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A Generalized Fibonacci Sequence and the Diophantine Equations $x^2 \pm kxy - y^2 \pm x = 0$

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Abstract. In this paper some properties of a generalization of Fibonacci sequence are investigated. Then we solve the Diophantine equations $x^2 \pm kxy - y^2 \pm x = 0$, where k is a positive integer, and describe the structure of solutions.

Keywords: Diophantine equation, Generalized Fibonacci sequence, Pell equation

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1. Introduction

Several authors (e.g. [3], [1], [6]) have discussed the conics whose equations are satisfied by pairs of successive terms of generalized Fibonacci sequences.

The second order recurrence $W_n(a, b; p, q)$, is defined by

$$W_0 = a, \ W_1 = b, \ W_{n+1} = pW_n - qW_{n-1}, \ n \geq 1$$

where a, b, p and q are arbitrary integers.

In [7], McDaniel proved that, if x and y are positive integers, then the pair (x, y) is a solution of $y^2 - Pxy - x^2 = \pm 1$ iff there exists a positive integer n such that $x = U_n$ and $y = U_{n+1}$, where $U_n = W_n(0, 1; P, -1)$.

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McDaniel's results was generalized by Melham in [9], where for example he proved that if m is even, then the points with integer coordinates on the conics $y^2 - V_m xy + x^2 \mp U_m^2 = 0$, are precisely the pairs $\mp(U_n, U_{n+m})$, where $V_n = W_n(2, P; P, -1)$.

Kilic and Omur in [5] considered all given results on special conics, and then gave more general results, generalizing work of Melham and McDaniel. For example they proved the following theorem which is the combination of Theorems 3 and 4 of [5].

Theorem 1.1. *The points with integer coordinates on the conics* $y^2 - V_{km}xy + (-1)^m x^2 \mp U_{km}^2 = 0$ *are precisely the pairs* $\mp(U_{kn}, U_{k(n+m)})$.

Marlewski and Zarzycki in [8] proved that the equation $x^2 - kxy + y^2 + x = 0$ with $k \in N$ has an infinite number of positive integer solutions x and y if and only if $k = 3$.

Now let

$$\varphi_0 = 0,\ \varphi_1 = 1,\ \varphi_{n+1} = k\varphi_n + \varphi_{n-1},\ n \geq 1. \tag{1.1}$$

In this paper we will prove that the equation

$$x^2 \pm kxy - y^2 \pm x = 0 \tag{1.2}$$

with $k \in N$ has infinite number of positive integer solutions x and y, and describe the structure of solutions, using Eq. (1.1).

In general, the standard approach to solve these equations is via reduction of quadratic forms and a parallel approach which uses pell equation and generalized pell equation. The generalized pell equation is the Diophantine equation $x^2 - Dy^2 = N$, where D and N are integers and $D > 0$ is not a perfect square. This equation is usually solved using continued fractions.

2. Some Preliminary Results

Let k be a positive integer. It is possible to rewrite Eq. (1.1) as a matrix equation. To do this, we assume that $Q := \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}$. Then $\begin{pmatrix} \varphi_1 \\ \varphi_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix} = Q \begin{pmatrix} \varphi_n \\ \varphi_{n-1} \end{pmatrix}$.

Since $\begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix} = Q^{n-1} \begin{pmatrix} k \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \varphi_n \\ \varphi_{n-1} \end{pmatrix} = Q^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, one can see that

$$Q^n = \begin{pmatrix} \varphi_{n+1} & \varphi_n \\ \varphi_n & \varphi_{n-1} \end{pmatrix}, n \geq 1.$$

Using $Q^0 = I, Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}, Q^{-n} = (Q^n)^{-1}$ we can extend the sequence (1.1) to $\mathbb{Z}$ by setting $\begin{pmatrix} \varphi_{-n+1} & \varphi_{-n} \\ \varphi_{-n} & \varphi_{-n-1} \end{pmatrix} = (-1)^n \begin{pmatrix} \varphi_{n-1} & -\varphi_n \\ -\varphi_n & \varphi_{n+1} \end{pmatrix}$. Thus $\varphi_{-n} = (-1)^{n+1}\varphi_n$.

Since $det(Q^n) = (-1)^n$ we have $\varphi_{n+1}\varphi_{n-1} - \varphi_n^2 = (-1)^n$. Therefore

$$k\varphi_n\varphi_{n+1} + \varphi_n^2 + (-1)^n = \varphi_{n+1}^2. \tag{2.1}$$

The equality $Q^{m+n} = Q^m Q^n$ gives

$$\varphi_{m+n} = \varphi_{m+1}\varphi_n + \varphi_m\varphi_{n-1} = \varphi_m\varphi_{n+1} + \varphi_{m-1}\varphi_n, \tag{2.2}$$

for all integers m and n.

Now let D be a positive integer not a perfect square and suppose that $\sqrt{D}$ is written as an infinite simple continued fraction $\sqrt{D} = [a_0, a_1, a_2, ...]$. For each nonnegative n the rational number $[a_0, a_1, ..., a_n] = h_n/k_n$ is called the n^{th} convergent to the infinite simple continued fraction $[a_0, a_1, a_2, ...]$.

It is easy to see that for any integer $k > 1$ the number $k^2 + 4$ is not a square, and we have the following infinite simple continued fraction of $\sqrt{k^2+4}$

$$\sqrt{k^2+4} = \begin{cases} [k, \overline{(k-1)/2, 1, 1, (k-1)/2, 2k}] & \text{k is odd,} \\ [k, \overline{k/2, 2k}] & \text{k is even.} \end{cases} \tag{2.3}$$

If $[a_0, a_1, a_2, ...]$ is an infinite simple continued fraction and h_n/k_n is the n^{th} convergent to it, then

$$\begin{matrix} h_0 = a_0, \ h_1 = a_1a_0 + 1, \ h_n = a_nh_{n-1} + h_{n-2}, \\ k_0 = 1\ , k_1 = a_1, \ k_n = a_nk_{n-1} + k_{n-2}, \end{matrix} \quad n \geq 2. \tag{2.4}$$

To simplify our calculations we define $h_{-1} = 1$ and $k_{-1} = 0$.

The next two theorems give the convergents of $\sqrt{k^2+4}$ in terms of the sequence φ_n. These results are proved in [7] using different arguments.

We first assume that k is odd. Then by Eq. (2.3) we have

$a_0 = k, \ a_{5n-4} = (k-1)/2, \ a_{5n-3} = 1, \ a_{5n-2} = 1, \ a_{5n-1} = (k-1)/2, \ a_{5n} = 2k, \ n \geq 1.$

It is easy to see that Eq. (2.4) can be written as

$$\begin{pmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_{n-1} & k_{n-1} \\ h_{n-2} & k_{n-2} \end{pmatrix}, n \geq 1.$$

Let $A_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$ and $P_n = \begin{pmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{pmatrix}$. Then we have

$$P_n = A_nP_{n-1}, \ (n \geq 1). \tag{2.5}$$

If we set $N = A_5A_4A_3A_2A_1$, then using Eq. (2.2) and by induction, we can prove the following Lemma.

Lemma 2.1. *For any positive integer t,* $N^t = \begin{pmatrix} \varphi_{3t+1} & 2\varphi_{3t} \\ 1/2\varphi_{3t} & \varphi_{3t-1} \end{pmatrix}$.

Theorem 2.2. *Let k be a positive odd integer and h_n/k_n be the n^{th} convergent to the infinite simple continued fraction of $\sqrt{k^2+4}$. Then for all nonnegative integer n*

a) $h_{10n} = \varphi_{6n} + \varphi_{6n+2}$,
b) $k_{10n} = \varphi_{6n+1}$,
c) $h_{10n+4} = 1/2(\varphi_{6n+2} + \varphi_{6n+4})$,
d) $k_{10n+4} = 1/2(\varphi_{6n+3})$,
e) $h_{10n+8} = \varphi_{6n+4} + \varphi_{6n+6}$,
f) $k_{10n+8} = \varphi_{6n+5}$.

Proof. To calculate h_{10n} and k_{10n}, using Eq. (2.5) we have

$$\begin{aligned} P_{5n} = A_{5n}P_{5n-1} = \cdots &= A_{5n}A_{5n-1}A_{5n-2}A_{5n-3}A_{5n-4}P_{5n-5} \\ &= A_5A_4A_3A_2A_1P_{5n-5} \\ &= NP_{5n-5}. \end{aligned}$$

Hence $P_{10n} = N^{2n}P_0$, and by Lemma 2.1, we derive

$$\begin{aligned} P_{10n} &= \begin{pmatrix} \varphi_{6n+1} & 2\varphi_{6n} \\ 1/2\varphi_{6n} & \varphi_{6n-1} \end{pmatrix} \begin{pmatrix} h_0 & k_0 \\ h_{-1} & k_{-1} \end{pmatrix} \\ &= \begin{pmatrix} \varphi_{6n+1} & 2\varphi_{6n} \\ 1/2\varphi_{6n} & \varphi_{6n-1} \end{pmatrix} \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} &h_{10n} = k\varphi_{6n+1} + 2\varphi_{6n} = \varphi_{6n+2} + \varphi_{6n}, \\ &k_{10n} = \varphi_{6n+1}. \end{aligned}$$

Also

$$\begin{aligned} P_{10n+4} &= A_{10n+4}A_{10n+3}A_{10n+2}A_{10n+1}N^{2n}P_0 \\ &= \begin{pmatrix} 1/2\varphi_3 & \varphi_2 \\ \varphi_2 & 2 \end{pmatrix} \begin{pmatrix} \varphi_{6n+1} & 2\varphi_{6n} \\ 1/2\varphi_{6n} & \varphi_{6n-1} \end{pmatrix} \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/2\varphi_{6n+3} & \varphi_{6n+2} \\ \varphi_{6n+2} & 2\varphi_{6n+1} \end{pmatrix} \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

and so,

$$\begin{aligned} &h_{10n+4} = 1/2(k\varphi_{6n+3} + 2\varphi_{6n+2}) = 1/2(\varphi_{6n+4} + \varphi_{6n+2}), \\ &k_{10n+4} = 1/2\varphi_{6n+3}. \end{aligned}$$

Finally

$$P_{10n+8} = A_{10n+8}A_{10n+7}A_{10n+6}N^{2n+1}P_0.$$

Hence

$$h_{10n+8} = k\varphi_{6n+5} + 2\varphi_{6n+4} = \varphi_{6n+6} + \varphi_{6n+4},$$
$$k_{10n+8} = \varphi_{6n+5}.$$

□

We now assume that k is even. By Eq. (2.3) we have

$$a_0 = k,\ a_{2n-1} = k/2,\ a_{2n} = 2k,\ n \geq 1.$$

Then Eq. (2.4) implies that

$$\begin{pmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_{n-1} & k_{n-1} \\ h_{n-2} & k_{n-2} \end{pmatrix},\ n \geq 1.$$

Using previous notations, this equation can be written as $P_n = A_n P_{n-1}$ where $A_{2n} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$ and $A_{2n-1} = \begin{pmatrix} k/2 & 1 \\ 1 & 0 \end{pmatrix}$ for all $n \geq 1$. Now

$$P_{2n} = A_{2n} P_{2n-1} = A_{2n} A_{2n-1} P_{2n-2}. \tag{2.6}$$

Let $M = A_{2n} A_{2n-1}$. Then again using Eq. (2.2) and induction, we can prove the following lemma.

Lemma 2.3. *For all positive integer t,* $M^t = \begin{pmatrix} \varphi_{2t+1} & 2\varphi_{2t} \\ 1/2\varphi_{2t} & \varphi_{2t-1} \end{pmatrix}$.

Theorem 2.4. *Let k be a positive even integer and h_n/k_n be the n^{th} convergent to the infinite simple continued fraction of $\sqrt{k^2+4}$. Then for all nonnegative integer n*

a) $h_{2n} = \varphi_{2n} + \varphi_{2n+2}$,
b) $k_{2n} = \varphi_{2n+1}$.

Proof. Using Eq. (2.6) and Lemma 2.3, we have

$$P_{2n} = MP_{2n-2} = M^n P_0 = \begin{pmatrix} \varphi_{2n+1} & 2\varphi_{2n} \\ 1/2\varphi_{2n} & \varphi_{2n-1} \end{pmatrix} \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix},$$

and so

$$h_{2n} = k\varphi_{2n+1} + 2\varphi_{2n} = \varphi_{2n+2} + \varphi_{2n},$$
$$k_{2n} = \varphi_{2n+1}.$$

□

3. The Diophantine Equations $x^2 \pm kxy - y^2 \pm x = 0$

In this section we show that the Diophantine equations $x^2 \pm kxy - y^2 \pm x = 0$ are solvable in integers for all positive integer k.

We first consider the equation

$$x^2 - kxy - y^2 + x = 0. \tag{3.1}$$

We will prove that all the solutions of Eq. (3.1) are the followings

$$
\begin{gathered}
(\varphi_{2n}^2, \varphi_{2n-1}\varphi_{2n}),\\
(\varphi_{2n}^2, -\varphi_{2n}\varphi_{2n+1}),\\
(-\varphi_{2n+1}^2, \varphi_{2n+1}\varphi_{2n+2}),\\
(-\varphi_{2n+1}^2, -\varphi_{2n}\varphi_{2n+1}).
\end{gathered} \tag{3.2}
$$

Lemma 3.1. *If (x, y) is a solution of Eq. (3.1), then the followings are also solutions of the same equation*

a) $(x, -kx - y)$,
b) $(ky - x - 1, y)$.

Proof. The proof is straightforward. □

Clearly $(0, 0)$ is a solution of Eq. (3.1). Using the above Lemma, and starting from $(0, 0)$ we get a sequence of solutions of Eq. (3.1) as

$$(0, 0), (-1, 0), (-1, k), (k^2, k), (k^2, -k(k^2 + 1)), (-(k^2 + 1)^2, -k(k^2 + 1)), \cdots .$$

These solutions can be written as

$$
\begin{aligned}
&(0, 0) = (\varphi_0^2, -\varphi_0\varphi_1), (-1, 0) = (-\varphi_1^2, -\varphi_0\varphi_1), (-1, k) = (-\varphi_1^2, \varphi_1\varphi_2),\\
&(k^2, k) = (\varphi_2^2, \varphi_1\varphi_2), \cdots .
\end{aligned}
$$

Theorem 3.2. *For all nonnegative integers n, the pairs in Eq. (3.2) satisfy Eq. (3.1).*

Proof. Induction on n. Clearly $(0, 0) = (\varphi_0^2, -\varphi_0\varphi_1)$ is a solution of Eq. (3.1). Suppose that $(x, y) = (\varphi_{2n}^2, -\varphi_{2n}\varphi_{2n+1})$ satisfies Eq. (3.1). Then by Lemma 3.1(b) and Eq. (2.1),

$$
\begin{aligned}
(x, y) &= (k(-\varphi_{2n}\varphi_{2n+1}) - \varphi_{2n}^2 - 1, -\varphi_{2n}\varphi_{2n+1})\\
&= (-(k\varphi_{2n}\varphi_{2n+1} + \varphi_{2n}^2 + 1), -\varphi_{2n}\varphi_{2n+1})\\
&= (-\varphi_{2n+1}^2, -\varphi_{2n}\varphi_{2n+1})
\end{aligned}
$$

is a solution of Eq. (3.1).

Now since $(x, y) = (-\varphi_{2n+1}^2, -\varphi_{2n}\varphi_{2n+1})$ is a solution of Eq. (3.1), by Lemma 3.1(a) and Eq. (1.1),

$$
\begin{aligned}
(x, y) &= (-\varphi_{2n+1}^2, -k(-\varphi_{2n+1}^2) - (-\varphi_{2n}\varphi_{2n+1}))\\
&= (-\varphi_{2n+1}^2, k\varphi_{2n+1}^2 + (\varphi_{2n}\varphi_{2n+1}))\\
&= (-\varphi_{2n+1}^2, \varphi_{2n+1}\varphi_{2n+2})
\end{aligned}
$$

is a solution of Eq. (3.1).

Similar reasoning shows that if $(-\varphi_{2n+1}^2, \varphi_{2n+1}\varphi_{2n+2})$ satisfies Eq. (3.1), then $(\varphi_{2n+2}^2, \varphi_{2n+1}\varphi_{2n+2})$ satisfies Eq. (3.1), and if $(\varphi_{2n+2}^2, \varphi_{2n+1}\varphi_{2n+2})$ is a solution of Eq. (3.1), then $(\varphi_{2n+2}^2, -\varphi_{2n+2}\varphi_{2n+3})$ satisfies Eq. (3.1). □

Next we prove that the solutions described in Eq. (3.2) are all the solutions of Eq. (3.1). We first consider the positive solutions.

A similar argument to that of ([8] Theorem 1), proves the following theorem.

Theorem 3.3. *If positive integers k, x and y satisfy the equation $x^2 - kxy - y^2 + x = 0$, then there exist positive integers c,e such that $x = c^2$, $y = ce$ and $gcd(c, e) = 1$.*

We need some properties of the Pell equation

$$x^2 - Dy^2 = M \tag{3.3}$$

where D is a given positive integer not a perfect square and M is a given integer.

Clearly, if $x_0^2 - Dy_0^2 = M$ is fulfilled for some integers x_0, y_0 and $\varphi_0^2 - Dv_0^2 = 1$, for some integers φ_0, v_0, then for any integer n, the pair (x_n, y_n) defined as

$$x_n + y_n\sqrt{D} = (x_0 + y_0\sqrt{D})(\varphi_0 + v_0\sqrt{D})^n,$$

which satisfies Eq. (3.3).

Theorem 3.4. *Let the integer M satisfies $|M| < \sqrt{D}$. Then, any positive integer solution (s, t) of Eq. (3.3) with $gcd(s, t) = 1$ satisfies $s = h_n, t = k_n$ for some positive integer n, where $\frac{h_n}{k_n}$ is the n^{th} convergent to the infinite simple continued fraction of $\sqrt{D} = [a_0, a_1, ...]$.*

Proof. see ([10], Theorem 7.24). □

Theorem 3.5. *Let $[a_0, a_1, ...]$ be the infinite simple continued fraction of $\sqrt{D}$ and suppose that m_n and q_n are two sequences given by*

$$m_0 = 0,\ q_0 = 1,\ m_{n+1} = a_nq_n - m_n,\ q_{n+1} = (D - m_{n+1}^2)/q_n.$$

Then

a) *m_n and q_n are integers for any positive integer n,*
b) *$h_n^2 - Dk_n^2 = (-1)^{n+1}q_{n+1}$ for any integer $n \geq -1$.*

Proof. see ([10], Theorem 7.22). □

Now we are ready to prove that all positive solutions of the Eq. (3.1) are in the form $(x, y) = (\varphi_{2n}^2, \varphi_{2n-1}\varphi_{2n})$. Using Theorem 3.3, there exist positive integers c and e such that $x = c^2$, $y = ce$ and $gcd(c, e) = 1$. Substituting in Eq. (3.1),we have

$$c^2 - kce - e^2 + 1 = 0.$$

We can consider this equation as a quadratic equation with respect to the indeterminate c. This equation has integer solutions if and only if $\Delta = (k^2 + 4)e^2 - 4$, is a square. i.e, there exists an integer t such that

$$t^2 - (k^2 + 4)e^2 = -4. \tag{3.4}$$

And in this case

$$c = (ke \pm t)/2. \tag{3.5}$$

Now we solve Eq. (3.4). We first assume that k is odd. From Eq. (2.3),

$$\sqrt{k^2+4} = [k, \overline{(k-1)/2, 1, 1, (k-1)/2, 2k}].$$

Applying Theorem 3.5, with

$a_0 = k,\ a_{5n-4} = (k-1)/2,\ a_{5n-3} = 1,\ a_{5n-2} = 1,\ a_{5n-1} = (k-1)/2, a_{5n} = 2k,\ n \geq 0,$

we get two eventually periodic sequences

$$\{m_n\}_{n=0}^{\infty} = \{0, \overline{k, k-2, 2, k-2, k}\},$$

and

$$\{(-1)^{n+1} q_{n+1}\}_{n=-1}^{\infty} = \{1, \overline{-4, k, -k, 4, -1, 4, -k, k, -4, 1}\}. \tag{3.6}$$

Now we assume that (t, e) is a positive solution of Eq. (3.4). Then from Eq. (3.4) we deduce that $gcd(t, e) = 1$ or 2. The sequence in Eq. (3.6) and Theorem 3.5, imply that

$$\begin{aligned} h_{10n}^2 - (k^2+4)k_{10n}^2 &= -4, \\ h_{10n+4}^2 - (k^2+4)k_{10n+4}^2 &= -1, \\ h_{10n+8}^2 - (k^2+4)k_{10n+8}^2 &= -4 \end{aligned} \tag{3.7}$$

for all $n \geq 0$.

Now from Eq. (3.7) we deduce that

$$(2h_{10n+4})^2 - (k^2+4)(2k_{10n+4})^2 = -4.$$

Moreover all of the solutions of Eq. (3.4) are as follows

$$(t, e) = (h_{10n}, k_{10n}),\ (2h_{10n+4}, 2k_{10n+4}),\ (h_{10n+8}, k_{10n+8})\ n \geq 0.$$

From Eq. (3.5), the solutions (c, e) are

$$\begin{aligned} &((kk_{10n} + h_{10n})/2, k_{10n}), \\ &(kk_{10n+4} + h_{10n+4}, 2k_{10n+4}), \\ &((kk_{10n+8} + h_{10n+8})/2, k_{10n+8}), \end{aligned} \tag{3.8}$$

for all $n \geq 0$. Now using Theorem 2.2, and rearranging Eq. (3.8), we have

$$\begin{aligned} (c, e) &= ((k\varphi_{6n+1} + \varphi_{6n} + \varphi_{6n+2})/2, \varphi_{6n+1}) = (\varphi_{6n+2}, \varphi_{6n+1}), \\ (c, e) &= (k(\tfrac{1}{2}\varphi_{6n+3}) + \tfrac{1}{2}(\varphi_{6n+2} + \varphi_{6n+4}), 2(\tfrac{1}{2}\varphi_{6n+3})) = (\varphi_{6n+4}, \varphi_{6n+3}), \\ (c, e) &= ((k\varphi_{6n+5} + \varphi_{6n+4} + \varphi_{6n+6})/2, \varphi_{6n+5}) = (\varphi_{6n+6}, \varphi_{6n+5}), \end{aligned}$$

and finally from Theorem 3.3, $(x, y) = (c^2, ce)$. So

$$\begin{aligned} (x, y) = &(\varphi_{6n+2}^2, \varphi_{6n+1}\varphi_{6n+2}), \\ &(\varphi_{6n+4}^2, \varphi_{6n+3}\varphi_{6n+4}), \\ &(\varphi_{6n+6}^2, \varphi_{6n+5}\varphi_{6n+6}), \end{aligned}$$

and therefore $(x, y) = (\varphi_{2n}^2, \varphi_{2n-1}\varphi_{2n})$ for every positive integer n. Thus we proved

Theorem 3.6. *If k is an odd positive integer, then every positive solutions of $x^2 - kxy - y^2 + x = 0$ is of the form $(x, y) = (\varphi_{2n}^2, \varphi_{2n-1}\varphi_{2n})$.*

Now we consider the case when k is even. In this case from Eq. (2.3), $\sqrt{k^2+4} = [k, \overline{k/2, 2k}]$. Let

$$a_0 = k,\ a_{2n+1} = k/2,\ a_{2n+2} = 2k,$$

for all $n \geq 0$. We get two eventually periodic sequences

$$\{m_n\}_{n=0}^{\infty} = \{0, \overline{k}\}$$

and

$$\{(-1)^{n+1} q_{n+1}\}_{n=-1}^{\infty} = \{\overline{1, -4}\}.$$

From this and Theorem 3.5, we have

$$h_{2n}^2 - (k^2+4)k_{2n}^2 = -4 \tag{3.9}$$

for all $n \geq 0$. Moreover in this case all solutions of Eq. (3.4) are $(t, e) = (h_{2n}, k_{2n})$, and using Eq. (3.5), $(c, e) = ((kk_{2n} + h_{2n})/2, k_{2n})$. But from Theorem 2.4, $h_{2n} = \varphi_{2n} + \varphi_{2n+2}$ and $k_{2n} = \varphi_{2n+1}$. Substituting in Eq. (3.5) we get $(c, e) = ((k\varphi_{2n+1} + \varphi_{2n} + \varphi_{2n+2})/2, \varphi_{2n+1})$. Therefore $(x, y) = (c^2, ce) = (\varphi_{2n+2}^2, \varphi_{2n+1}\varphi_{2n+2})$. Thus we proved,

Theorem 3.7. *If k is a positive even integer, then every positive solution of the equation $x^2 - kxy - y^2 + x = 0$ is in the form $(x, y) = (\varphi_{2n}^2, \varphi_{2n-1}\varphi_{2n})$.*

Now we find all (not necessarily positive) solutions of the equation $x^2 - kxy - y^2 + x = 0$. First assume that $x > 0$ and $y < 0$. By substituting $x \to x$ and $y \to -y$, it is enough to consider the new equation $x^2 + kxy - y^2 + x = 0$ and its positive solutions

$$(x, y) = (\varphi_{2n}^2, \varphi_{2n}\varphi_{2n+1}).$$

Similarly if $x < 0$ and $y > 0$, then substituting $x \to -x$
and $y \to y$ and considering the equation $x^2 + kxy - y^2 - x = 0$, we have

$$(x, y) = (\varphi_{2n+1}^2, \varphi_{2n+1}\varphi_{2n+2}).$$

Finally if $x < 0$ and $y < 0$, then by substituting $x \to -x$
and $y \to -y$, it is enough to consider the equation $x^2 - kxy - y^2 - x = 0$ and find its positive solutions, that are

$$(x, y) = (\varphi_{2n+1}^2, \varphi_{2n}\varphi_{2n+1}).$$

In general and using the above discussions we have

Theorem 3.8. *If k is an integer, then all the solutions of the equation $x^2 - kxy - y^2 + x = 0$ are*

i) $(\varphi_{2n}^2, \varphi_{2n-1}\varphi_{2n})$,
ii) $(\varphi_{2n}^2, -\varphi_{2n}\varphi_{2n+1})$,
iii) $(-\varphi_{2n+1}^2, \varphi_{2n+1}\varphi_{2n+2})$,
iv) $(-\varphi_{2n+1}^2, -\varphi_{2n}\varphi_{2n+1})$,

where $n \geq 0$ is an integer.

Finally any solution of the equations $x^2 \pm kxy - y^2 \pm x = 0$, corresponds to the solution of the equation $x^2 - kxy - y^2 + x = 0$. In the following table, we summarize our calculation

Table 1. solution of the equations $x^2 \pm kxy - y^2 \pm x = 0$

equation	solutions
$x^2 - kxy - y^2 + x = 0$	$(\varphi_{2n}^2, \varphi_{2n-1}\varphi_{2n})$ $(\varphi_{2n}^2, -\varphi_{2n}\varphi_{2n+1})$ $(-\varphi_{2n+1}^2, \varphi_{2n+1}\varphi_{2n+2})$ $(-\varphi_{2n+1}^2, -\varphi_{2n}\varphi_{2n+1})$
$x^2 + kxy - y^2 + x = 0$	$(\varphi_{2n}^2, \varphi_{2n}\varphi_{2n+1})$ $(\varphi_{2n}^2, -\varphi_{2n-1}\varphi_{2n})$ $(-\varphi_{2n+1}^2, \varphi_{2n}\varphi_{2n+1})$ $(-\varphi_{2n+1}^2, -\varphi_{2n+1}\varphi_{2n+2})$
$x^2 - kxy - y^2 - x = 0$	$(\varphi_{2n+1}^2, \varphi_{2n}\varphi_{2n+1})$ $(\varphi_{2n+1}^2, -\varphi_{2n+1}\varphi_{2n+2})$ $(-\varphi_{2n}^2, \varphi_{2n}\varphi_{2n+1})$ $(-\varphi_{2n}^2, -\varphi_{2n-1}\varphi_{2n})$
$x^2 + kxy - y^2 - x = 0$	$(\varphi_{2n+1}^2, \varphi_{2n+1}\varphi_{2n+2})$ $(\varphi_{2n+1}^2, -\varphi_{2n}\varphi_{2n+1})$ $(-\varphi_{2n}^2, \varphi_{2n-1}\varphi_{2n})$ $(-\varphi_{2n}^2, -\varphi_{2n}\varphi_{2n+1})$

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