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Weak complete parts in semihypergroups

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Abstract. In this article we generalize the notion of complete parts, by introducing a weaker condition in definition. Using this generalization we define and analyse a new class of semihypergroups, which are called weak complete semihypergroups. Complete parts were introduced about 40 years ago by M. Koskas and they represent a basic notion of hyperstucture theory, utilized in constructing an important class of subhypergroups of a hypergroup and also they are used to define complete hypergroups.

Keywords: (semi)Hypergroup, (strongly) Regular relation, Complete parts, γ-part.

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1. Introduction

Hyperstructure theory was born in 1934 at the 8th congress of Scandinavian Mathematicions, where Marty [14] introduced the hypergroup notion as a generalization of groups and after, he proved its utility in solving some problems of groups, Algebraic functions and Rational fractions. Surveys of the theory can be found in the book of Corsini [3], Vougiouklis [19], Corsini and Leoreanu [4]. Complete parts were introduced by Koskas [10] and studied then

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by Miglirato [15], Corsini and Sureau [3, 4]. In [6], Davvaz and Karimian introduced the concept of γ-parts and studied some properties of $C_\gamma(A)$, where $C_\gamma(A)$ is the intersection of all γ-parts of a semihypergroup H, which include a subset A. See [7, 11]. In [16] Mousavi, Leoreanu-Fotea and Jafarpour introduced the notion of a $\Re$-part in semihypergroups and studied its properties. H. Babaie, M. Jafarpour and S.Sh. Mousavi in [1] introduced and investigated $\Re$-part in hyperrings. In this article we introduce and analyse the notion of *weak complete part* of a semihypergroup which is a generalization of the notion of a complete part in semihypergroups. This generalization leads to a new class of semihypergroups.

We recall here some basic notions of hypergroup theory.

Let H be a nonempty set and $P^*(H)$ the set of all non-empty subsets of H. Let $\circ$ be a *hyperoperation* (or *join operation*) on H, that is, $\circ$ is a function from $H \times H$ into $P^*(H)$. If $(a, b) \in H \times H$, its image under $\circ$ in $P^*(H)$ is denoted by $a \circ b$ or ab. The join operation is extended to subsets of H in a natural way, that is $A \circ B = \bigcup\{ab \mid a \in A, b \in B\}$. The notation aA is used for $\{a\}A$ and Aa for $A\{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \circ)$ is called a *semihypergroup* if $a(bc) = (ab)c$ for all $a, b, c \in H$ and is called a *hypergroup* if it is a semihypergroup and $aH = Ha = H$ for all $a \in H$. A non-empty subset A of a semihypergroup $(H, \circ)$ is called *subsemihypergroup* of H if $A \circ A \subseteq A$ and it is called a *complete part* of H if for all $n \geqslant 2$ and for all $(x_1, x_2, ..., x_n) \in H^n$ the following implication holds:

$$\prod_{i=1}^{n} x_i \bigcap A \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subseteq A.$$

The *complete closure* of A in H is the intersection of all complete parts which contain A and it will be denoted by $\mathcal{C}(A)$. A semihypergroup $(H, \circ)$ is called complete if for all $(x, y) \in H^2$, $\mathcal{C}(x \circ y) = x \circ y$.
In what follows, we mention some results concerning complete parts and complete semihypergroups which can be found in [4].

Theorem 1.1. *If A is a non-empty subset of a semihypergroup $(H, \circ)$, then* $\mathcal{C}(A) = \cup_{a \in A} \mathcal{C}(a)$.

Theorem 1.2. *A semihypergroup $(H, \circ)$ is complete if $H = \cup_{s \in S} A_s$, where S and A_s satisfy the conditions:*

(i) *$(S, \cdot)$ is a semigroup;*
(ii) *for all $(s, t) \in S^2$, where $s \neq t$ we have $A_s \cap A_t = \emptyset$;*
(iii) *if $(a, b) \in A_s \times A_t$, then $a \circ b = A_{s \cdot t}$.*

Theorem 1.3. *Let $(H, \circ)$ be a semihypergroup. The following conditions are equivalent:*

(i) *for all $(x, y) \in H^2$ and for all $a \in x \circ y$, $\mathcal{C}(a) = x \circ y$;*

(ii) *for all* $(x, y) \in H^2$ $\mathcal{C}(x \circ y) = x \circ y$.

For all $n > 1$ define the relation β_n on a semihypergroup H, as follows:

$$a\ \beta_n\ b \Leftrightarrow \exists (x_1, ..., x_n) \in H^n\ :\ \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

and $\beta = \bigcup_{i=1}^{n} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. Suppose that β^* is the transitive closure of β so β^* is an equivalence relation, see [3]. β^* is the least equivalence strongly regular relation on a hypergroup H, such that the quotient H/β^* is a group with respect to the following operation,

$$\beta^*(x) \otimes \beta^*(y) = \beta^*(z), \qquad \forall z \in x \circ y.$$

The *heart* ω_H of a hypergroup H is the set of all elements x of H, for which the equivalence class $\beta^*(x)$ is the identity of the group H/β^*.

Theorem 1.4. ω_H *is a complete part of* H.

Theorem 1.5. *If* $(H, \circ)$ *is a semihypergroup and* A *is a non-empty subset of* H *then* $\mathcal{C}(A) = \omega_H \circ A$.

2. Weak complete parts

In this section first we generalize the notion of complete parts and then we introduce the class of weak complete semihypergroups which contains the class of complete semihypergroups.

Definition 2.1. Let $(H, \circ)$ be a semihypergroup. For a non-empty subset A of H we say that:

A is a *weak complete part* of H, if for all $x_1, ..., x_n \in H$ the following implication is valid:

$$\prod_{i=1}^{n} x_i \bigcap A \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subseteq A^m,$$

where $A^m = A \circ A ... \circ A$ (m times), for some $m \in \mathbb{N}$.

Remark 2.2. Every complete part is a weak complete part but the following example shows that the converse is not true.

Example 2.3. *Suppose that* $H = \{e, a, b\}$. *Define the hyperoperation* $\circ$ *on* H *as follows:*

$\circ$	e	a	b
e	H	$\{e, a\}$	$\{e, a\}$
a	$\{e, a\}$	H	H
b	H	$\{e, a\}$	$\{e, a\}$

Now let $A = \{a\}$. In above table we have $b \circ b = \{e, a\}$ thus $b \circ b \cap A \neq \emptyset$ but $b \circ b \nsubseteq A$ hence A is not a complete part. Since $a \circ a = H$ we conclude that A is a weak complete part of H.

Example 2.4. *Suppose that $H = \{e, a, b, c\}$. Consider the hypergroup $(H, \circ')$, where $\circ'$ is defined on H as fallows:*

$\circ'$	e	a	b	c
e	e	a	b	c
a	a	a	H	c
b	b	$\{e, a, b\}$	b	$\{b, c\}$
c	c	$\{a, c\}$	c	H

In this case we can see that $A = \{e, a, b\}$ and $B = \{b, c\}$ are weak complete parts while $A \cap B = \{b\}$ is not a weak complete part of H.

Denote by $\mathcal{W}(A)$ the intersection of all weak complete parts which contain A. Notice that $\mathcal{W}(A)$ is not a weak complete part of H necessarily.

In what follows, we present a manner to construct weak complete parts of H. Let H be a semihypergroup, $\mathcal{U}$ be the set of finite products of elements of H. If $u = \prod_{i=1}^{n} x_i$ and $1 \leq k \leq m \leq n$ we denote $u_{k,m} = \prod_{i=k}^{m} x_i$ and we call it a *subproduct* of u. Moreover we denote by $M(u)$ the set of all subproducts of u.

Now suppose that B is a non-empty subset of H, $1 = m_0 \leq m_1 \leq m_2 \leq \ldots m_j \leq n = m_{j+1}$, $(j \in \mathbb{N})$ and $u = \prod_{i=0}^{j} u_{m_i, m_{i+1}} \cap B \neq \emptyset$. We define

$$\lambda(u, B) = \{(u_{1,m_1}, \ldots, u_{m_j, n}) | u_{m_s, m_{s+1}} \cap B \neq \emptyset, \text{ for every } 0 \leq s \leq j\}.$$

Denote $|\lambda| = max\{j | (u_{1,m_1}, \ldots, u_{m_j, n}) \in \lambda(u, B)\}$ and

$$\Lambda(u, B) = \{(u_{1,m_1}, \ldots, u_{m_{|\lambda|}, n}) | (u_{1,m_1}, \ldots, u_{m_j, n}) \in \lambda(u, B)\}$$

and $\Lambda(B) = \cup_{u \in \mathcal{U}} \Lambda(u, B)$. Notice that $\Lambda(u, B) \subseteq \lambda(u, B)$ and $0 \leq |\lambda|$.

Definition 2.5. Let H be a semihypergroup and A, B be non-empty subsets of H.

Set $K_1(A) = A$, $K_{t+1}(A) = \{x \mid \exists s : \ 0 \leq s \leq |\lambda|, \ x \in u_{m_s, m_{s+1}}$, where $(u_{1,m_1}, \ldots, u_{m_{|\lambda|}, n}) \in \Lambda(K_t(A))\}$,

and $K(A) = \cup_{n \geq 1} K_n(A)$.

Let us consider what exactly the above notation means.

For all $0 \leq s \leq |\lambda|$, we say that a hyperproduct $u_{m_s, m_{s+1}}$ is *irreducible with respect to* A if $u_{m_s, m_{s+1}} \cap A \neq \emptyset$ and all proper subproducts of $u_{m_s, m_{s+1}}$ have empty intersection to A. For instance, if we consider $A = \{a\}$ in Example 2.4, then $a \circ' b$ is an irreducible with respect to A, while $a \circ' b \circ' a$ contains a, but it is not irreducible with respect to A.

Hence $K_2(A)$ is the union of all hyperproducts, which are irreducible with respect to A. Generally, $K_{t+1}(A)$ is the union of all hyperproducts, which are irreducible with respect to $K_t(A)$.

Notice that the complete closure $\mathcal{C}(A)$ of A is the union of the sets $\mathcal{C}_t(A)$, where t is a nonzero natural number and $\mathcal{C}_1(A) = A$, while $\mathcal{C}_{t+1}(A)$ is the union of all hyperproducts, which have nonempty intersection to $\mathcal{C}_t(A)$, see [3, 4]. Since we clearly have $K_t(A) \subseteq \mathcal{C}_t(A)$, for all t, it follows that $K(A) \subseteq \mathcal{C}(A)$.

Proposition 2.6. *Let H be a semihypergroup and A be a non-empty subset of H. Then $K(A)$ is a weak complete part of H which contains A.*

Proof. For proving our claim suppose that $u = \prod_{i=1}^{n} x_i$ and $u \bigcap K(A) \neq \emptyset$ so there exists $t \in \mathbb{N}$ such that $u \bigcap K_{t+1}(A) \neq \emptyset$, hence there exists $(u_{1,m_1}, ..., u_{m_{|\lambda|},n})$ in $\Lambda(K_t(A))$, whence $u_{m_s,m_{s+1}} \subseteq K_{t+1}(A)$, $\forall s:\ 0 \leq s \leq |\lambda|$, thus

$$u = u_{1,m_1} ... u_{m_{|\lambda|},n} \subseteq [K_{t+1}(A)]^{|\lambda|+1} \subseteq [K(A)]^{|\lambda|+1}.$$

□

Proposition 2.7. *Let H be a semihypergroup and A be a non-empty subset of H. Then $K(C(A)) = C(A)$, where $C(A)$ is the complete closure of A in H.*

Proof. According previous proposition we have $C(A) \subseteq K(C(A))$ so it is necessary to prove that $K(C(A)) \subseteq C(A)$. By induction on t we prove that $K_t(C(A)) \subseteq C(A)$, for every $t \in \mathbb{N}$. It is clear that $K_1(C(A)) \subseteq C(A)$. Suppose that $K_t(C(A)) \subseteq C(A)$. We prove that $K_{t+1}(C(A)) \subseteq C(A)$. If $z \in K_{t+1}(C(A))$, then there exists $(u_{1,m_1}, ..., u_{m_{|\lambda|},n}) \in \Lambda(K_t(C(A)))$ such that $z \in u_{m_s,m_{s+1}}$ for some s, $0 \leq s \leq |\lambda|$ and $u_{m_s,m_{s+1}} \cap K_t(C(A)) \neq \emptyset$. Since $K_t(C(A)) \subseteq C(A)$ and $C(A)$ is complete, it follows that $z \in u_{m_s,m_{s+1}} \subseteq C(A)$ hence $K_{t+1}(C(A)) \subseteq C(A)$ and so $K(C(A)) \subseteq C(A)$. □

Corollary 2.8. *If H is a semihypergroup and A is a non-empty subset of H, then $K(C(A)) = \mathcal{W}(C(A))$ and $\mathcal{W}(A) \subseteq C(A)$.*

Example 2.9. *Suppose $H = \{e, a\}$. Consider the hyperoperation $\circ$ on H as follows:*

$\circ$	e	a
e	e	a
a	a	$\{e,a\}$

It is easy to see that $\{a\}$ is a weak complete part of H and $K(a) = \{a\} = \mathcal{W}(a)$ while $C(a) = H$.

Proposition 2.10. *If H is a semihypergroup and A is a subsemihypergroup of H then A is a weak complete part if and only if A is a complete part.*

Proof. Let A be a weak complete part of H and $\prod\limits_{i=1}^{n} x_i \bigcap A \neq \emptyset$, therefore $\prod\limits_{i=1}^{n} x_i \subseteq A^m$. Since A is a subsemihypergroup of H we have $A^m \subseteq A$ hence $\prod\limits_{i=1}^{n} x_i \subseteq A$ and so A is a complete part of H. □

Definition 2.11. A semihypergroup $(H, \circ)$ is called *weak complete* if

$$\forall (x, y) \in H^2,\ K(x \circ y) = x \circ y.$$

Remark 2.12. Every complete semihypergroup is a weak complete semihypergroup but the hypergroup in Example 2.3 is a weak complete semihypergroup which is not a complete semihypergroup.

Example 2.13. *Let $(S, \cdot)$ be a semigroup and $\{A_s\}_{s\in S}$ be a family of nonempty sets, such that the following condition holds:*

For all $s \in S$, the set $T_s = (\{A_{s^k} \mid k$ is a nonzero natural number$\}, \subseteq)$ has a maximum, denoted by $A_{s^{M_s}}$, such that if $A_t \cap A_s \neq \emptyset$ then $A_t \subseteq A_{s^{M_s}}$.

Clearly, if $A_t \cap A_s \neq \emptyset$, then we also have $A_s \subseteq A_{t^{M_t}}$.

Let $H = \cup_{s\in S} A_s$. For all $x \in H$, we denote $S_x = \{A_s \mid x \in A_s\}$.

By hypothesis, if $A_s \in S_x$ then $S_x \subseteq T_s$. Since for all nonzero natural number k, $A_{s^k} \subseteq A_{s^{M_s}}$, it follows that $A_{s^{M_s}} \in S_x$. That is why we shall write sometimes A_{s_x} instead of $A_{s^{M_s}}$.

We define on H the following hyperoperation: for all $x, y \in H$,

$$x \circ y = A_{s_x \cdot s_y}.$$

Then $(H, \circ)$ is a weak complete semihypergroup.

Indeed, if $\prod_{i=1}^{k} x_i \cap \prod_{j=1}^{r} y_j \neq \emptyset$, then $A_{\prod_{i=1}^{k} s_{x_i}} \cap A_{\prod_{j=1}^{r} s_{y_j}} \neq \emptyset$. If we denote $\prod_{j=1}^{r} s_{y_j} = s_0$ then by hypothesis $A_{\prod_{i=1}^{k} s_{x_i}} \subseteq A_{{s_0}^{M_{s_0}}}$. This means that $\prod_{i=1}^{k} x_i \subseteq (\prod_{j=1}^{r} y_j)^{M_{s_0}}$.

Hence $(H, \circ)$ is a weak complete semihypergroup.

Starting with a weak complete semihypergroup, we can construct other weak complete semihypergroups, by constructing the so-called K_H-semihypergroups, see [3, 4]. We recall that a K_H-semihypergroup is a semihypergroup constructed from a semihypergroup $(H, \circ)$ and a family $\{A(x)\}_{x\in H}$ of nonempty and mutually disjoint subsets of H. Set $K_H = \bigcup\limits_{x\in H} A(x)$ and define the hyperoperation $*$ on K_H as follows:

$$\forall (a, b) \in K_H^2;\ \ a \in A(x),\ b \in A(y), \quad a * b = \bigcup_{z\in x\circ y} A(z)$$

$(H, \circ)$ is a hypergroup if and only if $(K_H, *)$ is a hypergroup (see [7]).

For all $P \in P^*(H)$, set $A(P) = \bigcup\limits_{x\in P} A(x)$. Then we obtain the following

Theorem 2.14. *P is a weak complete part of H if and only if $A(P)$ is a weak complete part of K_H.*

Proof. Suppose that $A(P)$ is a weak complete part of K_H and $\prod_{i=1}^{m} y_i \bigcap P \neq \emptyset$. So we have

$$\begin{aligned}\prod_{i=1}^{m} y_i \cap P \neq \emptyset &\Rightarrow \exists p \in P, \text{ such that } p \in \prod_{i=1}^{m} y_i \\ &\Rightarrow \exists p \in P, \text{ such that } A(p) \subseteq \bigcup_{u \in \prod_{i=1}^{m} y_i} A(u) \\ &\Rightarrow \bigcup_{u \in \prod_{i=1}^{m} y_i} A(u) \cap A(P) \neq \emptyset \\ &\Rightarrow \exists n \in \mathbb{N}, \text{ such that } \bigcup_{u \in \prod_{i=1}^{m} y_i} A(u) \subseteq [A(P)]^n.\end{aligned}$$

For all $t \in \prod_{i=1}^{m} y_i$, there exists $(q_1, ..., q_n) \in P^n$ such that $A(t) \subseteq \bigcup_{s \in q_1 \circ \ldots \circ q_n} A(s)$. Thus $A(t) \cap A(s) \neq \emptyset$ and therefore $t = s$ and hence $t \in P^n$, thus $\prod_{i=1}^{m} y_i \subseteq P^n$.

Conversely, let $(z_1, ..., z_m) \in K_H^m$ be such that $* \prod_{i=1}^{m} z_i \bigcap A(P) \neq \emptyset$, where $* \prod$ denotes a hyperproduct of elements in K_H. There exists $(x_1, ..., x_m) \in H^m$ such that for all $1 \leqslant i \leqslant m$, $z_i \in A(x_i)$. Suppose that $u \in \bigcup_{y \in \prod_{i=1}^{m} x_i} A(y) \cap A(P)$. Thus $u \in A(y_0)$ for some $y_0 \in \prod_{i=1}^{m} x_i$. Since $u \in A(P)$, there exists $y_1 \in P$ such that $u \in A(y_1)$. Therefore $A(y_0) \cap A(y_1) \neq \emptyset$, which implies that $y_0 = y_1 \in \prod_{i=1}^{m} x_i \cap P$. Since P is weak complete part of H there exists $n \in \mathbb{N}$ such that $\prod_{i=1}^{m} x_i \subseteq P^n$. Hence $* \prod_{i=1}^{m} z_i \subseteq [A(P)]^n$. □

Corollary 2.15. *H is a weak complete semihypergroup if and only if K_H is a weak complete semihypergroup.*

Proof. We consider $P = x \circ y$ so $a * b = A(x \circ y)$, where $a \in A(x),\ b \in A(y)$ and we apply the above theorem. □

3. Conclusion

In this paper we have introduced a generalization of the notion of complete parts, which is useful in order to analyse a new class of semihypergroups: the weak complete semihypergroups. Complete parts are connected to a special subhypergroup of a hypergroup, which is the heart of a hypergroup. Our study can be continued in order to analyse other subhypergroups, which can be connected to weak complete parts.

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