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Block Diagonal Majorization on $\mathbf{C}_0$

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Abstract. Let $\mathbf{c}_0$ be the real vector space of all real sequences which converge to zero. For every $x, y \in \mathbf{c}_0$, it is said that y is block diagonal majorized by x (written $y \prec_b x$) if there exists a block diagonal row stochastic matrix R such that $y = Rx$. In this paper we find the possible structure of linear functions $T : \mathbf{c}_0 \to \mathbf{c}_0$ preserving $\prec_b$.

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1. Introduction

Let V and W be two linear spaces and let $\sim$ be a relation on both of V and W. A linear function $T : V \to W$ is said to be a linear preserver (strong linear preserver) of $\sim$ if for every $x, y \in V$, $Tx \sim Ty$ whenever $x \sim y$ ($Tx \sim Ty$ if and only if $x \sim y$). The topic of linear preservers is of interest to a large group of matrix theorists, see [8] for a survey of linear preserver problems. In this paper we shall designate by $\mathbf{M}_n$, $\mathbb{R}_m$ and $\mathbb{R}^n$ the set of all $n \times n$, $1 \times m$ and $n \times 1$ real matrices respectively. We recall that a matrix $R \in \mathbf{M}_n$ is row stochastic if all its entries are nonnegative and $Re = e$, where $e = (1, \ldots, 1)^t \in \mathbb{R}^n$. For vectors $x, y \in \mathbb{R}^n$, it is said that x is left matrix majorized by y (respectively x^t is right matrix majorized by y^t) and write $x \prec_l y$ (respectively $x^t \prec_r y^t$) if for some row stochastic matrix $R \in \mathbf{M}_n$; $x = Ry$ (respectively $x^t = y^t R$). It is known

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that, for $x, y \in \mathbb{R}^n$, $x \prec_l y$ if and only if $\min y \leq \min x \leq \max x \leq \max y$, here the maximum and the minimum are taken over all components of x and y. A characterization of linear functions $T : \mathbb{R}^n \to \mathbb{R}^p$ which preserve left matrix majorization $\prec_l$, can be found in [5, 6]. Note that the right and left matrix majorizations are essentially different and no characterization is known for functions $T : \mathbb{R}_n \to \mathbb{R}_p$ preserving right matrix majorization. There has been a great deal of interests in studying linear maps preserving or strongly preserving some special kinds of majorizations on some matrix spaces; for more information about types of majorizations see [4] and [7], and for their preservers see [1]-[3], and [5]-[6].

Definition 1.1. *(a) Let $\{n_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{N}$ and let $R_i \in \mathbf{M}_{n_i}$ be a row stochastic matrix for every $i \in \mathbb{N}$. Then $R = \bigoplus_{i=1}^{\infty} R_i$ is called a block diagonal row stochastic matrix.*
(b) For every $x, y \in \mathbf{c}_0$, it is said that y is block diagonal majorized by x (written $y \prec_b x$) if there exists a block diagonal row stochastic matrix R such that $y = Rx$. We write $x \sim_b y$ if $x \prec_b y$ and $y \prec_b x$.

In this paper, we find the possible structure of linear functions T on $\mathbf{c}_0$ preserving block diagonal majorization.

2. Block diagonal majorization

This section studies some properties of the notion of block diagonal majorization and we obtain some equivalent conditions for this concept.

Proposition 2.1. *Let $x, y \in \mathbf{c}_0$. If $x \prec_b y$, then $\inf y \leq \inf x \leq \sup x \leq \sup y$. Furthermore if $\inf x = \min x$ (respectively $\sup x = \max x$) then $\inf y = \min y$ (respectively $\sup y = \max y$).*

Proof. Assume that $x \prec_b y$, then there exists a block diagonal matrix $R = (r_{ij})$ such that $x = Ry$, and so for every $i \in \mathbb{N}$, $x_i = \sum_{j=1}^{\infty} r_{ij} y_j$ where $r_{ij} \geq 0$ and $\sum_{j=1}^{\infty} r_{ij} = 1$. It follows that $\inf y \leq x_i$ and hence $\inf y \leq \inf x$. One can show that $\sup x \leq \sup y$ with a similar argument. Suppose that $\inf x = \min x$. We just consider the case $x, y \geq 0$. Since $x \in \mathbf{c}_0$, there exists an integer $i \geq 1$ such that $\inf x = \min x = x_i = 0$. In the other hand $x_i = \sum_{j=1}^{\infty} r_{ij} y_j$, $r_{ij} \geq 0$ and $\sum_{j=1}^{\infty} r_{ij} = 1$, it follows that $y_k = 0$ for some $k \in \mathbb{N}$ and hence $\inf y = \min y$. $\square$

For $x, y \in \mathbb{R}^n$, one can easily show that $x \prec_l y$ if and only if $\min y \leq \min x \leq \max x \leq \max y$, but the following example shows that this is not true for $\prec_b$ on $\mathbf{c}_0$ (the converse of Proposition 2.1 is not true).

Example 2.2. *Let $x = (0, \frac{1}{2}, \frac{1}{3}, \ldots)^t$ and $y = (\frac{1}{2}, \frac{1}{3}, \ldots)^t$. Then $x, y \in \mathbf{c}_0$ and*

$$\inf y \leq \inf x \leq \sup x \leq \sup y,$$

but it is easy to verify that $x \not\prec_b y$.

The following proposition gives two equivalent conditions for $\prec_b$ on $\mathbf{c}_0$.

Proposition 2.3. *Let $x, y \in \mathbf{c}_0$. Then the following conditions are equivalent.*
(i) $x \prec_b y$.
(ii) There exists a subsequence $\{k_n\}_{n=0}^{\infty}$ *of sequence* $\{k\}_{k=0}^{\infty}$ *with* $k_0 = 0$ *such that for every* $j \in \mathbb{N}$, $(x_{k_j+1}, \ldots, x_{k_{j+1}})^t \prec_l (y_{k_j+1}, \ldots, y_{k_{j+1}})^t$.

(iii) There exists a subsequence $\{k_n\}_{n=0}^{\infty}$ *of sequence* $\{k\}_{k=0}^{\infty}$ *with* $k_0 = 0$ *such that for every* $j \in \mathbb{N}$,

$$\min_{k_j+1 \le i \le k_{j+1}} y_i \le \min_{k_j+1 \le i \le k_{j+1}} x_i \le \max_{k_j+1 \le i \le k_{j+1}} x_i \le \max_{k_j+1 \le i \le k_{j+1}} y_i \ .$$

Proof. $(i) \to (ii)$. Let $x \prec_b y$. Then there exists a diagonal block matrix $R = \bigoplus_{i=1}^{\infty} R_i$ such that $x = Ry$ and R_i is a $m_i \times m_i$ row stochastic matrix. Put $k_0 = 0$ and $k_n = \sum_{i=1}^{n} m_i$ then $(x_{k_j+1}, \ldots, x_{k_{j+1}})^t = R_{j+1}(y_{k_j+1}, \ldots, y_{k_{j+1}})^t$ and hence $(x_{k_j+1}, \ldots, x_{k_{j+1}})^t \prec_l (y_{k_j+1}, \ldots, y_{k_{j+1}})^t$.

$(ii) \to (i)$. Suppose that there exists a subsequence $\{k_n\}_{n=0}^{\infty}$ with $k_0 = 0$ and $(x_{k_j+1}, \ldots, x_{k_{j+1}})^t \prec_l (y_{k_j+1}, \ldots, y_{k_{j+1}})^t$ for every $j \in \mathbb{N}$. Then for every $j \in \mathbb{N}$ there exists a row stochastic $R_{j+1} \in \mathbf{M}_{k_{j+1}-k_j}$ such that $(x_{k_j+1}, \ldots x_{k_{j+1}})^t = R_{j+1}(y_{k_j+1}, \ldots, y_{k_{j+1}})^t$ and hence $x = Ry$, where $R = \bigoplus_{i=1}^{\infty} R_i$.

$(ii) \leftrightarrow (iii)$ is a direct consequence of definition of $\prec_l$.

$\square$

3. Linear preservers of $\prec_b$

In this section we will find the possible structure of linear functions $T : c_0 \to c_0$ which preserve $\prec_b$. The symbol e_i is used for the sequence $(0, \ldots, 0, 1, 0, \ldots)$ in c_0, where 1 is in the ith place.

Proposition 3.1. *Let $T : c_0 \to c_0$ preserve $\prec_b$. Suppose that $a := \sup Te_1$ and $b := \inf Te_1$, then for every $i \in \mathbb{N}$, $a = \sup Te_i = \max Te_i \ge 0$ and $b = \inf Te_i = \min Te_i \le 0$.*

Proof. First we show that $\sup Te_i = \sup Te_j$ and $\inf Te_i = \inf Te_j$ for every $i, j \in \mathbb{N}$. By Proposition 2.3, we have $e_i \prec_b e_j$ and $e_j \prec_b e_i$ for every $i, j \in \mathbb{N}$, it follows that $Te_i \prec_b Te_j$ and $Te_j \prec_b Te_i$. By Proposition 2.1, $\inf Te_j \le \inf Te_i \le \sup Te_i \le \sup Te_j$ and $\inf Te_j \le \inf Te_i \le \sup Te_j \le \sup Te_i$. This would imply $\inf Te_i = \inf Te_j$ and $\sup Te_i = \sup Te_j$. To complete the proof it is enough to show that $\sup Te_1 = \max Te_1$ and $\inf Te_1 = \min Te_1$. We just consider the case that Te_1 has only nonnegative components, so $Te_1 = (t_{11}, t_{21}, \ldots)^t \ge 0$. Since $\lim_{i \to \infty} t_{i1} = 0$ and $t_{i1} \ge 0$, $\sup Te_1 = \max Te_1$. Let $k \in \mathbb{N}$ be such that $t_{k1} = a$. By Proposition 2.3, $e_1 + e_2 \prec_b e_1$ and hence $Te_1 + Te_2 \prec_b Te_1$. By Proposition 2.1, $\max(Te_1 + Te_2) \le \max Te_1$ and consequently $t_{k1} + t_{k2} \le \max(Te_1 + Te_2) \le \max Te_1 = t_{k1}$. It follows that $t_{k2} = 0$, and hence $\inf Te_2 = \min Te_2 = 0$. Since $e_1 \prec_b e_2$, by Proposition 2.1 we conclude that $\inf Te_2 = \min Te_2 = \inf Te_1 = \min Te_1 = 0$. It is clear that $b \le 0 \le a$, as desired. $\square$

Lemma 3.2. *Let $T : c_0 \to c_0$ be a linear preserver of $\prec_b$. Suppose that a and b are as in Proposition 3.1. If $t_{ij} = a$ (respectively $t_{ij} = b$) for some $i, j \in \mathbb{N}$, then $t_{ik} \leq 0$ (respectively $t_{kj} \geq 0$) for all $k \in \mathbb{N} \setminus \{j\}$.*

Proof. Let $t_{ij} = a$ for some $i, j \in \mathbb{N}$. For every $k \in \mathbb{N} \setminus \{j\}$, $e_k + e_j \prec_b e_1$ and hence $(Te_k+Te_j) \prec_b Te_1$. Use Lemma 2.1 to write $\max(Te_k+Te_j) \leq \max Te_1$. Therefore $t_{ik}+a = t_{ik}+t_{ij} \leq \max(Te_k+Te_j) \leq \max Te_1 = a$ and consequently $t_{ik} \leq 0$. □

Note that a linear function $T : c_0 \to c_0$ preserves $\prec_b$ if and only if $\alpha T : c_0 \to c_0$ preserves $\prec_b$ for every nonzero $\alpha \in \mathbb{R}$. Let $T : c_0 \to c_0$ be a nonzero linear preserver of $\prec_b$. Assume that a and b are as in Proposition 3.1. Now, we consider two cases:
Case 1; If $|b| \leq |a|$. Then $T' := \frac{1}{a}T : c_0 \to c_0$ preserves $\prec_b$ and $0 \leq -b' := \min T'e_i \leq 1 = a' = \max T'e_i$.
Case 2; If $|b| > |a|$. Then $T' := \frac{-1}{b}T : c_0 \to c_0$ preserves $\prec_b$ and $0 \leq -b' := \min T'e_i \leq 1 = a' = \max T'e_i$.
Consequently, without loss of generality for every linear function $T : c_0 \to c_0$ preserving $\prec_b$ we may assume that $0 \leq -b \leq a = 1$.

Definition 3.3. *Let $T : c_0 \to c_0$ be a linear preserver of $\prec_b$. Assume that $a(=1)$ and b are as in Proposition 3.1. For every $k \in \mathbb{N}$, $\mathbf{I}_k := \{i \in \mathbb{N} : t_{ik} = 1\}$ and $\mathbf{J}_k := \{j \in \mathbb{N} : t_{jk} = b\}$.*

Theorem 3.4. *Let $T : c_0 \to c_0$ be a linear preserver of $\prec_b$. Assume that $0 \leq -b \leq a = 1$ are as in Proposition 3.1. Then for every $k \in \mathbb{N}$, there exist*
(i) $i_k \in \mathbf{I}_k$, such that for every $j \neq k$, $t_{i_k\ j} = 0$.
(ii) $j_k \in \mathbf{J}_k$, such that for every $j \neq k$, $t_{j_k\ j} = 0$.

Proof. Let $k \in \mathbb{N}$. For every $m \in \mathbb{N}$ there exists a large enough $N \in \mathbb{N}$ and there exist some $i_m \in \mathbf{I}_k$ such that $\min T(-Ne_k + e_{k+m}) = -N + t_{i_m k+m}$. It is clear that $(-Ne_k + e_{k+m}) \sim_b (-Ne_k + e_{k+m} + e_j)$ for every $j \in \mathbb{N} \setminus \{k, k+m\}$. Then $T(-Ne_k + e_{k+m}) \sim_b T(-Ne_k + e_{k+m} + e_j)$ and hence $(-N + t_{i_m k+m}) = \min T(-Ne_k + e_{k+m}) = \min T(-Ne_k + e_{k+m} + e_j) \leq (-N + t_{i_m k+m} + t_{i_m j})$. Consequently $t_{i_m j} \geq 0$ for every $j \in \mathbb{N} \setminus \{k, k+m\}$. Use Lemma 3.2 to conclude that $t_{i_m j} = 0$ for every $j \in \mathbb{N} \setminus \{k, k+m\}$. Since I_k is a finite set, there exist two distinct number $m, n \in \mathbb{N}$ such that $i_m = i_n$. Therefore $t_{i_m j} = 0$ for every $j \in (\mathbb{N} \setminus \{k, k+m\}) \cup (\mathbb{N} \setminus \{k, k+n\})$. Since $m \neq n$, $t_{i_m j} = 0$ for every $j \neq k$. With an argument same as the above one may prove (ii). □

Let A be an infinite matrix. Then the row indices of A and the column indices of A are $\{1, 2, \ldots\}$. Let α and β be nonempty sets of indices $\{1, 2, \ldots\}$. A submatrix $A[\alpha, \beta]$ is a matrix whose rows have indices α among the row indices of A, and whose columns have indices β among the column indices of A. Now, we can prove the main theorem of this paper. For a linear operator

$T : c_0 \to c_0$, we use the symbol $[T]$ for the infinite matrix with Te_j as jth column, i.e. $[T] = \big[Te_1\big|Te_2\big|\ldots\big|Te_j\big|\ldots\big]$.

Theorem 3.5. *Let $T : c_0 \to c_0$ be a linear preserver of $\prec_b$. Assume that $0 \leq -b \leq a = 1$ are as in Proposition 3.1. Then one of the following holds;*
(i) There exist infinite permutations P and Q such that P and bQ are submatrices of $[T]$.
(ii) $[T]$ is a row substochastic matrix and there exists an infinite permutation P such that P is a submatrix of $[T]$.

Proof. We consider two cases.
Case 1; Let $b \neq 0$. By Theorem 3.4 for every $k \in \mathbb{N}$ there exist $i_k, j_k \in \mathbb{N}$ such that
(i) $t_{k\ i_k} = 1$ and $t_{k\ j} = 0$ for every $j \in \mathbb{N} \setminus \{i_k\}$,
(ii) $t_{k\ j_k} = 1$ and $t_{k\ j} = 0$ for every $j \in \mathbb{N} \setminus \{j_k\}$.
Put $\alpha = \{i_1, i_2, \ldots\}$, $\beta = \{j_1, j_2, \ldots\}$, $P = [T][\alpha, \mathbb{N}]$ and $Q = \frac{1}{b}[T][\beta, \mathbb{N}]$. It is clear that P and Q are infinite permutations and P and bQ are submatrices of $[T]$, therefore (i) holds.
Case 2; Let $b = 0$. Then $t_{i\ j} \geq 0$ for all $i, j \in \mathbb{N}$. For every $m \in \mathbb{N}$, put $X_m = e_1 + \ldots + e_m$, it is clear that $X_m \prec_b e_1$ and hence $TX_m \prec_b Te_1$. By using Lemma 2.1, it follows that

$$0 \leq \sum_{j=1}^{m} t_{i\ j} = (TX_m)_i \leq \max Te_1 = 1, \ \forall\ i \in \mathbb{N},$$

where $(TX_m)_i$ is the i^{th} component of TX_m. Therefore the nonnegative infinite series $\sum_{j=1}^{\infty} t_{i\ j}$ is convergent and $\sum_{j=1}^{\infty} t_{i\ j} \leq 1$ for every $i \in \mathbb{N}$. Consequently $[T]$ is a row substochastic matrix. Since $a = 1$, with an argument same as the above $[T]$ has a submatrix which is an infinite permutation, therefore (ii) holds. □

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