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Frames in 2-inner Product Spaces

Ali Akbar Arefijamaal^{*} and Ghadir Sadeghi Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

> E-mail: arefijamaal@hsu.ac.ir E-mail: ghadir54@gmail.com

ABSTRACT. In this paper, we introduce the notion of a frame in a 2inner product space and give some characterizations. These frames can be considered as a usual frame in a Hilbert space, so they share many useful properties with frames.

Keywords: 2-inner product space, 2-norm space, Frame, Frame operator.

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1. INTRODUCTION AND PRELIMINARIES

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [12] in 1952 to study some deep problems in nonharmonic Fourier series. Various generalizations of frames have been proposed; frame of subspaces [2, 6], pseudo-frames [18], oblique frames [10], continuous frames [1, 4, 14] and so on. The concept of frames in Banach spaces have been introduced by Grochenig [16], Casazza, Han and Larson [5] and Christensen and Stoeva [11].

The concept of linear 2-normed spaces has been investigated by S. Gahler in 1965 [15] and has been developed extensively in different subjects by many authors [3, 7, 8, 13, 14, 17]. A concept which is related to a 2-normed space is 2-inner product space which have been intensively studied by many mathematicians in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive

^{*}Corresponding author

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list of the related references can be found in the book [7]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let \mathcal{X} be a linear space of dimension greater than 1 over the field \mathbb{K} , where \mathbb{K} is the real or complex numbers field. Suppose that (.,.|.) is a \mathbb{K} -valued function defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ satisfying the following conditions:

 $(I1)(x, x|z) \ge 0$ and (x, x|z) = 0 if and only if x and z are linearly dependent, (I2)(x, x|z) = (z, z|x),

 $(I3)(y, x|z) = \overline{(x, y|z)},$

 $(\mathrm{I4})(\alpha x, y|z) = \alpha(x, y|z) \text{ for all } \alpha \in \mathbb{K},$

$$(I5)(x_1 + x_2, y|z) = (x_1, y|z) + (x_2, y|z)$$

(.,.|.) is called a 2-*inner product* on \mathcal{X} and $(\mathcal{X}, (.,.|.))$ is called a 2-*inner product* space (or 2-pre Hilbert space). Some basic properties of 2-inner product (.,.|.) can be immediately obtained as follows ([8, 13]):

- (0, y|z) = (x, 0|z) = (x, y|0) = 0,
- $(x, \alpha y|z) = \overline{\alpha}(x, y|z),$
- $(x, y | \alpha z) = |\alpha|^2 (x, y | z)$, for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{K}$.

Using the above properties, we can prove the Cauchy-Schwarz inequality

(1.1)
$$|(x,y|z)|^2 \le (x,x|z)(y,y|z).$$

Example 1.1. If $(\mathcal{X}, \langle ., . \rangle)$ is an inner product space, then the standard 2-inner product (., .|.) is defined on \mathcal{X} by

(1.2)
$$(x,y|z) = \begin{vmatrix} \langle x,y \rangle & \langle x,z \rangle \\ \langle z,y \rangle & \langle z,z \rangle \end{vmatrix} = \langle x,y \rangle \langle z,z \rangle - \langle x,z \rangle \langle z,y \rangle,$$

for all $x, y, z \in \mathcal{X}$.

In any given 2-inner product space $(\mathcal{X}, (., .|.))$, we can define a function $\|., .\|$ on $\mathcal{X} \times \mathcal{X}$ by

(1.3)
$$||x,z|| = (x,x|z)^{\frac{1}{2}},$$

for all $x, z \in \mathcal{X}$.

It is easy to see that, this function satisfies the following conditions:

(N1) $||x, z|| \ge 0$ and ||x, z|| = 0 if and only if x and z are linearly dependent, (N2) ||x, z|| = ||z, x||,

(N3) $\|\alpha x, z\| = |\alpha| \|x, z\|$ for all $\alpha \in \mathbb{K}$,

(N4) $||x_1 + x_2, z|| \le ||x_1, z|| + ||x_2, z||.$

Any function $\|.,.\|$ defined on $\mathcal{X} \times \mathcal{X}$ and satisfying the conditions (N1)-(N4) is called a 2-norm on X and $(\mathcal{X}, \|.,.\|)$ is called a *linear 2-normed space*. Whenever a 2-inner product space $(\mathcal{X}, (.,.|.))$ is given, we consider it as a linear 2-normed space $(\mathcal{X}, \|.,.\|)$ with the 2-norm defined by (1.3). In the present paper, we shall introduce 2-frames for a 2-inner product space and describe some fundamental properties of them. This implies that each element in the underlying 2-inner product space can be written as an unconditionally convergent infinite linear combination of the frame elements.

2. FRAMES IN THE STANDARD 2-INNER PRODUCT SPACES

Throughout this paper, we assume that \mathcal{H} is a separable Hilbert space, with the inner product $\langle ., . \rangle$ chosen to be linear in the first entry. We first review some basic facts about frames in \mathcal{H} , then try to define them in a standard 2-inner product space.

Definition 2.1. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *frame* for \mathcal{H} if there exist A, B > 0 such that

(2.1)
$$A \|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \le B \|f\|^2, \quad (f \in \mathcal{H}).$$

The numbers A, B are called *frame bounds*. The frame is called *tight* if A = B. Given a frame $\{f_i\}_{i=1}^{\infty}$, the frame operator is defined by

$$Sf = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i.$$

The series defining Sf converges unconditionally for all $f \in \mathcal{H}$ and S is a bounded, invertible, and self-adjoint operator. This leads to the frame decomposition:

$$f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle f_i, \qquad (f \in \mathcal{H}).$$

The possibility of representing every $f \in \mathcal{H}$ in this way is the main feature of a frame. The coefficients $\{\langle f, S^{-1}f_i \rangle\}_{i=1}^{\infty}$ are called frame coefficients. A sequence satisfying the upper frame condition is called a *Bessel sequence*. A sequence $\{f_i\}_{i=1}^{\infty}$ is Bessel sequence if and only if the operator $T : \{c_i\} \mapsto \sum_{i=1}^{\infty} c_i f_i$ is a well-defined operator from l^2 into \mathcal{H} . In that case T, which is called the *pre-frame operator*, is automatically bounded. When $\{f_i\}_{i=1}^{\infty}$ is a frame, the pre-frame operator T is well-defined and $S = TT^*$. For more details see [9, Section 5.1]. Also see [19] for a class of finite frames.

Let \mathcal{X} be a 2-inner product space. A sequence $\{a_n\}_{n=1}^{\infty}$ of \mathcal{X} is said to be convergent if there exists an element $a \in \mathcal{X}$ such that $\lim_{n\to\infty} ||a_n - a, x|| = 0$, for all $x \in \mathcal{X}$. Similarly, we can define a Cauchy sequence in \mathcal{X} . A 2-inner product space \mathcal{X} is called a 2-*Hilbert space* if it is complete. That is, every Cauchy sequence in \mathcal{X} is convergent in this space [17]. Clearly, the limit of any convergent sequence is unique and if $(\mathcal{X}, (., .|.))$ is the standard 2-inner product, then this topology is the original topology on \mathcal{X} .

Now we are ready to define 2-frames on a 2-Hilbert space.

Definition 2.2. Let $(\mathcal{X}, (., .|.))$ be a 2-Hilbert space and $\xi \in \mathcal{X}$. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements in \mathcal{X} is called a 2-frame (associated to ξ) if there exist A, B > 0 such that

(2.2)
$$A \|x, \xi\|^2 \le \sum_{i=1}^{\infty} |(x, x_i | \xi)|^2 \le B \|x, \xi\|^2, \quad (x \in \mathcal{X}).$$

A sequence satisfying the upper 2-frame condition is called a 2-*Bessel sequence*. In (2.2) we may assume that every x_i is linearly independent to ξ , by (1.1) and the property (I1).

The following proposition shows that in the standard 2-inner product spaces every frame is a 2-frame.

Proposition 2.3. Let $(\mathcal{H}, \langle ., . \rangle)$ be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ is a frame for \mathcal{H} . Then it is a 2-frame with the standard 2-inner product on \mathcal{H} .

Proof. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a frame with the bounds A, B and $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$. Then by using (2.1) and (1.2) for every $x \in \mathcal{H}$ we have

$$\sum_{i=1}^{\infty} |\langle x, x_i | \xi \rangle|^2 = \sum_{i=1}^{\infty} |\langle x - \langle x, \xi \rangle \xi, x_i \rangle|^2$$

$$\leq B ||x - \langle x, \xi \rangle \xi ||^2$$

$$\leq B(||x||^2 - |\langle x, \xi \rangle|^2)$$

$$= B(x, x|\xi).$$

The argument for lower bound is similar.

The converse of the above proposition is not true. In fact, by the following proposition, every 2-frame is a frame for a closed subspace of \mathcal{H} with codimension 1. For each $\xi \in \mathcal{H}$ we denote by L_{ξ} the subspace generated with ξ .

Proposition 2.4. Let $(\mathcal{H}, \langle ., . \rangle)$ be a Hilbert space and $\xi \in \mathcal{H}$. Every 2-frame associated to ξ is a frame for L_{ξ}^{\perp} .

Proof. If $\{x_i\}_{i=1}^{\infty}$ is a 2-frame with the bounds A, B then (2.2) implies that there exist A, B > 0 such that

$$A(\|x\|^2 - |\langle x,\xi\rangle|^2) \le \sum_{i=1}^{\infty} |\langle x - \langle x,\xi\rangle\xi, x_i\rangle|^2 \le B(\|x\|^2 - |\langle x,\xi\rangle|^2), \qquad (x \in \mathcal{H}).$$

Therefore, $\{x_i\}_{i=1}^{\infty}$ is a frame for the Hilbert space L_{ξ}^{\perp} .

Remark 2.5. Let \mathcal{H} be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ is a frame for \mathcal{H} with the frame operator S. If $\langle x_j, S^{-1}x_j \rangle = 1$ for some $j \in \mathbb{N}$, then $\{x_i\}_{i \neq j}$ is incomplete

and therefore it is not a frame for \mathcal{H} [9, Theorem 5.3.9]. Assume that $||x_j|| = 1$ and consider the standard 2-inner product on \mathcal{H} . It is not difficult to see that

$$\sum_{i=1}^{\infty} |(x, x_i | x_j)|^2 = \sum_{i=1, i \neq j}^{\infty} |(x, x_i | x_j)|^2.$$

Now the proof of Proposition 2.3 shows that $\{x_i\}_{i\neq j}$ is a 2-frame for \mathcal{H} associated to x_j .

3. Some properties of 2-frames

This section is devoted to establishing pre-frame and frame operator for a 2-frame. To extend a well-known result in Hilbert spaces to 2-inner product spaces.

Lemma 3.1. Let $(\mathcal{X}, (., .|.))$ be a 2-inner product space and $x, z \in \mathcal{X}$. Then

(3.1)
$$||x,z|| = \sup\{|(x,y|z)|; y \in \mathcal{X}, ||y,z|| = 1\}$$

Proof. By the Cauchy-Schwarz inequality (1.1) we observe that

$$(x, y|z) \le ||x, z|| ||y, z|| = ||x, z||$$

for every $y \in \mathcal{X}$ such that ||y, z|| = 1. Moreover, if $y = \frac{1}{||x, z||}x$, then ||y, z|| = 1and therefore (x, y|z) = ||x, z||.

For the remainder, we assume $(\mathcal{X}, (., .|.))$ is a 2-Hilbert space and L_{ξ} the subspace generated with ξ for a fix element ξ in \mathcal{X} . Denote by \mathcal{M}_{ξ} the algebraic complement of L_{ξ} in \mathcal{X} . So $L_{\xi} \oplus \mathcal{M}_{\xi} = \mathcal{X}$.

We first define the inner product $\langle ., . \rangle_{\xi}$ on \mathcal{X} as following:

$$\langle x,z\rangle_{\xi}=(x,z|\xi).$$

A straightforward calculations shows that $\langle ., . \rangle_{\xi}$ is a semi-inner product on \mathcal{X} . It is well-known that this semi-inner product induces an inner product on the quotient space \mathcal{X}/L_{ξ} as

$$\langle x + L_{\xi}, z + L_{\xi} \rangle_{\xi} = \langle x, z \rangle_{\xi}, \qquad (x, z \in \mathcal{X}).$$

By identifying \mathcal{X}/L_{ξ} with \mathcal{M}_{ξ} in an obvious way, we obtain an inner product on \mathcal{M}_{ξ} . Define

(3.2)
$$||x||_{\xi} = \sqrt{\langle x, x \rangle_{\xi}} \qquad (x \in \mathcal{M}_{\xi}).$$

Then $(\mathcal{M}_{\xi}, \|.\|_{\xi})$ is a norm space.

Now if $\{x_i\}_{i=1}^{\infty} \subseteq \mathcal{X}$ is a 2-frame associated to ξ with bounds A and B, then we can rewrite (2.2) as

$$A||x||_{\xi}^{2} \leq \sum_{i=1}^{\infty} |\langle x, x_{i} \rangle_{\xi}|^{2} \leq B||x||_{\xi}^{2}, \qquad (x \in \mathcal{M}_{\xi}).$$

That is, $\{x_i\}_{i=1}^{\infty}$ is a frame for \mathcal{M}_{ξ} . Let \mathcal{X}_{ξ} be the completion of the inner product space \mathcal{M}_{ξ} . Due to Lemma 5.1.2 of [9] the sequence $\{x_i\}_{i=1}^{\infty}$ is also a frame for \mathcal{X}_{ξ} with the same bounds. To summarize, we have the following theorem.

Theorem 3.2. Let $(\mathcal{X}, (., .|.))$ be a 2-Hilbert space. Then $\{x_i\}_{i=1}^{\infty} \subseteq \mathcal{X}$ is a 2-frame associated to ξ with bounds A and B if and only if it is a frame for the Hilbert space \mathcal{X}_{ξ} with bounds A and B.

By the above theorem, every question about 2-frames in a 2-Hilbert space can be solved as a question about frames in a Hilbert space.

Lemma 3.3. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-Bessel sequence in \mathcal{X} . Then the 2-pre frame operator $T_{\xi} : l^2 \to \mathcal{X}_{\xi}$ defined by

$$(3.3) T_{\xi}\{c_i\} = \sum_{i=1}^{\infty} c_i x_i$$

is well-defined and bounded.

Proof. Suppose $\{c_i\}_{i=1}^{\infty} \in l^2$, then by using (3.1) and (3.2) we have

$$\begin{split} \|\sum_{i=1}^{m} c_{i}x_{i} - \sum_{i=1}^{n} c_{i}x_{i}\|_{\xi}^{2} &= \|\sum_{i=1}^{m} c_{i}x_{i} - \sum_{i=1}^{n} c_{i}x_{i}, \xi\|^{2} \\ &= \sup\{|(\sum_{i=n}^{m} c_{i}x_{i}, y|\xi)|^{2}, y \in \mathcal{X}, \|y, \xi\| = 1\} \\ &\leq \sum_{i=n}^{m} |c_{i}|^{2} \sup\{\sum_{i=n}^{m} |(x_{i}, y|\xi)|^{2}, y \in \mathcal{X}, \|y, \xi\| = 1\} \\ &\leq B\sum_{i=n}^{m} |c_{i}|^{2} \end{split}$$

where B is the (upper) bound of $\{x_i\}_{i=1}^{\infty}$. This implies that $\sum_{i=1}^{\infty} c_i x_i$ is welldefined as an element of \mathcal{X}_{ξ} . Moreover, if $\{c_i\}_{i=1}^{\infty}$ is a sequence in l^2 , then an argument as above shows that $||T_{\xi}\{c_i\}||_{\xi} \leq \sqrt{B}||\{c_i\}||_2$. In particular, $||T_{\xi}|| \leq \sqrt{B}$.

Next, we can compute T_{ξ}^* , the adjoint of T_{ξ} as

$$T_{\xi}^* : \mathcal{X}_{\xi} \to l^2; \qquad T_{\xi}^* x = \{(x, x_i | \xi)\}_{i=1}^{\infty}.$$

It is easy to check that T_{ξ}^* is well-defined. Moreover, it follows by (2.2) that $||T_{\xi}^*|| \leq \sqrt{B}$.

Definition 3.4. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame associated to ξ with bounds A and B in a 2-Hilbert space \mathcal{X} . The operator $S_{\xi} : \mathcal{X}_{\xi} \to \mathcal{X}_{\xi}$ defined by

(3.4)
$$S_{\xi}x = \sum_{i=1}^{\infty} (x, x_i | \xi) x_i$$

is called the 2-frame operator for $\{x_i\}_{i=1}^{\infty}$.

Clearly, $S_{\xi} = T_{\xi}T_{\xi}^*$ and therefore $||S_{\xi}|| \leq B$. We can conclude the boundedness of S_{ξ} directly. Indeed, we see from (I3),(I4),(I5) and (3.1) that

$$\begin{split} \|S_{\xi}x\|_{\xi}^{2} &= \|S_{\xi}x,\xi\|^{2} \\ &= sup\{|(S_{\xi}x,y|\xi)|^{2}, \quad y \in \mathcal{X}, \|y,\xi\| = 1\} \\ &\leq sup\{\sum_{i=1}^{\infty} |(x,x_{i}|\xi)|^{2} \sum_{i=1}^{\infty} |(y,x_{i}|\xi)|^{2}, \quad y \in \mathcal{X}, \|y,\xi\| = 1\} \\ &\leq B^{2}\|x\|_{\xi}^{2}. \end{split}$$

Now we state some of the important properties of S_{ξ} .

Theorem 3.5. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame associated to ξ for a 2-Hilbert space $(\mathcal{X}, (., .|.))$ with 2-frame operator S_{ξ} and frame bounds A, B. Then S_{ξ} is invertible, self-adjoint, and positive.

Proof. Obviously, the operator S_{ξ} is self-adjoint. The inequality (2.2) means that

$$A \|x\|_{\xi}^{2} \leq \langle S_{\xi}x, x \rangle_{\xi} \leq B \|x\|_{\xi}^{2}, \qquad (x \in \mathcal{X}_{\xi}).$$

Hence, S_{ξ} is a positive element in the set of all bounded operators on the Hilbert space \mathcal{X}_{ξ} . More precisely, with symbols $AI \leq S_{\xi} \leq BI$ where I is the identity operator on \mathcal{X}_{ξ} . Furthermore,

$$||I - B^{-1}S_{\xi}|| = \sup_{||x||_{\xi}=1} |\langle (I - B^{-1}S_{\xi})x, x \rangle_{\xi}| \le \frac{B - A}{B} < 1.$$

This shows that S_{ξ} is invertible.

Corollary 3.6. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame in a 2-Hilbert space \mathcal{X} with frame operator S_{ξ} . Then each $x \in \mathcal{X}_{\xi}$ has an expansion of the following

$$x = S_{\xi} S_{\xi}^{-1} x = \sum_{i=1}^{\infty} (S_{\xi}^{-1} x, x_i | \xi) x_i.$$

Remark 3.7. If $\{x_i\}_{i=1}^{\infty}$ is a 2-frame associated to ξ , then every $x \in \mathcal{X}$ has a representation as

$$x = \alpha \xi + \sum_{i=1}^{\infty} c_i x_i,$$

for some $\alpha \in \mathbb{C}$ and $\{c_i\}_{i=1}^{\infty} \in l^2$. The coefficients $\{c_i\}_{i=1}^{\infty}$ are not unique, but the frame coefficients $\{(S_{\xi}^{-1}x, x_i|\xi)\}_{i=1}^{\infty}$ introduced in the Corollary 3.6 have minimal l^2 -norm among all sequences representing x, see Lemma 5.3.6 of [9].

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