$$

which implies that $\max\{\delta(y,Sy), \delta(y,Ty)\} = 0$, hence $\delta(Ty,y) = \delta(Sy,y) = 0$.
Now we have

$$N(z,y) = \max\{d(z,y), \delta(z,Tz), \delta(y,Sy), \frac{D(y,Tz)+d(z,Sy)}{2}\} = d(z,y)$$

and

$$\psi(d(z,y)) = \psi(\delta(Sz,Ty)) \leq \psi(N(z,y)) - \phi(N(z,y))$$
$$= \psi(d(z,y)) - \phi(d(z,y))$$

. That imply $d(z,y)=0$ i.e, $z=y$. Hence z is the unique common fixed point of S and T. □

Example 2.2. Let $E = [0,1]$ and $d(x,y) = |x-y|$. For all $x \in E$ define $S, T : E \to B(E)$ by

$$Tx = [\frac{x}{4}, \frac{x}{2}], \qquad Sx = [0, \frac{x}{5}].$$

Then

$$\delta(Tx, Sy) = \begin{cases} \frac{x}{2} & 0 \le \frac{y}{5} \le \frac{x}{2} \\ \max\{\frac{y}{5} - \frac{x}{4}, \frac{x}{2}\} & \frac{x}{2} \le \frac{y}{5} \le 1. \end{cases}$$

and

$$\delta(x, Tx) = \frac{3x}{4}, \qquad \delta(y, Sy) = y.$$

We also consider $\psi(t) = 2t$ and $\phi(t) = \frac{t}{2}$. We note that if $\frac{y}{5} \le \frac{x}{2}$, then

$$\psi(\delta(Tx, Sy) = x \le \frac{9x}{8} = \frac{3}{2}\delta(x, Tx) \le \frac{3}{2}(N(x,y)) = \psi(N(x,y)) - \phi(N(x,y))$$

and if $\frac{x}{2} \le \frac{y}{5}$, then

$$\psi(\delta(Tx, Sy) = 2.(\frac{y}{5} - \frac{x}{4}) \le \frac{2y}{5} \le \frac{3y}{2} = \frac{3}{2}\delta(y, Sy) \le \frac{3}{2}(N(x,y)) = \psi(N(x,y)) - \phi(N(x,y)).$$

This arguments show that the mappings T and S satisfy the conditions of Theorem 2.1. Now it is easy to see that 0 is the only common fixed point of these two mappings.

In the following we shall see that Theorems 1.4 and 1.5 are easily derived from our main result.

Remark 2.3. In Theorem 2.1, if E is bounded and $T, S : E \to E$ are given, then we obtain Theorem 1.5. Furthermore if $\psi(t) = t$ for all $t \in [0, \infty)$ then we obtain Theorem 1.4.

Note that in the above theorems there are just two control functions; namely, ϕ and ψ. For instance, in Theorem 1.3 above due to Dutta and Choudhury [5], we have

$$\psi(d(Tx, Ty)) \le \psi(d(x,y)) - \phi(d(x,y))$$

for all $x, y \in E$. This can be generalized to the following theorem.

Theorem 2.4. Let (E, d) be a complete metric space, and $T : E \to E$ be a self-mapping satisfying

$$\psi_1(d(Tx, Ty)) \le \psi_2(d(x,y)) - \psi_3(d(x,y)) \qquad x, y \in E$$

where $\psi_1, \psi_2, \psi_3 : [0, \infty) \to [0, \infty)$ are functions satisfying the following conditions:

(i) ψ_1 is continuous and monotone nondecreasing,
(ii) ψ_2 is continuous,
(iii) ψ_3 is lower semicontinuous,
(iv) $\psi_1(t) = 0 = \psi_2(t) = 0 = \psi_3(t)$ if and only if $t = 0$,
(v) $\psi_1(t) - \psi_2(t) + \psi_3(t) > 0$ for $t > 0$.

Then T has a unique fixed point.

For a proof and an illustrative example satisfying all the conditions of the theorem, we refer the reader to a preprint by the current authors [6].

Acknowledgement. We wish to express our thanks to the referee for his useful comments which were very helpful to improve this paper.

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